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hand in on 26th June 2015 before lecture

Advanced Statistical Physics II – Problem Sheet 10

Problem 1 – **Residue theorem**

The residue theorem says that for a complex variable z:

$$\oint_\gamma f(z)\,dz = 2\pi i \mathrm{Res}(f, z_0), \tag{1}$$

where z_0 is a simple pole.

a) Assume $f(z)$ has a k-th pole, then show that the residue is given by

$$\mathrm{Res}(f, z_0) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z)|_{z=z_0} \tag{2}$$

Remark: If $f(z) = \frac{g(z)}{(z-z_0)^k}$*, then* z_0 *is a pole of order* k *(*k*-th pole).*

b) Solve the integral $\int_{-\infty}^{\infty} dx \frac{1}{1+x^2}$

c) Solve the integral $\int_{-\infty}^{\infty} dx \frac{\sin(x)}{x}$

Hint: A direct application of the residue theorem is not possible. How can $sin(x)$ *be expressed?*

d) Evaluate $\int_{-\infty}^{\infty} dx \frac{e^{ax}}{1+e^x}$, where $0 < a < 1$.

Problem 2 – **Kramers-Kronig relation: Harmonic oscillator**

Consider a harmonic oscillator with a damping coefficient γ in the zero mass limit, where the equation of motion is given by

$$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = F(t). \tag{3}$$

a) Find the real part of the response function $\tilde{\chi}'(\omega)$, and the imaginary part of the response function $\tilde{\chi}''(\omega)$. For this use the Fourier transform $\tilde{x}(\omega) = \int dt e^{i\omega t} x(t)$.

b) Find the poles of

$$\frac{\tilde{\chi}''(\omega')}{\omega' - \omega} \text{ and } \frac{\tilde{\chi}'(\omega')}{\omega - \omega'}. \tag{4}$$

Depict them in the complex ω' space, respectively.

c) Convince yourself that the Kramers-Kronig relations

$$\tilde{\chi}'(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}''(\omega')}{\omega' - \omega}, \tag{5}$$

$$\tilde{\chi}''(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}'(\omega')}{\omega - \omega'}, \tag{6}$$

hold for the system, where $\mathcal{P}$ is the principal value. For this use the decomposed expression of the above integrals

$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{\omega - \omega'} = \underbrace{\frac{1}{2\pi}\oint d\omega' \frac{f(\omega')}{\omega - \omega' - \varepsilon i}}_{\equiv A(\omega)} + \underbrace{\frac{1}{2\pi}\oint d\omega' \frac{f(\omega')}{\omega - \omega' + \varepsilon i}}_{\equiv B(\omega)}, \tag{7}$$

where $A(\omega)$ is the closed path integral in a domain of positive pole, and $B(\omega)$ is the closed path integral in a domain of negative pole, in the limit $\varepsilon \to 0$.

Problem 3 – **Kramers-Kronig relation: Charged system and dielectric function**

Consider a dilute solution (damping coefficient γ) of charged particles (mass m and charge ze per particle) with the number density n, subject to a time-dependent electric field $E(t)$. Suppose that the gain spectrum of the dielectric response function $\tilde{\chi}'$ is found as

$$\tilde{\chi}'(\omega) = \frac{-mnz^2e^2}{m^2\omega^2 + \gamma^2}. \tag{8}$$

a) Use the Kramers-Kronig relation to find $\tilde{\chi}''(\omega)$, and find the response function $\tilde{\chi}(\omega)$.

b) Starting from the equation of motion

$$m\ddot{x} + \gamma\dot{x} = zeE(t), \tag{9}$$

find the response function $\tilde{\chi}(\omega)$, where the linear response for the polarization $p(t) = nzex(t)$ is given by $\tilde{p}(\omega) = \tilde{\chi}(\omega)\tilde{E}(\omega)$.

c) Do the inverse Fourier transform of the dielectric function $\epsilon(\omega) = 1 + \chi(\omega)$. How does $\epsilon(t)$ depend on time? For calculation, if needed, use

$$\int_{-\infty}^{\infty} dx \frac{\cos(xt)}{a^2x^2 + b^2} = \frac{\pi}{ab} e^{-bt/a}, \tag{10}$$

$$\int_{-\infty}^{\infty} dx \frac{\sin(xt)}{a^2x^3 + b^2x} = \frac{\pi}{b^2}(1 - e^{-bt/a}). \tag{11}$$