Statistical Physics I
WS 2014/15, FUB

Sheet 13

Please hand in your solutions before Monday (9. Feb.) 10:15.
All the points from this sheet are bonus.

Problem 1 : Bose-Einstein condensation (15 points)

a) (5 points) For low temperature $T < T_c$ the pressure of an ideal Bose-Einstein gas is

$$p = k_B T \frac{g_{5/2}(\xi)}{\lambda^3}, \tag{1}$$

where λ is the thermal wave length $\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$, $\xi = \exp(\beta\mu)$, and g is the Bose function. Use the Clausius-Clapeyron equation for the phase equilibrium between particles in the condensate and particles in the gas and find the condensation enthalpy ("heat of condensation"). Hint: the condensate has (essentially) no volume.

b) (5 points) Instead of holding number of particles and volume fixed, and lowering the temperature, a Bose-Einstein condensation can also occur by reducing the volume at fixed temperature such that the density reaches a critical density ρ_c. Show that

$$\frac{\langle n_0 \rangle}{N} = \begin{cases} 1 - \frac{\rho}{\rho_c} & \text{for } \rho > \rho_c \\ 0 & \text{for } \rho < \rho_c \end{cases}. \tag{2}$$

c) (5 points) Write down the definition for occupation numbers for Bosons. From the occupation number in the ground state (at any temperature) solve for $\xi = \exp(\beta\mu)$ with finite chemical potential μ. Use this ξ to show that the ground state contribution to the pressure p_0

$$p_0 = -\frac{k_B T}{V} \ln(\xi - 1) \tag{3}$$

vanishes for large volume V and finite occupation numbers n_0.

Problem 2 : Photons (10 points)

The partition function of a Photon gas in logarithmic form is

$$\ln Z = \frac{V}{\hbar^3 c^3 \beta^3} \frac{\pi^2}{45}. \tag{4}$$

a) (3 points) Show that the photon pressure is

$$\mathrm{p} = \frac{1}{3}\frac{E}{V} = \frac{4}{3}\frac{\sigma}{c} T^4,$$

where c is the speed of light and σ is the Stefan-Boltzmann constant $\sigma = \frac{k_B^4 \pi^2}{60 \hbar^3 c^2}$.

b) (3 points) Show that the entropy of a photon gas is

$$S = \frac{16}{3c} \sigma V T^3.$$

c) (4 points) Use the Gibbs-Duhem relation

$$\langle N \rangle \mu = E - TS + \mathrm{p}V,$$

to show that the chemical potential must be zero if $\langle N \rangle \neq 0$ which is the case for photons .

Problem 3 : Debye solid (10 points)

The average energy U of a crystal of N particles in three dimension can be approximated as

$$U = \frac{9Nk_BT^4}{T_D^3} \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx, \tag{5}$$

where T_D is the Debye temperature.

a) (2 points) The particles in the crystal can be modelled as N harmonic oscillators in three dimension. Using the equipartition theorem find the average energy U.

b) (2 points) Using Eq. (5), show that U for $T \gg T_D$ reduces to the same result in a).

c) (3 points) Using Eq. (5), show that U for $T \ll T_D$ leads to $U = 3\pi^4 Nk_BT^4/(5T_D^3)$.

d) (3 points) Find the heat capacity C_V for $T \ll T_D$. The Debye temperature for Gold is 170 K, for Silver is 215 K, and for Copper is 344 K, respectively. Sketch a plot of C_V/T per mole for these metals, as function of T^2. [You can compare your plot with the experimental results (Fig. 5, 6, and 7) from the paper by William S. Corak et al., *Physical Review* **98** 1699 (1955) (https://userpage.fu-berlin.de/~wkkim/corak.pdf).]