Sheet 12

Please hand in your solutions before the Wednesday Feb 4 lecture at 10:15.

Problem 1 : Quantum Harmonic Oscillator (10 points)

Consider N independent quantum oscillators subject to a Hamiltonian

$$\mathcal{H}(n_i) = \sum_{i=1}^{N} \hbar\omega(n_i + \frac{1}{2}) \tag{1}$$

where $n_i = 0, 1, 2, \ldots$ is the quantum occupation number for the i^{th} oscillator.

a) (3 points) Find the partition function of the system and then determine internal energy and mean value of the oscillator's quantum number.

b) (4 points) Find the entropy of the system and investigate it in the limits $T \longrightarrow 0$ and $k_B T \gg \hbar\omega$.

c) (3 points) Calculate the heat capacity C_V, as functions of temperature T , and N .

Problem 2 : Ising Model (20 points)

Consider a one-dimensional lattice of N dipoles which interact only with their nearest neighbours. The interaction energy for j-th dipole is $U_j = -\epsilon(s_{j-1}s_j + s_j s_{j+1})$ where s_j is an integer whose value is 1 or -1.

a) (4 points) Construct the total Hamiltonian using U_j.

b) (6 points) Find the partition function Z in closed form.

c) (5 points) Calculate the average energy $\langle U \rangle$, and sketch a plot of $\langle U \rangle$ as a function of $0 < k_B T < 100$ J where $N = 10$ and $\epsilon = 1$ J.

d) (5 points) Calculate the heat capacity C_V, and sketch a plot of C_V as a function of $0 < k_B T < 20$ J where $N = 10$ and $\epsilon = 1$ J.

Problem 3 : Corrections to Fermi Gas (20 points)

Consider a quasi-degenerate Fermi gas at low temperature. The number of fermions N reads

$$N = \frac{3N}{2\epsilon_F^{3/2}} \int_0^\infty d\epsilon \sqrt{\epsilon}\, n_{FD}(\epsilon), \tag{2}$$

where N on both sides can be eliminated, $n_{FD}(\epsilon)$ is the Fermi-Dirac distribution, and ϵ_F is the Fermi energy.

a) (10 points) By use of the Sommerfeld expansion given in the Lecture note, Show that the temperature-dependent first correction term of the chemical potential $\mu(T = 0) = \epsilon_F$ is $-\pi^2(k_B T)^2/(12\epsilon_F)$. Use

$$\int_{-\infty}^{\infty} dx \frac{x^2 e^x}{(e^x + 1)^2} = \frac{\pi^2}{3}, \tag{3}$$

if needed.

b) (10 points) Calculate the grand potential Φ using the above calculated μ. For this, one may expand the grand partition function $\Xi(\epsilon_F, \mu, T)$ around $T = 0$ to the lowest order of T.