Statistical Mechanics WS 2013/14 Sheet 2

Please hand in your solutions before the Monday lecture

Problem 1 Phase space (20 points)

Sketch a typical path of a particle with fixed energy in $\{p, q\}$-space for the following one-dimensional systems:

a) free particle

b) free particle between two hard, reflecting walls at $\mp q_0$

c) particle in the homogeneously approximated gravitational field of the earth

d) harmonic oscillator

Problem 2 Classical harmonic oscillators (40 points)

A system consists of N weakly interacting one-dimensional harmonic oscillators that are confined to a volume V. The interaction between them is negligible so that the system is approximately described by the Hamiltonian:

$$H(\{q_i, p_i\}) = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 q_i^2 \right) \tag{1}$$

a) Calculate the phase space volume occupied by the microcanonical ensemble

$$\Omega(E) = \frac{1}{N! h^{3N}} \int_{E<H(\{q,p\})<E+\Delta} d^{3N}q d^{3N}p \quad \text{for } \Delta << E \tag{2}$$

b) Find the entropy $S_B(E, V) = k_B \log(\Omega(E))$, where k_B is Boltzmann's constant. Then obtain the temperature from $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V\text{const}}$.

c) Solve for the internal energy E in terms of T and N. What is the average energy per oscillator in the thermodynamic limit $N \to \infty$?

d) Compute the heat capacity $C_V = \left(\frac{\partial E}{\partial T}\right)_{V\text{const}}$.

Problem 3 Mixing entropy (20 points)

The entropy of an ideal gas is given by

$$S(E, V, N) = \frac{5}{2} N k_B + N k_B \log\left(\frac{V}{N} \left(\frac{4\pi m E}{3h^2 N} \right)^{3/2} \right) \tag{3}$$

Consider a gas container that is separated into two halves by a wall. Each half contains an ideal gas and let the temperature and pressure be the same in both compartments. Suppose the partition is removed and the gases are free to mix.

Calculate the entropy change ΔS for the cases that

a) the gas molecules in the left and right half are of different type.

b) the gas molecules are of the same type.

Problem 4 Liouville's theorem (Bonus exercise, 20 points)

The Boltzmann entropy is defined as the logarithm of the phase space volume $\Omega(E)$ that is compatible with a given macrostate (E, V, N) ($S_B = k_B \log \int_{H(\{q,p\})=E} d\Omega$). Let $\rho_N(q_1, \ldots, q_N; p_1, \ldots, p_N; t)$ be the normalized density of mircostates in an ensemble. Another way to define the entropy is

$$S_G = -k_B \int_\Omega \rho_N(\Omega, t) \log(\rho_N(\Omega, t)) d\Omega. \tag{4}$$

a) For which phase space distributions ρ_N are S_B and S_G equivalent?

b) Show that if ρ_N satisfies Liouville's equation $\frac{\partial \rho_N}{\partial t} = -\{\rho_N, H\}$ for a Hamiltonian H, then $\frac{dS_G}{dt} = 0$. (Replace the time derivative of ρ_N under the integral with the Poisson brackets and integrate by parts.)

c) Show that a phase space distribution of the form $\rho_N = f(E - H(\{q,p\}))$ is stationary, that is, $\frac{\partial \rho_N}{\partial t} = 0$.

d) A container with gas is prepared in such a way that initially all molecules are located in the right half. After some time the gas will fill the entire container. Say qualitatively how S_B and S_G change during the process (Think about the 2nd law of thermodynamics).

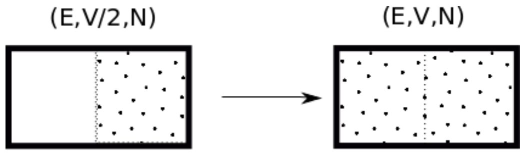

Fig. 1: