Contents

Chapter 5

Angular momentum and rotationally symmetric problems

In this chapter, we will consider a special and still very important type of problems: Ones that are rotationally symmetric. In quantum theory, one frequently encounters such problems. The most prominent example is the Hydrogen atom, which will also form our most elaborate example at the end of the chapter. We will see that the concept of angular momentum plays an important role, which is why we also start with that.

5.1 Angular momentum

5.1.1 Definition

Let us consider the spatial degree of freedom of particles in $\mathbb{R}^3$. So the position X and the momentum operator P have three components, as usual labeled X_1, X_2, X_3 and P_1, P_2, P_3. We now define the angular momentum operator:

Angular momentum operator: This operator is defined as

$$L = X \times P. \tag{5.1}$$

This is of course exactly the same definition as angular momentum is defined classically. In components, we have

$$L_i = \epsilon_{i,j,k} X_j P_k, \tag{5.2}$$

where ϵ is the completely antisymmetric tensor, with

$$\epsilon_{i,j,k} = \begin{cases} 1 & \text{for even permutations of } (1,2,3), \\ -1 & \text{for odd permutations of } (1,2,3), \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

The angular momentum operator satisfies the following commutation relations:

$$[L_i, L_j] = i\hbar\epsilon_{i,j,k} L_k, \tag{5.4}$$

$$[L_i, X_j] = i\hbar\epsilon_{i,j,k} X_k, \tag{5.5}$$

$$[L_i, P_j] = i\hbar\epsilon_{i,j,k} P_k. \tag{5.6}$$

I hope the overloading of i is not confusing here: In one instance, it is of course an index, in the other the complex unit.

5.1.2 Angular momentum operator as generator of rotations

Let us first discuss that the angular momentum operator is the generator of rotations. This will not be used later, so if this should not be entirely clear, do not worry. Let us denote with $R(n, \theta)$ the rotation operator, one that rotates around the axis defined by $n \in \mathbb{R}^3$ by the angle $\theta \in [0, 2\pi)$. For example if n is the unit vector along the z axis, then, this is just a rotation around the z axis. Such a rotation $R(n, \theta)$ acts as follows

$$|\psi\rangle \mapsto R(n, \theta)|\psi\rangle, \tag{5.7}$$

where $R(n, \theta)|\psi\rangle$ is the rotated state vector. A rotation is now reflected as the following transformation on wave functions,

$$(R(n, \theta)\psi)(x) = \psi(\mathcal{R}(n, \theta)^{-1} x). \tag{5.8}$$

Here, $\mathcal{R}$ is the rotation matrix actually acting in $\mathbb{R}^3$. Is this confusing? I hope not. R is the rotation operator actually acting in the Hilbert space. $\mathcal{R}$ is the rotation operator in $\mathbb{R}^3$. Mathematically, such a construction is called a representation. Intuitively, we turn the coordinates by $\mathcal{R}^{-1}$, and this is reflected by applying R in Hilbert space. Why the inverse? Well, this is actually a matter of definition. It makes a lot of sense for some reasons, but let us for the moment being just call it a convention. The angular momentum operator being the generator of rotations means the following:

Angular momentum as generator of rotations: For every $n \in \mathbb{R}^3$ and every angle $\theta \in [0, 2\pi)$, we have that

$$R(n, \theta) = \exp\left(-\frac{i}{\hbar}\theta n \cdot L\right). \tag{5.9}$$

In just the same way, it should be mentioned, the momentum operator is the generator of translations.

5.1.3 Spectrum of the angular momentum operator

The operator L^2 commutes with each component of L,

$$[L^2, L_j] = 0,\ j = 1, 2, 3. \tag{5.10}$$

The same is true, by the way, for any operator for which

$$R(n, \theta) A R(n, \theta)^\dagger \tag{5.11}$$

for any $n \in \mathbb{R}^3$ and any $\theta \in [0, 2\pi)$, so any rotationally invariant operator. Therefore, the observables L^2 and L_3 are compatible, and they can be simultaneously diagonalized. Why L_3? This is just a convention, any direction would have done the job. Let us be specific and choose this z direction. Let us now identify the eigenvalues of L^2 and L_3. Let us for that purpose define

$$L_\pm = L_1 \pm i L_2. \tag{5.12}$$

These operators have the following properties:

$$L_\pm^\dagger = L_\mp, \tag{5.13}$$

$$[L_3, L_\pm] = i\hbar L_2 \pm L_1 = \pm\hbar L_\pm, \tag{5.14}$$

$$[L_+, L_-] = -2i[L_1, L_2] = 2\hbar L_3, \tag{5.15}$$

$$[L^2, L_\pm] = 0. \tag{5.16}$$

It is not difficult to verify all of these commutation relations, use

$$L^2 = \frac{1}{2}(L_+ L_- + L_- L_+) + L_3^2. \tag{5.17}$$

Let us denote with $|l, m\rangle$ the normalized eigenvectors of L^2 and L_3: We write

