Freie Universität Berlin
Fachbereich Physik
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Statistical Physics and Thermodynamics (SS 2018)

Problem sheet 3

Hand in: Friday, May 18 during the lecture (note: one week later than usual)

http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre

1 Total differential (7 points)

The differential $\alpha(x, y) = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y$ is called *exact* if there is a function $F(x, y)$, so that $\mathrm{d}F(x, y) = (\partial F(x, y)/\partial x)\mathrm{d}x + (\partial F/\partial y)\mathrm{d}y$ is equal to $\alpha(x, y)$.

a) Show that if α is exact, then $A(x, y)$ and $B(x, y)$ satisfy the condition

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}. \quad \textbf{(2 points)} \tag{1}$$

Remark: The converse not necessarily holds.

Consider $\alpha(x, y) := (x^2 - xy)\mathrm{d}x + x^2\mathrm{d}y$

b) Does $\alpha(x, y)$ fullfill equation (1)? **(1 point)**

c) Determine the exponent $n \in \mathbb{N}$ so that $\beta(x, y) = \alpha(x, y)/x^n$ is an exact differential. **(2 points)**

d) Find a function $F(x, y)$ such that $\mathrm{d}F(x, y) = \beta(x, y)$ and $F(1, 0) = 0$. **(1 point)**

e) Find a curve $(x, y(x))$ in space where $F(x, y) = F(1, 0)$ is constant. Sketch $y(x)$ in the range $0 < x < 3$. **(1 point)**

2 Legendre transformation (3 points)

Consider a function $f(x)$ and the differential $\mathrm{d}f := u(x)\mathrm{d}x$ where $u(x) = \mathrm{d}f(x)/\mathrm{d}x$.

a) Show that you can define a function $g(u) := xu(x) - f(x)$ for which,

$$\frac{\mathrm{d}g}{\mathrm{d}u} = x \quad \textbf{(1 point)} \tag{2}$$

$g(u)$ is the Legendre transformed function of $f(x)$. We can also define the Legendre transform for a function of two variables $g(x, v) := yv(x, y) - f(x, y)$, where $v(x, y) = \partial f(x, y)/\partial y$.

b) Consider the two functions $f_1(x) = \alpha x^2$ and $f_2(x, y) = \alpha x^2 y^3$ where $\alpha \in \mathbb{R}$ and calculate their Legendre transformations $g_1(u)$ and $g_2(x, v)$. **(2 points)**

3 Phase space and 1D harmonic oscillator (10 points)

Consider a 1D harmonic oscillator with mass m and position q in a potential $V(q) = kq^2/2$.

a) Write down the Lagrange function $L(q, \dot{q}) = T - V$, where T is the kinetic energy and V the potential energy, and derive the equation of motion for q using the Euler-Lagrange equation. **(1 point)**

b) For the 1D harmonic oscillator, explicitly calculate the Legendre transformation

$$\mathcal{H}(q, p) = p\dot{q}(q, p) - L(q, \dot{q}(q, p)), \tag{3}$$

where the canonical momentum is defined by $p = \partial L/\partial \dot{q}$. **(1 point)**

c) Calculate Hamilton's equations of motion, and solve them to obtain the phase space trajectory $(q(t), p(t))$ with initial conditions $q(0) = q_0$, $p(0) = p_0$. **(2 points)**

d) Use your result from c) to calculate $\mathcal{H}(q(t), p(t))$ and $L(q(t), \dot{q}(t))$ along a solution of the equations of motion. Are they conserved along a trajectory? **(2 points)**

e) Sketch a solution from c) with initial energy $E = \mathcal{H}(q_0, p_0)$ in phase space, i.e. in the (q, p)-plane. Is the resulting curve closed? What geometric shape does the trajectory have? At which points does it intersect the q- and p-axes? **(2 points)**

f) Assume the oscillator starts at rest with energy E. Calculate the running averages of kinetic and potential energy, i.e.

$$\bar{E}_{\text{kin}}(t) = \frac{1}{t}\int_0^t T(t')\,\mathrm{d}t', \qquad \bar{E}_{\text{pot}}(t) = \frac{1}{t}\int_0^t V(t')\,\mathrm{d}t' \tag{4}$$

and express them in terms of E. What are the limits for $t \to 0$, $t \to \infty$? **(2 points)**