

Statistical Physics Exam

13th April 2016

Name	Student Number

Problem 1	Problem 2	Problem 3	Problem 4	Total	Percentage	Mark
25	25	25	25	100		

Useful constants

Gas constant R	$8.31 J / (K \cdot mol)$
Boltzmann constant k_B	$1.38 \cdot 10^{-23} J / K$
Avogadro number N_A	$6.02 \cdot 10^{23} mol^{-1}$
Speed of light c	$300 \cdot 10^6 m / s$

(25 points) Problem 1

- (a) (3 points) In which of those ensembles is the temperature constant?
 - (i) micro-canonical **no, but number of particles N , volume, V , and energy, E**
 - (ii) canonical **yes, and number of particles N , volume, V**
 - (iii) grand-canonical **yes, and number of particles N , chemical potential, μ**

In case the temperature is not constant in any of those ensembles, what else is constant?

- (b) (3 points) Write down the definition of the partition function for a microcanonical, canonical and grand-canonical ensemble in the classical case.

$$\begin{aligned}\Omega(N, V, E) &= \sum_{r: E_r = E} 1 \\ Z(N, V, T) &= \sum_{E_r} \Omega(E_r) \exp(-\beta E_r) \\ \Xi(\mu, V, T) &= \sum_{N_r} Z(N_r) \exp(\beta \mu N_r)\end{aligned}$$

- (c) (1 point) In a phase diagram, what is meant by “triple point”?
At the triple point, that is a point (p_0, T_0) in the $p(T)$ phase diagramme, three phases, e.g. gas, solid, and liquid, coexist.
- (d) (3 points) In equilibrium, what is extremised in the microcanonical, the canonical, and the grand canonical ensemble?
Entropy is maximal, in canonical and grand-canonical ensembles, s.t. boundary conditions, free energy is minimal, grand-potential is minimal.
- (e) (6 points) Which quantities are limited to a range for an arbitrary system of fermions with spin S ? Write down the range.
 - (i) Total number of states **unlimited**, $N = \sum_{s_z = \pm \frac{1}{2}} \sum_{|p| \leq p_F} 1$ occupied at $T = 0$
 - (ii) Number of particles per state $n_p = \{0, 1\}$
 - (iii) Number of states per energy **$2S + 1$ spin states per energy state**, i.e. $\{\varepsilon_p^\uparrow, \varepsilon_p^\downarrow\}$ for $S = \frac{1}{2}$
 - (iv) Chemical potential **$\mu \leq \varepsilon_F$** , equal at $T=0$

- (f) (3 points) When does quantum statistics become important (break-down of classical statistical mechanics)?
Quantum statistics becomes important when the thermal wave lengths approaches the average distance between particles

$$\lambda = \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{1}{2}} = v^{\frac{1}{3}}$$

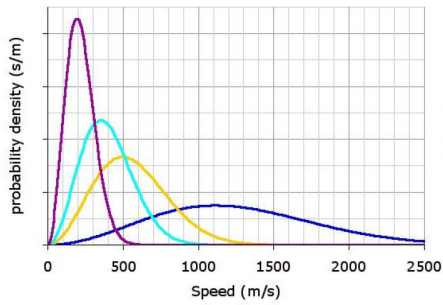
that is for low masses, high densities, low temperatures

- (g) (4 points) What does the probability distribution for the absolute value of the velocity look like in a system of constant temperature? What does the probability distribution of each component of the velocity look like? In both cases, just give the dependence of the probability distribution on the velocity and the temperature without prefactors.

Maxwell-Boltzmann distribution:
absolute value:

$$w(|v|) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi |v|^2 \exp\left(-\frac{m|v|^2}{2k_B T}\right)$$

$$\propto T^{-3/2} |v|^2 \exp\left(-\frac{|v|^2}{T}\right)$$

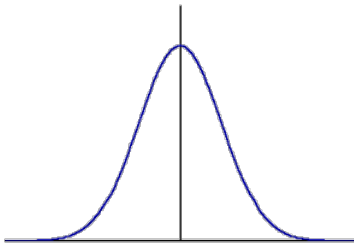


$$T_{red} < T_{cyan} < T_{yellow} < T_{blue}$$

component ($x = y = z$):

$$w(v_x) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{mv_x^2}{2k_B T}\right)$$

$$\propto \sqrt{T} \exp\left(-\frac{v_x^2}{T}\right)$$

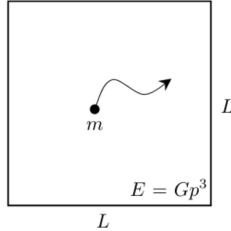


- (h) (1 point) How does the heat capacity of a classical ideal monoatomic gas depend on temperature?

