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Advanced Statistical Physics (WS11/12)
Problem sheet 14

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Problem 1: 1D Debye Solid

A one-dimensional lattice of a linear array of N particles ($N \gg 1$) interacting via spring-like nearest neighbor forces. The normal mode frequencies (radians/sec) are given by

$$\omega_k = \bar{\omega}\sqrt{2(1-\cos(2\pi k/N))} = 2\bar{\omega}|\sin(\pi k/N)|,$$

where $\bar{\omega} > 0$ is a constant and k an integer ranging from $-N/2$ to $+N/2$. The system is in thermal equilibrium at temperature T. Let C_V be the constant "volume" (length) specific heat.
The problem is to be treated quantum mechanically. So the internal energy is $U = \sum_k \hbar\omega_k \langle n_k\rangle$, where $\langle n_k\rangle = [\exp(\beta\hbar\omega_k) - 1]^{-1}$ is the occupation number of the k-th mode.

a) Calculate C_V for the high temperature regime.

b) Calculate C_V as a function of T and $\bar{\omega}$ for the low temperature regime.

Hints for part b): You can use $\sum_{k=0}^{N/2} f(k) \approx \int_0^{N/2} f(x)\,\mathrm{d}x$ for a smooth function $f(k)$, since $N \gg 1$ is large.
For a large parameter $a \gg 1$ you can approximate the following integral:

$$\int_0^1 \frac{y^2 e^{-ay}}{\sqrt{1-y^2}}\,\mathrm{d}y \approx \int_0^1 y^2 e^{-ay}\,\mathrm{d}y \approx \int_0^\infty y^2 e^{-ay}\,\mathrm{d}y$$

Convince yourself that the approximations are justified.

To obtain the final result you can use:

$$\int_0^\infty y^2 e^{-y}\,\mathrm{d}y = 2$$

Problem 2: Density of States

Take a system of N electrons in a box of volume V. The walls of the box are infinitely high potential barriers.

a) The density of states (DOS) describes the number of states in the energy interval ϵ and $\epsilon+\mathrm{d}\epsilon$. It is convenient to calculate the DOS in p-space, where it corresponds to the volume of the sperical shell $p, p+\mathrm{d}p$:

$$D(p)\,\mathrm{d}p = 2\,\frac{V}{h^3}\,4\pi p^2\,\mathrm{d}p$$

The additional factor of 2 arises due to the nature of electrons, where one energy level has two spin states. The number of particles in the volume V may be written as $N = \int_0^\infty D(\epsilon)\,f(\epsilon)\,\mathrm{d}\epsilon$, where $f(\epsilon)$ is the Fermi distribution.

Calculate $D(\epsilon)\,\mathrm{d}\epsilon$. Analogously define $D(p)$ for the 2D and 1D electron gas and calculate $D(\epsilon)\,\mathrm{d}\epsilon$.

b) Calculate the internal energy at $T = 0\,\mathrm{K}$ in terms of ϵ_F, where ϵ_F is the Fermi energy.

c) Calculate the specific heat capacity at low temperatures $k_B T \ll E_F$, where ϵ_F is the Fermi energy. Show first that the heat capacity can be written as:

$$C = \int_0^\infty (\epsilon - \epsilon_f)\, D(\epsilon)\, \frac{\mathrm{d}f}{\mathrm{d}T}\, \mathrm{d}\epsilon$$

Use this result to calculate C as a function of ϵ_F.

Hint:

$$\int_{-\infty}^{\infty} \frac{x^2\, e^x}{(e^x + 1)^2} = \pi^2/3$$

Problem 3: 2D Photon Gas

Consider a photon gas enclosed in a 2 dimensional layer of volume V and in equilibrium at temperature T. The photon is a massless particle, so that $\epsilon = pc$.

a) What is the chemical potential of the gas? Explain.

b) Determine how the number of photons in the volume depends upon the temperature.

c) One may write the energy density in the form

$$\frac{\bar{U}}{V} = \int_0^\infty u(\omega)\, \mathrm{d}\omega,$$

where ω is the angular frequency.

Determine the form of $u(\omega)$, the spectral density of the energy.
What is the temperature dependence of the energy $\bar{E}$?

Hint: Use the density of states as defined in problem 2 for part b.
The following integrals are helpful:

$$A = \int_0^\infty \frac{x}{e^x - 1}\, \mathrm{d}x = \frac{\pi^2}{6}$$
$$B = \int_0^\infty \frac{x^2}{e^x - 1}\, \mathrm{d}x \approx 2.4$$

Problem 4: Bose-Einstein Condensation

Consider a gas of non-interacting, non-relativistic bosons. Explain whether and why the Bose-Einstein condensation effect that applies to a three-dimensional gas applies also to a two-dimensional gas and to a one-dimensional gas.

Hint: Write down an expression for the mean particle number $n = N/V$. At T_C the chemical potential is near to zero and may be neglected. Consider that BEC only occurs, when n stays finite for $\mu \to 0$.