Contents

Chapter 7

Addition of angular momenta

We know turn to the problem of "adding angular momenta". This is a basic problem that is encountered whenever one has a spin degree of freedom and an orbital angular momentum operator. Then it often makes sense to think of a total angular momentum operator. There are some subtleties involved then, however, which we will take care of in this chapter.

7.1 Spin and the general problem of adding angular momenta

7.1.1 Spin

We already have a clear understanding of the spin degree of freedom. We know that the Hilbert space $\mathcal{H}$ of a particle is given by

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2. \tag{7.1}$$

This means, of course, if we define the *spin operator* S as

$$S = \frac{\hbar}{2}\sigma, \tag{7.2}$$

with σ being the vector of Pauli matrices, we have that

$$[S, X] = 0, \tag{7.3}$$

$$[S, P] = 0, \tag{7.4}$$

$$[S, L] = 0. \tag{7.5}$$

This is obvious from one perspective, but may take a moment of thought from another. S on the one hand and X, P, and L on the other act on different degrees of freedom, so on different factors in the tensor product. Hence, they clearly commute, by the very definition, and we do not have to compute anything. Also, we have that

$$[S_i, S_j] = i\hbar \sum_k \varepsilon_{i,j,k} S_k, \tag{7.6}$$

for all i, j, so the spin operator satisfies the commutation relation of an angular momentum operator.

7.1.2 General problem

Let us hence assume that a particle has a spatial degree of freedom as well as one associated with spin. Therefore, we have a L and a S operator. It makes sense to think of the *total angular momentum operator*

$$J = L + S. \tag{7.7}$$

Or, think of two electrons with a spin: Then one would like to again consider the angular momentum operator

$$J = S^{(1)} + S^{(2)}. \tag{7.8}$$

Generally, let us consider the problem of adding

$$J = J^{(1)} + J^{(2)} \tag{7.9}$$

where $J^{(1)}$ and $J^{(2)}$ satisfy the commutation relations of Eq. (7.11). If $J^{(1)}$ and $J^{(2)}$ belong to different degrees of freedom (such as above in Eq. (7.7, 7.8), then they will surely commute

$$[J^{(1)}, J^{(2)}] = 0. \tag{7.10}$$

It also follows that the components of J satisfy

$$[J_i, J_j] = i\hbar \sum_k \varepsilon_{i,j,k} J_k, \tag{7.11}$$

for all i, j, so J is again an angular momentum operator. And all properties that we know of angular momenta apply to J, equally as they have applied to $J^{(1)}$ and $J^{(2)}$ individually.

Let us denote the eigenvectors of $J^{(1)}$ and $J^{(2)}$ by

$$\{|j^{(1)}, m^{(1)}\rangle : m^{(1)} = -j^{(1)}, \dots, j^{(1)}\rangle\}, \tag{7.12}$$

$$\{|j^{(2)}, m^{(2)}\rangle : m^{(2)} = -j^{(2)}, \dots, j^{(2)}\rangle\}, \tag{7.13}$$

$$\tag{7.14}$$

respectively. From these we can form the tensor products

$$\{|j^{(1)}, m^{(1)}; j^{(2)}, m^{(2)}\rangle = |j^{(1)}, m^{(1)}\rangle \otimes |j^{(2)}, m^{(2)}\rangle\}. \tag{7.15}$$