Not at all.

(25 points) Problem 2 - Two-dimensional classical particle

A classical 2-dimensional system of N non-interacting particles with mass m is confined in a square of side L . The energy momentum relationship is $E = Gp^3$ (rather than $E \sim p^2$) that G being a constant of appropriate dimensions.



- (a) (9 points) Show that the one-particle partition function looks like

$$Z_1 = \int_0^\infty g(E)e^{-\beta E} dE \quad (1)$$

In the classical limit, the partition function for one particle in such a system can be calculated as follows

$$Z_1 = \frac{2}{h^2} \int d^2x \int d^2p e^{-\beta E} = \frac{2}{h^2} \int_0^L dx \int_0^L dy \int d^2p e^{-\beta Gp^3} = \frac{2L^2}{h^2} \int d^2p e^{-\beta Gp^3}$$

Now considering a circle of radius p , the above integral becomes

$$Z_1 = \frac{2L^2}{h^2} \int_0^\infty 2\pi p dp e^{-\beta Gp^3} = \frac{4\pi L^2}{h^2} \int_0^\infty p dp e^{-\beta Gp^3}$$

Now using the substitution that

$$E = Gp^3 \quad \rightarrow \quad p = \left(\frac{E}{G}\right)^{1/3} \quad \Rightarrow \quad dE = 3Gp^2 dp = 3G^{1/3} E^{2/3} dp$$

we have that

$$Z_1 = \frac{4\pi L^2}{h^2} \int_0^\infty p dp e^{-\beta Gp^3} = \frac{4\pi L^2}{h^2} \int_0^\infty \left(\frac{E}{G}\right)^{1/3} \frac{1}{3G^{1/3} E^{2/3}} dE e^{-\beta E} = \frac{4\pi L^2}{3h^2 G^{2/3}} \int_0^\infty e^{-\beta E} E^{-1/3} dE$$

Here we see that this takes on the form

$$Z_1 = \int_0^\infty g(E)e^{-\beta E} dE$$

where $g(E)$ is the energy density of states, and so we have that the density of states is

$$g(E) = \frac{4\pi L^2}{3h^2 G^{2/3}} E^{-1/3}$$

- (b) (6 points) Show that using a suitable substitution the integral of N particle partition function is of the form $Z_N = \frac{1}{N!} (\alpha \beta^{-2/3})^N$ where α includes an integral of the form $\int_0^\infty r^{-x} x^{-1/3} dx = \Gamma(\frac{2}{3})$. [Hint: $\Gamma(\frac{2}{3})$ is just a constant number]

Now, in order to calculate the heat capacity, we find the average energy of the system, by first finding the partition function of the system as follows

$$\begin{aligned} Z_N = \frac{Z_1^N}{N!} &= \frac{1}{N!} \left(\frac{4\pi L^2}{3h^2 G^{2/3}} \int_0^\infty e^{-\beta E} E^{-1/3} dE \right)^N \\ &= \frac{1}{N!} \left(\frac{4\pi L^2}{3h^2 G^{2/3}} \beta^{-2/3} \int_0^\infty e^{-\beta E} (\beta E)^{-1/3} d(\beta E) \right)^N \\ &= \frac{1}{N!} \left(\frac{4\pi L^2}{3h^2 G^{2/3}} \beta^{-2/3} \Gamma(2/3) \right)^N = \frac{1}{N!} (\alpha \beta^{-2/3})^N \end{aligned}$$

where

$$\alpha = \frac{4\pi L^2}{3h^2 G^{2/3}} \Gamma(2/3)$$

- (c) (4 points) Calculate the average energy.
The average energy is then found as

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z_N = -\frac{1}{Z_N} \frac{\partial Z_N}{\partial \beta}$$

Then

$$\langle E \rangle = -\frac{N!}{\left(\frac{4\pi L^2}{3h^2 G^{2/3}} \Gamma(2/3) \beta^{-2/3} \right)^N} \frac{1}{N!} \left(\frac{4\pi L^2}{3h^2 G^{2/3}} \Gamma(2/3) \right)^N \left(\frac{-2N}{3} \beta^{-\frac{2N}{3}-1} \right) = \frac{2}{3} N \beta^{-1} = \frac{2}{3} N k_B T$$

- (d) (4 points) Find the entropy.

$$A = -k_B T \ln Z_N = -k_B T \ln \left(\frac{1}{N!} (\alpha \beta^{-2/3})^N \right) \quad (2)$$

$$S = -\left(\frac{\partial A}{\partial T} \right)_{N,V} = -k_B \ln \left(\frac{1}{N!} (\alpha \beta^{-2/3})^N \right) - k_B T \frac{2}{3} T^{-1/3} \ln \left(\frac{1}{N!} (\alpha \beta^{-2/3})^N \right) = \frac{A}{T} + \frac{2}{3} \frac{A}{T^{1/3}} \quad (3)$$

