Statistical Mechanics WS 2013/14 Sheet 12

Please hand in your solutions (in pairs) before the Monday lecture.

Problem 1 : Photon Gas (20 points)

Consider a photon gas in a volume V at temperature T. The energy of the photon is $\epsilon = pc$ (or $= \hbar\omega$), where p is the momentum and ω is the frequency.

a) (4 points) Using canonical ensemble, calculate the average energy. Use $\epsilon = pc$.

b) (4 points) Does the above result obey the "equipartition theorem"? If not, explain what is the condition for the Hamiltonian to satisfy the equipartition theorem.

c) (4 points) Find the chemical potential of the photon.

d) (4 points) Find the integral form for the average number of photons. How does this depend on T?

e) (4 points) Find the integral form for the average energy of photons. How does this depend on T?

[Hint: For d) and e) use $\epsilon = \hbar\omega$ and make the integrals dimensionless.]

Problem 2 : The Planck Distribution (20 points)

The Planck distribution is photon's radiated power R per unit area per unit wavelength λ:

$$\frac{dR}{d\lambda} = \frac{2\pi hc^2}{\lambda^5(e^{hc/(\lambda k_B T)} - 1)}. \tag{1}$$

a) (10 points) Using the above distribution, express the number of photons per unit area per second per unit wavelength. [Hint: Energy per photon is hc/λ.]

b) (5 points) Using the sun's temperature $T = 6000$ K and wavelength range of the visible light 400 nm $\leq \lambda \leq$ 700 nm, Calculate the number of photons radiated from the sun per second per unit area [meter^2].

c) (5 points) What is number of photons radiated from the "cosmic background" per second per unit area [meter^2]? Use $T = 2.7$ K.

Problem 3 Stoner ferromagnetism (20 points)

The conduction electrons in a metal can be treated as a gas of fermions of spin 1/2 (with up/down degeneracy), and density $n = N/V$. The Coulomb repulsion favors wave functions that are anti-symmetric in position coordinates, thus keeping the electrons apart. Because of the full (position *and* spin) anti-symmetry of fermionic wave functions, this interaction may be approximated by an effective spin-spin coupling that favors states with parallel spin. In this simple approximation, the net effect is described by an interaction energy

$$U = \alpha\frac{N_+ N_-}{V} \tag{2}$$

where N_+ and $N_- = N - N_+$ are the numbers of electrons with up and down spins, and V is the volume.

a) The ground state has two fermi seas filled by the spin-up and spin-down electrons. Express the corresponding fermi wavevectors $k_{F\pm}$ in terms of the densities $n_\pm = N_\pm/V$.

b) Calculate the kinetic energy density of the ground state as a function of the densities $n_\pm$, and fundamental constants.

c) Assuming small deviations $n_\pm = \frac{n}{2} \pm \delta$ from the symmetric state, expand the kinetic energy to fourth order in δ.

d) Express the spin-spin interaction density U/V in terms of n and δ. Find the critical value of α_c, such that for $\alpha > \alpha_c$ the electron gas can lower its total energy by spontaneously developing a magnetization. (This is known as the *Stoner instability.*)

e) Explain qualitatively, and sketch the behaviour of, the spontaneous magnetization as a function of α.

Problem 4 Spin 1/2 fermions in an external magnetic field (20 points)

Consider an ideal gas of N spin 1/2 fermions confined to a volume V at zero temperature. The fermions are in an external magnetic field B. The energy of a particle is $\epsilon = \frac{p^2}{2m} \pm \mu_B B$, where μ_B is the Bohr magneton.

a) Give an expression for the chemical potential μ_0 for vanishing magnetic field as a function of the particle density N/V.

b) Calculate the average particle energy as a function of μ_0 for weak external magnetic fields.

c) Calculate the pressure $P = -\frac{\partial E}{\partial V}$ for vanishing magnetic fields.

d) Calculate the susceptibility $\chi = \frac{\partial m}{\partial B}$ for weak external magnetic fields.