Sheet 3

Please hand in your solutions before the Monday lecture at 10:15.

Problem 1 : Entropy (30 points)

a) (10 points) Show that in the microcanonical ensemble (N,V,E=const.) the uniform distribution, i.e. all microstates have equal probability, maximises the thermodynamic entropy S. [Hint: Represent the probability of a microstate by the number of systems in that microstates where the total number of systems is fixed.]

b) Suppose that there are n_0 non-interacting O_2 gas molecules (without any internal degrees of freedom) of a volume v_0 in a room of a volume V.

1) (10 points) Find the rescaled entropy S/k_B in terms of $V(\gg v_0)$, v_0 and $n_0(\gg 1)$, in equilibrium. And sketch a plot of the rescaled entropy $S/k_B(n_0)$ as a function of $100 \leq n_0 \leq 900$ at $V = 1000v_0$.

2) (10 points) Suppose that now you are in the room (so large that your volume is negligible) full of O_2 gas molecules ($n_0 = V/v_0$) and you start to consume the O_2 molecules with a rate α sec^{-1} (and also you expel certain amount of CO_2 but ignore it for simplicity). Calculate the rate α^* which maximises the rescaled entropy S/k_B, in terms of n_0 and time t.

Problem 2 : Harmonic Oscillators (20 points)

Consider N classical harmonic oscillators with coordinates and momenta $\{p_i, q_i\}$, and subject to Hamiltonian

$$\mathcal{H}(\{p_i, q_i\}) = \sum_{i=1}^{N} \left(\frac{1}{2m} p_i^2 + \tfrac{1}{2} m\omega^2 q_i^2 \right)$$

a) (7 points) Calculate the phase space volume occupied by the microcanonical ensemble.

$$\Omega(E) = \frac{1}{N! h^N} \int_{E<H(\{q,p\})<E+\Delta} d^N q d^N p \quad \text{for } \Delta \ll E$$

b) (6 points) Calculate the entropy S, as a function of the total energy E.
(*Hint.* By appropriate change of scale, the surface of constant energy can be deformed into a sphere. You may then ignore the difference between the surface area and volume for $N \gg 1$.)

c) (7 points) Find the joint probability density $P(p, q)$ for a single oscillator. Hence calculate the mean kinetic energy, and mean potential energy for each oscillator.