- (e) (2 points) Find the heat capacity.
The heat capacity at constant volume is

$$C_V = \left(\frac{\partial U}{\partial T} \right) = \frac{2}{3} N k_B$$

(25 points) Problem 3 - Relativistic Bosons

Consider a two-dimensional Bose-Einstein gas in a domain of area a . This gas is characterized by a fixed average number $\langle N \rangle$ of ultrarelativistic particles with spin 0 and single particle energy $E = cp$ with p the absolute value of the momentum and c the speed of light. Here, we show that unlike the non-relativistic case we find a critical temperature T_c for the Bose-Einstein condensation in two dimensions.

- (a) (2 points) Using the Boson statistic for the number of particles per state, write down an integral expression for the average number of particles $\langle N \rangle$.

The average number of particles is

$$\langle N \rangle = \frac{1}{h^2} \int \frac{1}{\xi^{-1} \exp^{\beta cp} - 1} dp^2 dq^2,$$

- (b) (4 points) Show that the number of Bose-Einstein gas particles per area is given by

$$n = \frac{\langle N \rangle}{a} = \frac{2\pi}{h^2} \int_0^\infty \xi p \exp^{-\beta cp} \frac{1}{1 - \xi \exp^{-\beta cp}} dp, \quad (4)$$

where $\xi = \exp^{\beta\mu}$ is the fugacity.

Hint: In 2D the Jacobi determinant for spherical coordinates reads as $dx dy = r dr d\phi$.

Starting from the average number of particles in (a) we can use spherical polar coordinates in the momentum space and perform exactly the integral in the spatial coordinates

$$\langle N \rangle = \frac{2\pi A}{h^2} \int_0^\infty \frac{p}{\xi^{-1} \exp^{\beta cp} - 1} dp$$

Rearranging leads to the upper expression.

- (c) (7 points) Use $x = \beta cp$ and $\xi \exp^{-x} < 1$. Expand the fraction in the integral of eq. 4 in an infinite series. Interchange summation and integration and solve the integral to show that the result reads as

$$n = \frac{2\pi}{\beta^2 c^2 h^2} \sum_{k=1}^{\infty} \frac{\xi^k}{k^2}. \quad (5)$$

Using the substitution, we now expand the denominator as a geometric series

$$\begin{aligned} n &= \frac{2\pi}{\beta^2 c^2 h^2} \int_0^\infty \xi x \exp^{-x} \frac{1}{1 - \xi \exp^{-x}} dx = \frac{2\pi}{\beta^2 c^2 h^2} \int_0^\infty \left(\xi x \exp^{-x} \sum_{k=0}^{\infty} \xi^k \exp^{-kx} \right) dx \\ &= \frac{2\pi}{\beta^2 c^2 h^2} \sum_{k=1}^{\infty} \xi^k \int_0^\infty x \exp^{-kx} dx \end{aligned}$$

where we have interchanged the series with the integral. We then solve the integral and obtain

$$n = \frac{2\pi}{\beta^2 c^2 h^2} \sum_{k=1}^{\infty} \frac{\xi^k}{k^2}$$

- (d) (6 points) With

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

and eq. 5, determine the critical temperature $T_c(n)$ for condensation. How does it scale with the area a , i.e. is it higher or lower for a gas in a larger area a_2 ?

The critical temperature T_c is found by setting $\xi = 1$ and $T = T_c$ in the equation for n :

$$n = \frac{2\pi}{\beta_c^2 c^2 \hbar^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^3}{3\beta_c^2 c^2 \hbar^2}$$

The above equation, for a given density, determines the critical temperature

$$\beta_c = \frac{1}{kT_c} = \frac{1}{3ch} \sqrt{\frac{\pi^3}{n}}$$

In a larger area, the density of particles $n = \frac{\langle N \rangle}{a}$ would decrease, hence β_c would increase or T_c decrease as well

- (e) (6 points) Also, for below T_c , find an expression for the way the particles density changes in the condensed phase as a function of the temperature T . Describe the meaning in your own words.

When β increases above β_c , a significant fraction (say n_0 of the total number of bosons) condenses in the ground state, so that the total density is

$$n = n_0 + \frac{\pi}{12\beta^2 c^2 \hbar^2} = n_0 + n \left(\frac{\beta_c}{\beta} \right)^2$$

and hence

$$n_0 = n \left(1 - \left(\frac{\beta_c}{\beta} \right)^2 \right)$$

(25 points) Problem 4 - Carnot cycle - van der Waals gas

Consider a Carnot cycle which is run by a van der Waals gas with the equation of state

$$\left(p + \frac{a}{V^2}\right)(V - b) = Nk_B T.$$

At all your calculations, assume that the number of particles is constant.

