Freie Universität Berlin
Fachbereich Physik
Prof. Dr. Roland Netz
http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre

hand in on 10th July 2015 before lecture

Advanced Statistical Physics II – Problem Sheet 12

Problem 1 – **Colored noise**

In the lecture we calculated the expectation value $\langle \delta F(t) \rangle$ and auto-correlation $\langle \delta F(t) \delta F(t') \rangle$ for white noise.

Here you will do this for the colored-noise case which was defined as

$$\delta F_x(t) = \int_0^\infty ds \frac{\Gamma(s)}{K} \delta F(t-s) \tag{1}$$

with $\Gamma(s) = \frac{K}{m} e^{-\frac{-\gamma s}{m}}$

a) Calculate the expectation value $\langle \delta F_x(t) \rangle$
b) Evaluate the auto-correlation $\langle \delta F_x(t) \delta F_x(t') \rangle$
c) Interpret the results for white and colored noise. Which case is more *physical*?

Problem 2 – **Linear Fokker-Planck equation in one dimension**

Consider the one-dimensional Fokker-Planck equation

$$\frac{d}{dt} f(x,t) = -\frac{d}{dx} \Theta x f(x,t) + B \frac{d^2}{dx^2} f(x,t), \tag{2}$$

where Θ, B are constants.

a) Show that the spatial Fourier transform of equation (2) yields

$$\frac{d}{dt} f(k,t) = k\Theta \frac{d}{dk} f(k,t) - Bk^2 f(k,t). \tag{3}$$

To arrive at the result, plug

$$f(x,t) = \frac{1}{2\pi} \int dk \, e^{-ikx} f(k,t) \tag{4}$$

into equation (2) and express everything as one inverse Fourier transform. Assume that $f(k,t) \longrightarrow 0$ for $|k| \longrightarrow \infty$.

b) Solve equation (3) by the ansatz

$$f(k,t) = A \exp\left[ia(t)k - \sigma(t)k^2/2\right], \tag{5}$$

and obtain

$$\frac{da}{dt} = \Theta a, \tag{6}$$

$$\frac{d\sigma}{dt} = 2\Theta\sigma + 2B. \tag{7}$$

c) Solve the equations (6) and (7) with initial conditions $a(0) = a_0$ and $\sigma(0) = \sigma_0$.

d) Calculate f(x,t) from equation (5). You will need the Gaussian integral

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{x^2}{2} - bx} = \sqrt{2\pi} e^{b^2/2}. \tag{8}$$

Problem 3 – **Fokker-Planck equation**

a) Show that the Boltzmann distribution $\rho_{eq} = e^{-\beta H(x,p)}/Z$ solves the x and p dependent Fokker-Planck equation,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{p\rho}{m}\right) + \frac{\partial}{\partial p}\left(\frac{dU(x)}{dx} + \frac{\gamma p}{m}\right)\rho + \gamma k_B T \frac{\partial^2 \rho}{\partial p^2}, \tag{9}$$

as a stationary solution. Here $U(x)$ denotes the potential and thus the Hamiltonian is $H(x,p) = \frac{p^2}{2m} + U(x)$.

b) Derive the Schrödinger-like equation

$$\frac{\partial \psi}{\partial t} = -D\left[-\frac{\partial^2}{\partial x^2} + U_{eff}\right]\psi, \tag{10}$$

and find U_{eff}. For this start from the x dependent Fokker-Planck equation (Smoluchowski equation)

$$\frac{\partial \rho}{\partial t} = D\frac{\partial}{\partial x}e^{-\beta U(x)}\frac{\partial}{\partial x}e^{\beta U(x)}\rho, \tag{11}$$

by using the wave function form $\rho = \sqrt{\rho_{eq}}\psi(x,t)$ where $\rho_{eq} = e^{-\beta U(x)}/Z$.