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Hand in on January 9th 2017 before the lecture

Advanced Statistical Physics II – Problem Sheet 9

Problem 1 – **Harmonic oscillator**

From the lecture we know that the response function $\tilde{\chi}(\omega)$[1] for the driven harmonic oscillator described by the equation of motion

$$m\ddot{x}(t) = -Kx(t) - \gamma\dot{x}(t) + F(t) \tag{1}$$

reads as

$$\tilde{\chi}(\omega) = \frac{1}{K - i\gamma\omega - m\omega^2} \tag{2}$$

with the particle mass m, spring constant K, friction coefficient γ and external force F.

a) (2P) The absorption coefficient is given by $\sigma(\omega) = \omega\tilde{\chi}''(\omega) = \omega\,\mathrm{Im}\,(\tilde{\chi}(\omega))$. Calculate the frequency ω_0 at which it becomes maximal. Sketch $\gamma\sigma$ as a function of ω/ω_0 for several values of γ, including the limiting case $\gamma \to 0$. How does $\sigma(\omega)$ look like for $m \to 0$?

b) (2P) Now consider $|\chi(\omega)| = \sqrt{\chi'(\omega)^2 + \chi''(\omega)^2}$ with $\tilde{\chi}'(\omega) = \mathrm{Re}\,(\tilde{\chi}(\omega))$. Determine the frequency $\omega^\star$ at which $|\chi(\omega)|$ is maximal. What happens to $|\chi(\omega)|$ for $\omega \to 0$ and $\omega \to \infty$? Sketch $|\chi(\omega)|$ for the three cases $m \ll \gamma$, $m \approx \gamma$ and $m \gg \gamma$.

c) (2P)Compute the phase angle ϕ between $\tilde{\chi}'(\omega)$ and $\tilde{\chi}''(\omega)$. Sketch ϕ as a function of ω/ω_0 for several values of γ, including the limiting case $\gamma \to 0$. How does $\phi(\omega)$ look like for $m \to 0$?

Problem 2 – **Einstein relation for diffusion**

Now we consider the harmonic oscillator from problem 1 in the massless case, hence $m = 0$.

a) (3P) Use residue calculus to obtain an expression for the response function $\chi(t)$ which is defined by the equation $x(t) = \int_{-\infty}^{t} \mathrm{d}t' \chi(t - t')F(t')$.

b) (2P) Determine $C_{xx}(t) = \langle x(t)x(0)\rangle$ and expand your result up to first order in K around $K = 0$.

c) (1P) Show that the mean squared displacement $\Delta x(t)^2 = \left\langle (x(t) - x(0))^2 \right\rangle = 2Dt$ is proportional to t. Give an expression for the proportionality constant D, which is called the diffusion coefficient.

Problem 3 – **Viscoelasticity: Kelvin-Voigt model**

A viscoelastic material is a material showing both elastic (like a harmonic oscillator) and viscous (like a Newtonian viscous fluid) properties. As an example for how to describe such a material, we consider the Kelvin-Voigt model in one dimension.

Our system has an equilibrium extension L_0 and the extension at time t will be denoted by $L(t)$. We measure the deformation by the dimensionless strain $\epsilon(t) := \Delta L(t)/L_0 = (L(t) - L_0)/L_0$.

[1] Remember that our convention for Fourier transforms is $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega)e^{-i\omega t}\,\mathrm{d}\omega$, $\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{i\omega t}\,\mathrm{d}t$, where $\tilde{x}(\omega)$ is the Fourier transform of $x(t)$.

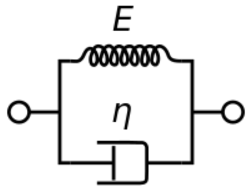

Figure 1: .

In the Kelvin-Voigt model, the relation between stress and strain is given by

$$\sigma(t) = E\epsilon(t) + \eta \frac{\partial \epsilon}{\partial t}(t)\,, \tag{3}$$

where $\sigma(t)$ is the stress the material experiences, E is the modulus of elasticity and η is the viscosity.

In the general linear theory of viscoelasticity it is customary to write the relation between stress and strain as

$$\epsilon(t) = \int_{-\infty}^{t} j(t-t') \frac{\mathrm{d}\sigma(t')}{\mathrm{d}t'}\,\mathrm{d}t'\,, \tag{4}$$

where the response function $j(t)$ is called the creep function and assumed to vanish for $t < 0$.

The goal of this exercise is to compare the Kelvin-Voigt model with the general formalism of response functions.

a) (2P) Show that in the Kelvin-Voigt model, the Fourier transform of the creep function is given by

$$\tilde{j}(\omega) = \begin{cases} \frac{1}{\eta} \frac{1}{i\omega(i\omega - E/\eta)} & \text{if } \omega \neq 0 \\ C\delta(\omega) & \text{if } \omega = 0, \end{cases} \tag{5}$$

here C is some constant (that will be identified by the boundary conditions for $j(t)$ in part b)) and $\delta(\omega)$ the delta distribution.

b) (3P) Transform your result from a) back into the time domain and choose C in such a way that the condition $j(t) = 0$ for $t < 0$ is satisfied. The response function you should get is

$$j(t) = \frac{\Theta(t)}{E}\left(1 - e^{-E/\eta\, t}\right), \tag{6}$$

where

$$\Theta(t) = \begin{cases} +1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{7}$$

is the Heaviside step function.

Hint: Use that, as we showed in the lecture, the Fourier transform of a convolution is the product of the individual Fourier transforms. You can use that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{-i\omega}\,\mathrm{d}\omega = \Theta(t) - \frac{1}{2} \tag{8}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{a - i\omega}\,\mathrm{d}\omega = e^{-at}\Theta(t) \qquad (a > 0). \tag{9}$$

c) (3P) Now assume a constant stress σ_0 is suddenly applied at a time $t_0 \geq 0$ and then released again at a later time $t_1 > t_0$, i.e.

$$\sigma(t) = \sigma_0 \Theta(t - t_0)\Theta(t_1 - t). \tag{10}$$

1. Use your result from b) to explicitly calculate the strain $\epsilon(t)$.
2. Plot $\sigma(t)$ and $\epsilon(t)$ schematically.
3. Does the Kelvin-Voigt material behave more like an elastic or a viscous material for small times? Does the model describe a liquid or a solid at short times?