Solution for problem 13.3b

In part a) we have obtained:

$$\mu_0 = \frac{1}{2m}\left(\frac{3h^3N}{8\pi V}\right)^{2/3} \tag{1}$$

Now we calculate the total energy of the fermions which are orientated opppositely to the magnetic field:

$$E_+ = \frac{V}{h^3}\int_0^{p_+} \epsilon_+ \mathrm{d}p^3 = \frac{4\pi V}{h^3}\int_0^{p_+}\left(\frac{p^2}{2m} + \mu_B H\right)p^2\mathrm{d}p = \frac{4\pi V}{h^3}\left(\frac{p_+^5}{10m} + \mu_B H\frac{p_+^3}{3}\right) \tag{2}$$

Analogously we find:

$$E_- = \frac{V}{h^3}\int_0^{p_-} \epsilon_- \mathrm{d}p^3 = \frac{4\pi V}{h^3}\left(\frac{p_-^5}{10m} - \mu_B H\frac{p_-^3}{3}\right) \tag{3}$$

The average energy is the sum of both terms:

$$\frac{E}{N} = \frac{E_+ + E_-}{N} = \frac{4\pi V}{h^3}\left(\frac{p_+^5 + p_-^5}{10m} + \mu_B H\frac{p_+^3 - p_-^3}{3}\right), \tag{4}$$

where $p_\pm = \sqrt{2m\mu_0}(1 \mp \mu_B H/\mu_0)^{1/2}$.
In order to insert the momenta we have to approximate them by a taylor expansion in $x = \mu_B H/\mu_0$:
p_+^5: $(1-x)^{5/2} \approx 1 - \frac{5}{2}x + \frac{15}{8}x^2$
p_-^5: $(1+x)^{5/2} \approx 1 + \frac{5}{2}x + \frac{15}{8}x^2$
p_+^3: $(1-x)^{3/2} \approx 1 - \frac{3}{2}x$
p_-^3: $(1+x)^{3/2} \approx 1 + \frac{3}{2}x$
Note that we expand the third order terms in p one order lower than the fifth order terms, since the third order terms have a prefactor in linear in x in equation (4):

$$\frac{p_+^5 + p_-^5}{10m} = \frac{(2m\mu_0)^{3/2}}{5}\mu_0\left[2 + \frac{15}{4}\left(\frac{\mu_B H}{\mu_0}\right)^2\right] = (2m\mu_0)^{3/2}\mu_0\left[\frac{2}{5} + \frac{3}{4}\left(\frac{\mu_B H}{\mu_0}\right)^2\right] \tag{5}$$

$$\frac{p_+^3 - p_-^3}{3}\mu_B H = -3\frac{\mu_B H}{\mu_0}(2m\mu_0)^{3/2}\frac{\mu_B H}{3} = -(2m\mu_0)^{3/2}\mu_0\left(\frac{\mu_B H}{\mu_0}\right)^2 \tag{6}$$

By inserting equations (5) and (6) in equation (4) we determine the final result:

$$\frac{E}{N} = \frac{4\pi V}{h^3}(2m\mu_0)^{3/2}\mu 0\left[\frac{2}{5} - \frac{1}{4}\left(\frac{\mu_B H}{\mu_0}\right)^2\right] \overset{(1)}{=} \frac{4\pi V}{h^3}\frac{3h^3N}{8\pi V}\mu_0\left[\frac{2}{5} - \frac{1}{4}\left(\frac{\mu_B H}{\mu_0}\right)^2\right] \tag{7}$$

$$= \frac{3}{5}\mu_0\left[1 - \frac{5}{8}\left(\frac{\mu_B H}{\mu_0}\right)^2\right] \tag{8}$$

The first correction term reduces the average particle energy in second order of the strength of the applied magnetic field H.

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