Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 2

Due date: 10:15 02/11/2018 J. Eisert

1. **Basic Quantum Mechanics Recap Continued** (16 points)
Consider the two-particle Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, and the Hamiltonian $H = X \otimes Z$, for Pauli operators X and Z. Suppose we prepare the states

$$|\psi_1\rangle = |00\rangle \tag{1}$$
$$|\psi_2\rangle = |11\rangle \tag{2}$$
$$|\psi_3\rangle = (1/\sqrt{3})|01\rangle + (\sqrt{2/3})|10\rangle, \tag{3}$$

with probabilities $p_1 = 1/2$, $p_2 = 1/8$ and $p_3 = 3/8$ respectively.

   (a) Write down the density operator ρ for this ensemble, both in bra-ket notation and explicitly a matrix, with respect to the basis for $\mathcal{H}$ built from the computational basis for a single particle, i.e. the basis $\{|0\rangle = (1,0)^T, |1\rangle = (0,1)^T\}$ for $\mathbb{C}^2$ (2 points).

   (b) Write down the unitary operator, both in bra-ket notation, and explicitly as a matrix, for a basis transformation from the computational basis for $\mathcal{H}$ to the basis built from the single particle basis $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$ (2 points).

   (c) Write down the density operator ρ for the above ensemble, again both in bra-ket notation and explicitly as a matrix, but with respect to the $\{|+\rangle, |-\rangle\}$ basis (2 points).

   (d) If $\rho(0) = \rho$ from (a), calculate $\rho(t)$ in the Schrödinger picture, using any basis (6 points).

   (e) Define $Y_1 = Y \otimes \mathbb{1}$. Working in the Heisenberg picture, what is $Y_1(t)$ if $Y_1(0) = Y_1$. (2 points)

   (f) Calculate $\langle Y_1(t)\rangle$ in both the Heisenberg and Schrödinger picture. (2 points)

2. **Identical Particles and Permutations 1** (8 points)
Consider the single particle Hilbert space $\mathcal{H}_s = \mathbb{C}^2$.

   (a) For both the two particle and three particle Hilbert spaces - i.e. $\mathcal{H}_s^{\otimes 2}$ and $\mathcal{H}_s^{\otimes 3}$ respectively - Write down an explicit basis for both the symmetric and anti-symmetric subspaces, if such a subspace exists. ($4\times$ 2 points)

3. **Identical Particles and Permutations 2** (14 points)
Consider a system formed by three identical particles, with wave functions $\psi(ijk)$, where $i \neq j \neq k$ take the values $1, 2, 3$. We denote the permutation group on 3 elements as S_3. Now consider the following three states:

$$\psi_a(123) = \frac{1}{\sqrt{(6)}}[\psi(123) + \psi(132) + \psi(231) + \psi(213) + \psi(312) + \psi(321)]$$
$$\psi_b(123) = \frac{1}{\sqrt{(6)}}[\psi(123) - \psi(132) + \psi(231) - \psi(213) + \psi(312) - \psi(321)]$$
$$\psi_c(123) = \frac{1}{2\sqrt{(3)}}[2\psi(123) + \psi(132) + 2\psi(213) - \psi(231) - \psi(312) - \psi(321)]$$

(a) Identify which of the above states are (i) fully symmetric, (ii) fully anti-symmetric (iii) of mixed symmetry, under the action of S_3. (2 points)

(b) Show explicitly that for the state of mixed symmetry, which we denote as ψ_m, there exists some permutation $P \in S_3$ such that $P\psi_m \neq \lambda\psi_m$. What are the physical implications of this? (2 points)

(c) Write down explicitly the relevant symmetrization and anti-symmetrization operator, S and A respectively. (2 points)

(d) Obtain $S\psi_x$ and $A\psi_x$ for all $x \in \{a, b, c\}$. (6 points)

(e) Given a Hamiltonian H, show that if $[H, P] = 0$ for all $P \in S_3$, then $[U(t), P] = 0$ for all $P \in S_3$, where $U(t)$ is the time evolution operator under H. (2 points)