Problem sheet 10

Please hand in your solutions before the lecture on Wednesday, 6th of January.

Problem I - Heat capacities

Consider a system with fixed number of particles N, that is characterized by the four state variables S, T, p, V. Further we are interested in the five susceptibilities

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p , \quad C_V = T\left(\frac{\partial S}{\partial T}\right)_V , \quad \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T , \quad \kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S , \quad \alpha_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p ,$$

which are the isobaric and isothermal heat capacity, the isothermal and the adiabatic compressibility and the thermal expansion coefficient.

(a) With the help of the Maxwell relations, prove the following identities:

$$\frac{C_p}{C_V} = \frac{\kappa_T}{\kappa_S} \qquad \text{and} \qquad C_p - C_V = T\frac{\alpha_p^2}{\kappa_T}V$$

Hint: To show the second identity, write the total differentials of $S(T, V)$ and $V(T, p)$. (6 points)

(b) For a non-ideal gas whose molecules interact with each other and hence obey the following equation of state:

$$\left(p + a\frac{N^2}{V^2}\right)V = Nk_BT$$

where a is an arbitrary parameter, find the function $\kappa_s(T, V, C_V)$. *(3 points)*

Problem II - Black body radiator

(a) Using $U = A + TS$, show the identity

$$\left(\frac{\partial U}{\partial V}\right)_T = -p + T\left(\frac{\partial p}{\partial T}\right)_V$$

(2 points)

(b) Using the identity of (a), calculate the temperature dependence $u(T)$ of a black body radiator, where $U = u(T)V$ and $p(T) = u(T)/3$. Also give an expression for the entropy $S(T, V)$ and the free energy $A(T, V)$. *(5 points)*

Problem III - Maxwell relations

Calculate the three Maxwell-relations for the grand-canonical ensemble.

$$\Phi(T, V, \mu) = E - TS - \mu N$$

(3 points)

Problem IV - Thermodynamic relations

(a) The chemical potential of a certain substance reads as

$$\mu(p, T) = \frac{(1-m)\,T}{1+m}\left(\frac{T^2}{(m+1)^2\,bp}\right)^{1/(m-1)}$$

with two constants m, b. Show that $E(S, V, N) \sim N^\alpha V^\beta S^\gamma$ and determine α, β, γ. *(7 points)*

(b) The heat capacity of some other substance is given by $C_V(T, V, N) = c_1 VT$. Further the volume is given by

$$V(S, p, N) = \frac{S}{\sqrt{2c_1\left(p + c_2 \ln\left(N/N_0\right)\right)}} \qquad \text{and} \qquad V(S, N, \mu) = \mu N/c_2$$

with constants c_1, c_2, N_0. Determine $E(S, V, N)$. *(4 points)*