

## Statistical Physics and Thermodynamics (SS 2018)

### Problem sheet 6

**Hand in: Friday, June 8th during the lecture**

<http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre/>

#### 1 Volume and surface in high dimensions (6 points)

An important and useful fact in statistical physics is that nearly all the volume of a high dimensional body is close to its surface. In this problem we investigate the properties of spheres and cubes in more detail.

- a) The volume of a  $N$ -dimensional sphere or cube is given by  $V_N(L) = L^N V_N(1)$ , where  $L$  is the radius or the edge length and  $V_N(1)$  the volume of the  $N$  dimensional body with unit radius or edge length. Show that the ratio of the volume of a thin shell and the volume of the full body can be written as

$$\lim_{N \rightarrow \infty} \frac{V_N(L) - V_N[L - \kappa \frac{L}{N}]}{V_N(L)} = 1 - e^{-\kappa}, \quad (1)$$

where  $\kappa$  is a constant. Discuss your result. **(2 points)**

For the relation between the volume  $V$  and the surface  $S$  of a sphere or cube we can write, respectively,

$$\frac{d}{dR} V_N(R) = S_N(R) \qquad \frac{d}{dL} V_N(L) = \frac{1}{2} S_N(L) \quad (2)$$

- b) Use that  $V_N(L) = L^N$  to calculate the surface of the  $N$ -dimensional unit cube. **(2 points)**  
c) Use that  $V_N(R) \propto R^N$  to calculate the relation between the volume  $V_N(R)$  and surface  $S_N(R)$  for a  $N$ -dimensional sphere. What does this mean for large  $N$ ? **(2 points)**

#### 2 Phase space of N harmonic oscillators (8 points)

Consider  $N$  uncoupled 1-dimensional indistinguishable harmonic oscillators with equal masses  $m$  and harmonic potential  $V(q_1, \dots, q_N) = k/2 \sum_i q_i^2$ . Their Hamiltonian is given by

$$\mathcal{H}(q, p) = \sum_{i=1}^N \left[ \frac{p_i^2}{2m} + \frac{k}{2} q_i^2 \right]. \quad (3)$$

Let  $\Omega(U, N)$  be the phase space volume of all states up to energy  $U$ ,

$$\Omega(U, N) = \frac{1}{N! h^3} \iint_{\mathcal{H} < U} d^N q d^N p = \frac{1}{N! h^3} \iint d^N q d^N p \Theta(U - \mathcal{H}(q, p)) \quad (4)$$

where  $h$  is the Planck constant and  $\Theta(x)$  is the Heaviside step function.

- a) Calculate  $\Omega(U, N)$ . **(5 points)**

*Hint: Use transformation of variables to arrive at the integral over a  $N$ -dimensional sphere. You can then use that the volume of a  $N$ -dimensional sphere of radius  $R$  is given by  $V_N(R) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^N$ , with  $\Gamma$  being the Gamma-function.*

- b) From your result for  $\Omega(U, N)$ , determine the entropy  $S = k_B \ln(\Omega(U, N))$ , the temperature  $T$  and the caloric equation of state  $U(T, N)$ . **(3 points)**

*Note: One can use the phase space volume  $\Omega(U, N)$  instead of the density of states  $\omega(U, N)$ , since almost all states contributing to  $\Omega(U, N)$  have the energy  $U$ .*

### 3 The Gibbs paradox (6 points)

Consider  $N$  particles, each of mass  $m$ , in a 3-dimensional volume  $V$  at temperature  $T$ . Each particle  $i$  has momentum  $p_i$ . Assume that the particles are non-interacting (ideal gas) and distinguishable (in contrast to the condition used in the lecture).

- a) Calculate the canonical partition function

$$Z = \prod_i \frac{1}{h^3} \left[ \int d^3 p_i \int d^3 q_i \right] \exp \left( -\frac{1}{k_B T} \sum_i \frac{p_i^2}{2m} \right) \quad (5)$$

for the  $N$ -particle system. Make sure to work out the integral. **(2 points)**

- b) Calculate the free energy  $F = -k_B T \ln(Z)$  from the partition function  $Z$ . Is  $F$  an extensive quantity? **(1 point)**
- c) Calculate the entropy  $S = -\partial F / \partial T$  from the free energy derived in part b). **(1 point)**

Next, consider two reservoirs, each having volume  $V$ , separated by a removable plate. Both reservoirs contain  $N$  particles.

- d) Using your result from part c), calculate the change in entropy  $\Delta S$  of the total system (defined as the difference between the entropies of the final and the initial states) when the separating plate is removed, and the system has returned to equilibrium (this is called the *entropy of mixing*). Explain your result in terms of reversibility. **(2 points)**