Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 8

Due date: 10:15 14/12/2018 J. Eisert

1. **Bogoliubov Theory of the Weakly Interacting Bose Gas**(2 + 4 points)

In lectures you utilized the following Bogoliubov transformation as a tool for studying the weakly interacting Bose gas:

$$b_k = u_k a_k + v_k a_{-k}^\dagger, \tag{1}$$
$$b_k^\dagger = u_k a_k^\dagger + v_k a_{-k}. \tag{2}$$

In order to ensure the operators b satisfy the Bose commutation relations, it is necessary to enforce

$$u_k^2 - v_k^2 = 1. \tag{3}$$

Additionally, we saw that in order to ensure that non-diagonal terms of the transformed Hamiltonian vanish, it is necessary to enforce

$$\Big(\frac{k^2}{2m} + nV_k\Big) u_k v_k + \frac{n}{2} V_k (u_k^2 + v_k^2) = 0. \tag{4}$$

(a) Derive explicitly the inverse of the Bogoliubov transformation given in eqns. (1) and (2).

(b) Equations (3) and (4) specify a system of equations which can be used to solve for u_k and v_k. Verify explicitly that

$$u_k^2 = \frac{w_k + \left(\frac{k^2}{2m} + nV_k\right)}{2w_k},$$
$$v_k^2 = \frac{-w_k + \left(\frac{k^2}{2m} + nV_k\right)}{2w_k} = \frac{(nV_k)^2}{2\omega_k(\omega_k + \frac{k^2}{2m} + nV_k)},$$
$$u_k v_k = -\frac{nV_k}{2\omega_k}$$

where w_k is defined as

$$w_k = \left(\Big(\frac{k^2}{2m} + nV_k\Big)^2 - \Big(nV_k\Big)^2\right)^{\frac{1}{2}}.$$

2. **Coherent States**(20 points)

Coherent states are a convenient basis when working with bosonic harmonic-oscillator-like Hamiltonians. The dynamics of these coherent states resemble the oscillatory behaviour of a classical harmonic oscillator. They are widely used in quantum optics and laser physics. In this exercise you will derive their properties. Consider the harmonic oscillator Hamiltonian:

$$\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}), \quad \text{with} \quad \hat{n} = a^\dagger a. \tag{5}$$

Coherent states are defined as the eigenstates of the annihilation operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{6}$$

where, sice a is not Hermitian, $\alpha = |\alpha|e^{i\phi}$ is complex.

(a) Find the expectation value $\bar{n}$ and $\bar{H}$ of $\hat{n}$ and $\hat{H}$ for the coherent state $|\alpha\rangle$. (2 Points)

(b) Show that the phase shifting operator $U(\theta) = e^{-i\theta\hat{n}}$ adds a phase to the annihilation operator a: (2 Points)

$$U(\theta)^\dagger a U(\theta) = a e^{-i\theta} \tag{7}$$

and that it shifts the phase of a coherent state:

$$U(\theta)|\alpha\rangle = |\alpha e^{-i\theta}\rangle. \tag{8}$$

(c) Consider now the so called displacement operator: (4 Points)

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \tag{9}$$

This operator is the 'creation' operator for the coherent states. Before proving this claim, show that this operator satisfies the following properties:

i. Show that it can also be written as:

$$D(\alpha) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}. \tag{10}$$

[Hint: use the property $e^{A+B} = e^A e^B e^{-[A,B]/2}$ valid when $[[A,B],B] = [[A,B],B] = 0$ (check that this is satisfied in our case)].

ii. Show that is a unitary operator:

$$D(\alpha)D(\alpha)^\dagger = \mathbb{1}, \tag{11}$$

Also check that $D(\alpha)^\dagger = D(-\alpha)$.

iii. Show that:

$$D(\alpha)^\dagger a D(\alpha) = a + \alpha. \tag{12}$$

iv. Finally show that:

$$D(\alpha)|0\rangle = |\alpha\rangle, \tag{13}$$

where $|0\rangle$ is the vacuum ($a|0\rangle = 0$).

(d) The coherent states can be expressed in terms of the eigenstates $|n\rangle$ of the number operator $\hat{n}$: (6 Points)

$$|\alpha\rangle = \sum |n\rangle\langle n|\alpha\rangle. \tag{14}$$

i. Find an expression for $\langle n|\alpha\rangle$ in terms of $\langle 0|\alpha\rangle$. [Hint: use the defining equation of the coherent states (6)].

ii. Find $\langle 0|\alpha\rangle$.

iii. Show that:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_0^\infty \frac{(\alpha a^\dagger)^n}{n!} |0\rangle. \tag{15}$$

(e) Calculate the probability of finding n bosons in the state $|\alpha\rangle$ and express that in terms of the expectation value $\bar{n}$. You should find a Poisson distribution. (2 Points)

(f) To finish show that the basis of coherent states is complete: (4 Points)

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}, \tag{16}$$

where the integration is done over the complex plane. [Hint: use the completeness of the number basis $\sum |n\rangle\langle n| = \mathbb{1}$]

Final remark: coherent states are not orthogonal $\langle\alpha|\beta\rangle \neq 0$. You can check this as an optional exercise.