Sheet 2

Please hand in your solutions before the Monday lecture at 10:15.

Problem 1 : Central Limit Theorem (20 points)

a) Suppose that all random variables $y_1, y_2, \cdots, y_N$ are independent and have the probability distribution $w(y_j) = 1/(2a)$ where $-a < y_j < a$. Here a is a positive real number.

1) (3 points) Calculate the characteristic function $G_j(k_j)$ for y_j.
2) (3 points) Calculate the characteristic function $G_N(Q)$ for $Y = \sum_{j=1}^{N} y_j/\sqrt{N}$.
3) (4 points) Find the the probability distribution for Y in the limit of $N \to \infty$. For this, perform the inverse Fourier transform of $G_N(Q)$.

b) 1) (6 points) Show that if all random variables $y_1, y_2, \cdots, y_N$ are independent and have the Lorentzian distribution $L(y_j) = \frac{1}{\pi}\left[\frac{1}{y_j^2+1}\right]$ where $-\infty < y_j < \infty$, then $Y = \sum_{j=1}^{N} y_j/N$ also has the same Lorentzian distribution, not the Gaussian.

2) (4 points) This seems to be in contradiction to the central limit theorem. Is the central limit theorem valid for any probability distribution functions? (Hint: What do the first moments look like?).

Problem 2 : Phase Space (14 points)

Sketch the potential $U(q)$ as a function of q, and typical paths of a particle in $\{p, q\}$-space (x axis for p and y axis for q) for the following one-dimensional systems. Here p is the momentum, and q is the position. In the systems energy E and mass m are fixed.

a) (3 points) a free particle

b) (3 points) a particle trapped in a monostable quartic potential $U = q^4$.

c) (4 points) a particle trapped in a bistable quartic potential $U = q^4 - 2q^2$.

d) (4 points) a particle under an attractive potential $U = -1/q^2$.

Problem 3 : Spin and Two state Systems (16 points)

a) Consider a sample which contains N_A distinguishable and independent hydrogen nuclei. Here N_A is the Avogadro's number. For each of the nucleus there are two possible spin states. Compute:

1) (3 points) The number of possible microstates for the N_A nuclei.
2) (3 points) The entropy of the system if all the microstates have the same energy (degenerated).
3) (2 points) Which ensemble is described by this sample?

b) n cars are parking at a free-parking lot with parking spaces for m ($n \le m$) cars in total. By parking at this free-parking lot each car driver can save the fee $-f$ per car that would have to be paid in other parking lots. Here f is a positive real number.

1) (2 points) Write down the rescaled entropy S/k_B of this parking lot, in terms of m and n. Use Stirling's formula for both m and n.
2) (2 points) Sketch a plot of S/k_B as a function of m at a fixed value of $n = 100$.
3) (2 points) Calculate the change of the rescaled entropy with respect to change of the total fee saving $F = -fn$, i.e., $b = \left(\frac{\partial S/k_B}{\partial F}\right)_m$.
4) (2 points) Find the fraction n/m of parked cars in the lot, in terms of f and b. And sketch a plot of n/m as a function of the fee saved at a certain fixed positive value of b.