Statistical Physics I
WS 2015/16, FUB

Problem sheet 13

Please hand in your solutions before the lecture on Wednesday, 27th of January.

Problem I - Summary Thermodynamics

(a) Assume we have the grand-canonical potential $\Xi(T, V, \mu)$ given, which of the following quantites are we able to compute:

$$S(T, V, \mu) \quad , \quad S(T, V, N) \quad , \quad \kappa_T(T, V, p) \quad , \quad E(T, V, \mu)$$

(2 points)

(b) Assume we have $p(T, V, N)$ and $E(T, V, N)$ given. Which of those functions has to be also known to be sufficient to obtain $A(T, V, N)$?

$$\left(\frac{\partial S}{\partial V}\right)_{T,N} \quad , \quad \mu(T, V, N) \quad , \quad \left(\frac{\partial S}{\partial N}\right)_{T,V} \quad , \quad E(S, V, N)$$

(2 points)

(c) Assume we have an isolated system with two chambers 1 and 2, each filled with the same ideal gas with N_1=N_2=N. Both chambers can exchange work and heat with each other, but no particles. Also, all processes will be quasistatic. Which of the following statements is true? Explain why.

i) If $T_1 = T_2$, then there will not be any heat flux between chamber 1 and 2.

ii) Exept from two systems in equilibrium, there are reversible and irreversible processes possible, depending on the inital conditions.

iii) In this system a Carnot cycle can be run in one chamber.

iv) There are processes possible such that the equilibrium temperature will not be $T_e = (T_1 + T_2)/2$.

(4 points)

Problem II - Pauli-Paramagnetism

A free electron gas is trapped in a Volume V which contains a magnetic field B. The energy of a single electron is thus given by

$$\epsilon_p = \frac{p^2}{2m} \pm \mu_B B$$

(a) Calculate the number of states $\Omega_\mu(E)$ as a function of the maximal electron energy in this system. Assume that the volume is large, thus we have a continuous spectrum of momenta (sum turns to integral). From this, calculate the density of states $\rho_\mu(E) = \mathrm{d}\Omega_\mu/\mathrm{d}E$. *(3 points)*

(b) Show that the number of particles with spin up $N_\uparrow$ or down $N_\downarrow$ is given by

$$N_{\uparrow/\downarrow} = \frac{1}{2}\int_0^\infty \mathrm{d}E\, \rho(E) n(E \pm \mu_B B)\,,$$

where we introduce the occupation per state $n(E)$ and the density of states without a magnetic field $\rho(E)$. Where does the prefactor 1/2 come from? *(3 points)*

(c) The magnetization is defined as $M = \frac{\mu_B}{V}(N_\downarrow - N_\uparrow)$. We assume the low temperature limit and small magnetic fields. Make a suitable expansion up to the first order and show that M is a functional of $\mathrm{d}n(E)/\mathrm{d}E$. Why is your approximation valid for the entire integration range (especially the lower boundary)? *(3 points)*

(d) Show that with the Sommerfeld expansion the magnetization can be written as

$$M = \frac{\mu_B^2 B}{V} \left(\rho(\mu) + \frac{\pi^2}{6\beta^2} \rho''(\mu) \right) ,$$

where μ is the chemical potential. *(2 points)*
Hint: You do not have to derive the Sommerfeld expansion again here.

(e) Expand your result up to the first order for $\mu \approx \epsilon_F$. Use the equations

$$\rho(\epsilon) = \frac{3N\epsilon^{1/2}}{2\epsilon_F^{3/2}} \qquad \text{and} \qquad \mu = \epsilon_F - \frac{\pi^2}{12\beta^2 \epsilon_F} ,$$

which were derived in the lecture, to show that

$$M = \frac{3N\mu_B^2 B}{2V\epsilon_F} \left(1 - \frac{\pi^2}{12\beta^2 \epsilon_F^2} \right) .$$

(2 points)

(f) Sodium has one valence electron per molecule. Plot the relative correction of its magnetization due to the temperature between 250 K and 350 K. *(2 points)*

Problem III - Entropy of the fermi gas

(a) Using the grand-canonical partition sum, calculate the entropy $S(T, V, \mu)$ of the fermi gas. Reexpress your result for the entropy as a function of the average occupation number $\langle n_i \rangle$ of an energy level i. *(5 points)*

(b) For the second expression, verify that for $T \to 0$ the entropy becomes zero. *(2 points)*