Freie Universität Berlin
Fachbereich Physik
June 7th 2016

Prof. Dr. Roland Netz
Douwe Bonthuis
Jan Daldrop
Sadra Kashef

Statistical Physics and Thermodynamics (SS 2016)

Problem Sheet 8

Hand in: Thursday, June 16th during the lecture

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/

1. Caloric equation of state for an ideal gas (7 points)

In this exercise we want to obtain the caloric equation of state, i.e. an expression for the internal energy U as a function of N, V and T, from the Sackur-Tetrode equation for the entropy of an ideal gas

$$S(N,V,U) = -k_\mathrm{B} N \ln\left(\frac{N}{V}\right) + \frac{3}{2} k_\mathrm{B} N \ln\left(\frac{U}{N}\right) + k_\mathrm{B} N \left(\frac{3}{2}\ln\left(\frac{4\pi m}{3h^2}\right) + \frac{5}{2}\right). \tag{1}$$

a) Invert eq. (1) to get an expression for $U(S,V,N)$. **(3 points)**

b) Calculate the temperature as a function of S,V,N by an appropriate derivative of $U(S,V,N)$. **(2 points)**

c) Use your results from (b) to find the caloric equation of state for the ideal gas. **(1 point)**

d) What is the heat capacity C_V of the ideal gas? **(1 point)**

2. Explicit evaluation of the Legendre transformation for an ideal gas (7 points)

a) Explain the idea of the Legendre transform in words. **(1 point)**

b) Start from the canonical form of the internal energy $U = U(S,V,N)$ for an ideal gas and find an expression for $T(S,V,N)$ by taking a suitable derivative. (**(1 point)**

Hint: Look at your results from exercise 1 (a/b).

c) Obtain an expression for $S(T,V,N)$ from $T(S,V,N)$. **(1 point)**

d) Write down the relation between the Helmholtz free energy F and the internal energy U. **(1 point)**

e) Eliminate the energy and the entropy to write down the free energy in its canonical form $F(T,V,N)$.
(2 points)

f) Why did we call this exercise 'Explicit evaluation of the Legendre transformation for an ideal gas'? **(1 point)**

3. Shannon entropy (6 points)

In information theory, the so-called *Shannon entropy* is a commonly employed concept, measuring the average information content of a (discrete) random variable X. It is given by

$$H_{\mathrm{s}}(X) = \langle -\ln\left(p(X)\right)\rangle = -\sum_{i=1}^{n} p(x_i)\ln p(x_i), \tag{2}$$

where p is the normalized probability.

In this exercise, we want to establish a connection between this abstract concept from information theory and statistical mechanics. Therefore, we assume the random variable X is realized as a statistical mechanics system with discrete energies H_i.

a) Introduce the non-normalized probabilities $\tilde{p}_i = e^{-\beta H_i}$ and argue that the partition function of this system is

$$Z = \sum_i \tilde{p}_i. \tag{3}$$

(1 point)

b) What is the free energy F of the system in terms of $\tilde{p}_i$? **(1 point)**

c) Show that the entropy of the system is equal to

$$S = k_{\mathrm{B}}\left[\ln\left(\sum_i \tilde{p}_i\right) - \frac{\sum_i \tilde{p}_i \ln \tilde{p}_i}{\sum_i \tilde{p}_i}\right]. \tag{4}$$

(2 points)

d) Now use the definition of the normalized probabilities

$$p_i = \frac{\tilde{p}_i}{\sum_j \tilde{p}_j} \tag{5}$$

to write the entropy eq. (4) in terms of p_i. Interpret your result. **(2 points)**