- (a) (2 points) Prove the following Maxwell relation:

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

The total differential of the free energy dA reads as $dA = SdT + pdV$. Taking the second mixed derivatives we obtain

$$\frac{\partial^2 A}{\partial V \partial T} = \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = \frac{\partial^2 A}{\partial T \partial V}.$$

- (b) (4 points) Use the upper Maxwell relation and the total differential of the entropy to show that

$$dU = C_V dT + \left[T \left(\frac{\partial p}{\partial T}\right)_V - p\right] dV \quad \text{with} \quad C_V = T \left(\frac{\partial S}{\partial T}\right)_V.$$

The total differential of $S(T, V)$ reads as

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

Plugging this in into the differential of $dU = TdS - pdV$ yields

$$dU = T \left(\frac{\partial S}{\partial T}\right)_V dT + \left[T \left(\frac{\partial S}{\partial V}\right)_T - p\right] dV,$$

which can be rewritten into the desired equation with the help of the Maxwell relation from a).

- (c) (6 points) Show explicitly that for a van der Waals gas $C_V(T)$ is a function of T only and indeed independent of V .

Hint: Derive Maxwell relations from the differential in (b) and calculate them explicitly using the equation of state.

Taking the mixed second derivative of dU from above yields

$$\left(\frac{\partial C_V}{\partial V}\right)_T = \left(\frac{\partial}{\partial T} \left[T \left(\frac{\partial p}{\partial T}\right)_V - p\right]\right)_V = T \left(\frac{\partial^2 p}{\partial T^2}\right)_V.$$

We immediately see by solving the equation of state for $p(T, V)$ that the second derivative vanishes, hence $C_V = C_V(T)$ only.

- (d) (3 points) Using the differentials for the energy and work and calculating all derivatives explicitly, show that the heat can be written as

$$dQ = C_V dT + \frac{Nk_B T}{V - b} dV \tag{6}$$

We know that $dU = dQ - pdV$, hence

$$dQ = C_V dT + T \left(\frac{\partial p}{\partial T}\right)_V dV = C_V dT + T \left(\frac{\partial p}{\partial T}\right)_V dV = C_V dT + \frac{Nk_B T}{V - b} dV$$

- (e) (4 points) Calculate the heat Q_H along an isothermal expansion at temperature T_H from V_1 to V_2 . Repeat this calculation for Q_C along the isothermal compression at temperature T_C from V_3 to V_4 .

As in a isothermal process $dT = 0$, we find

$$Q_H = \int_{V_1}^{V_2} \frac{Nk_B T_H}{V-b} dV = Nk_B T_H \ln \left(\frac{V_2-b}{V_1-b} \right).$$

Similarly,

$$Q_C = Nk_B T_C \ln \left(\frac{V_4-b}{V_3-b} \right).$$

- (f) (3 points) Write down the differential of the heat during adiabatic processes. Use eqn. 6 and integrate the resulting equation for an adiabatic expansion from V_2 to V_3 and an adiabatic compression from V_4 to V_1 .

Hint: One integral is not solvable explicitly.

In an adiabatic process we get $dQ = 0$, hence

$$C_V(T)dT = \frac{Nk_B T}{V-b} dV \quad \Rightarrow \quad \int_{T_H}^{T_L} \frac{C_V(T)}{T} dT = Nk_B \ln \left(\frac{V_3-b}{V_2-b} \right).$$

Similarly,

$$\int_{T_L}^{T_H} \frac{C_V(T)}{T} dT = Nk_B \ln \left(\frac{V_1-b}{V_4-b} \right)$$

- (g) (3 points) Use all relations you obtained in d) and e) to show that for the van der Waals gas the efficiency of a Carnot process is $1 - T_C/T_H$.

From e) we see that

$$Nk_B \ln \left(\frac{V_1-b}{V_4-b} \right) = -Nk_B \ln \left(\frac{V_3-b}{V_2-b} \right) \quad \Leftrightarrow \quad \ln \left(\frac{V_1-b}{V_2-b} \right) = -\ln \left(\frac{V_3-b}{V_4-b} \right)$$

We therefore obtain for the efficiency

$$\eta = 1 + \frac{Q_C}{Q_H} = 1 + \frac{T_C \ln \left(\frac{V_4-b}{V_3-b} \right)}{T_H \ln \left(\frac{V_2-b}{V_1-b} \right)} = 1 - \frac{T_C}{T_H}$$