These vectors are the eigenvectors of

$$(J^{(1)})^2,\ J_3^{(1)},\ (J^{(2)})^2,\ J_3^{(2)} \tag{7.16}$$

by definition, with eigenvalues

$$\hbar^2 j^{(1)}(j^{(1)}+1),\ \hbar m^{(1)},\ \hbar^2 j^{(2)}(j^{(2)}+1),\ \hbar m^{(2)}, \tag{7.17}$$

by definition: This notation might look a bit clumsy, but this is just what we already know. These vectors are also eigenvalues of the third component J_3 of J, with eigenvalue $\hbar(m^{(1)}+m^{(2)})$. But they are *not* eigenvectors of J^2, since

$$[J^2, J_3^{(1)}] \neq 0,\ [J^2, J_3^{(2)}] \neq 0. \tag{7.18}$$

However, for many applications, one would like to find eigenvectors of J^2 in a similar way as we had known eigenvectors of $(J^{(1)})^2$ and $(J^{(2)})^2$ individually. In yet other words, we want to find the simultaneous eigenvectors of

$$J^2,\ J_3,\ (J^{(1)})^2,\ (J^{(2)})^2. \tag{7.19}$$

We approach this problem by first looking at two spins, then the orbital angular momentum and a spin, and then have a look at the general problem.

7.2 Coupling of two spins

We have two spin degrees of freedom $S^{(1)}$ and $S^{(2)}$ and the total angular momentum operator

$$J = S^{(1)} + S^{(2)}. \tag{7.20}$$

The eigenvectors

$$\{|0,0\rangle,\ |0,1\rangle,\ |1,0\rangle,\ |1,1\rangle\} \tag{7.21}$$

form the eigenvectors of $(S^{(1)})^2$, $(S^{(2)})^2$, $S_3^{(1)}$, $S_3^{(2)}$. More specifically,

$$J_3|1,1\rangle = \hbar|1,1\rangle, \tag{7.22}$$
$$J_3|1,0\rangle = 0, \tag{7.23}$$
$$J_3|0,0\rangle = -\hbar|0,0\rangle, \tag{7.24}$$
$$J_3|0,1\rangle = 0. \tag{7.25}$$

What is more, we have that

$$\begin{aligned} J^2 &= (S^{(1)})^2 + (S^{(2)})^2 + 2S^{(1)}\cdot S^{(2)} \\ &= \frac{3}{2}\hbar^2 + 2S_3^{(1)}S_3^{(2)} + S_+S_- + S_-S_+. \end{aligned} \tag{7.26}$$

We also know the following

$$J^2|1,1\rangle = \left(\frac{3}{2}\hbar^2 + 2\left(\frac{\hbar}{2}\right)^2\right)|1,1\rangle = 2\hbar^2|1,1\rangle, \quad (7.27)$$

$$J^2|0,0\rangle = 2\hbar^2|0,0\rangle. \quad (7.28)$$

The vectors $|1,1\rangle$ and $|0,0\rangle$ therefore have total spin 1 and a z-component of $\pm\hbar$. The missing eigenvectors for total spin 1 we get by applying J_- to $|1,1\rangle$ and normalize

$$\frac{1}{\hbar\sqrt{2}}J_-|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right). \quad (7.29)$$

In this way, we have found all three eigenvectors with total spin 1. There is another eigenvector, for total spin 0, given by

$$\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right). \quad (7.30)$$

For this we have that

$$J_3\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) = 0,\ J^2\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) = 0. \quad (7.31)$$

Total angular momentum eigenvectors for two spins: In the notation $|J,m\rangle_J$ of the total spin operator we have that

$$|1,1\rangle_J = |1,1\rangle, \quad (7.32)$$

$$|1,0\rangle_J = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right), \quad (7.33)$$

$$|1,-1\rangle_J = |0,0\rangle, \quad (7.34)$$

$$|0,0\rangle_J = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right). \quad (7.35)$$

The first three eigenvectors are called *triplet vectors* – spanning the three-dimensional *triplet eigenspace* – for obvious reasons. The last one is the *singlet*. Two of them are product vectors, and the other two are maximally entangled.

7.3 Adding orbital angular momentum and a spin

We have already seen what strategy one can follow to construct eigenvectors of the total angular momentum. Before we turn to the structure of the general problem, let us look at the other most important problem, that of coupling an orbital angular momentum degree of freedom and a spin, so that now

$$J = L + S. \quad (7.36)$$