$$L^2 |l, m\rangle = \hbar l(l+1) |l, m\rangle, \tag{5.18}$$

$$L_3 |l, m\rangle = \hbar m |l, m\rangle, \tag{5.19}$$

for reasons that will become clear in a second, where we will also specify what values l and m can take. We find

$$L_3 L_\pm |l, m\rangle = L_\pm L_3 |l, m\rangle \pm \hbar L_\pm |l, m\rangle, \tag{5.20}$$

that is to say,

$$L_3 (L_\pm |l, m\rangle) = \hbar(m \pm 1)\, (L_\pm |l, m\rangle)\,. \tag{5.21}$$

That means that unless $L_\pm|l,m\rangle$ is the zero vector, it is again an eigenvector of L_3, but with eigenvalue $\hbar(m\pm 1)$. What is more

$$L^2(L_\pm|l,m\rangle) = L_\pm L^2|l,m\rangle = \hbar^2 l(l+1)(L_\pm|l,m\rangle). \tag{5.22}$$

Therefore, $(L_\pm|l,m\rangle)$ is an eigenvector of L^2 with the same eigenvalue as $|l,m\rangle$.

We can go further than that. Considering the norm

$$\begin{aligned} \|L_\pm|l,m\rangle\|^2 &= \langle l,m|L_\mp L_\pm|l,m\rangle = \langle l,m|(L^2 - L_3^2 \mp \hbar L_3)|l,m\rangle \\ &= \hbar^2\left(l(l+1) - m^2 \mp m\right). \end{aligned} \tag{5.23}$$

Therefore, making use of normalization

$$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l,m\rangle. \tag{5.24}$$

Strictly speaking, the latter relation only follows up to a phase factor $e^{i\phi}$ with $\phi \in [0, 2\pi)$, but we have taken the standard convention where this phase factor vanishes. Since this norm is non-negative, we must have that

$$l(l+1) - m(m\pm 1) \geq 0. \tag{5.25}$$

From this it follows that

$$-l \leq m \leq l. \tag{5.26}$$

We also find that

$$L_+|l,m\rangle = 0 \tag{5.27}$$

if and only if $m = l$ and

$$L_-|l,m\rangle = 0 \tag{5.28}$$

if and only if $m = -l$. One can recursively hence find all values of m: Starting from $|l,l\rangle$, one finds

$$L_-|l,l\rangle \sim |l,l-1\rangle, \tag{5.29}$$

$$(L_-)^2|l,l\rangle \sim |l,l-2\rangle, \tag{5.30}$$

and so on. In order to to have this lead to the value $-l$, we have to have a $k \in \mathbb{N}$ such that $l - k = -l$. Hence, $l = k/2$.

This is quite interesting. Purely from the algebraic structure, so using the commutation relations, we have found that l can take either the values

$$l = 0, 1, 2, \ldots \tag{5.31}$$

or

$$l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots. \tag{5.32}$$

The values for m are

$$m = -l, -l+1, \ldots, l-1, l. \tag{5.33}$$

In other words, the following is true:

Spectrum of the angular momentum operator: The two operators L^2 and L_3 commute. Their eigenvectors are denoted as $\{|l, m\rangle\}$. Here, l can be either a positive integer or positive half-integer, with m taking integer-spaced values between $-l$ and l.

5.1.4 Angular momentum operator in the position representation

After all this algebraic beauty, let us get dirty again and compute the angular momentum operator in the position representation. Unsurprisingly, polar coordinates are a useful coordinate system for that. To start with, in the position representation, the angular momentum operator takes the form

$$L = \frac{\hbar}{i} X \times \nabla. \tag{5.34}$$

We remind ourselves of the form of the ∇ differential operator in polar coordinates. This is

$$\nabla = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}. \tag{5.35}$$

We therefore get

$$L_3 = \frac{\hbar}{i}\left(-\sin(\phi)\frac{\partial}{\partial\theta} - \cos(\phi)\mathrm{ctg}(\theta)\frac{\partial}{\partial\phi}\right), \tag{5.36}$$

$$L_2 = \frac{\hbar}{i}\left(\cos(\phi)\frac{\partial}{\partial\theta} - \sin(\phi)\mathrm{ctg}(\theta)\frac{\partial}{\partial\phi}\right), \tag{5.37}$$

$$L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{5.38}$$

This means that,

$$L_\pm = \hbar e^{\pm i\theta}\left(\pm\frac{\partial}{\partial\theta} + i\mathrm{ctg}(\theta)\frac{\partial}{\partial\phi}\right), \tag{5.39}$$

$$L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right)\right). \tag{5.40}$$