Freie Universität Berlin
Fachbereich Physik
Prof. Dr. Roland Netz
http://www.physik.fu-berlin.de/en/einrichtungen/ag/ag-netz/lehre

3.12.2018
Due date: 10.12.2018

Advanced Statistical Physics II – Problem Sheet 7

Problem 1 – **Diffusion equation**

a) (5P) Consider the (inhomogeneous) diffusion equation in one dimension:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) c(x,t) = f(x,t) \tag{1}$$

The standard (homogeneous) diffusion equation is recovered for $f \equiv 0$. Using Fourier analysis, derive the expression for the Green's function $G(x,t|x_0,t_0)$ for eq. (1):

$$G(x,t|x_0,t_0) = \frac{\Theta(t-t_0)}{\sqrt{4\pi D(t-t_0)}} e^{-\frac{(x-x_0)^2}{4D(t-t_0)}}, \qquad \Theta(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{else} \end{cases} \tag{2}$$

i.e., $\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) G(x,t|x_0,t_0) = \delta(x-x_0)\delta(t-t_0)$.

b) (2P) Show that the solution of (1) is given by the convolution of the source term $f(x,t)$ and the Green's function $G(x,t|x_0,t_0)$

$$c(x,t) = \int dx' \int dt' G(x,t|x',t') f(x',t') \tag{3}$$

c) (6P) Use $G(x,t|x_0,t_0)$ to find the solution $c(x,t)$ for a box-shaped initial concentration profile:

$$c_0(x) = \begin{cases} 1/2l & |x| < l \\ 0 & \text{else} \end{cases} \tag{4}$$

i.e. $f(x,t) = \delta(t)c_0(x)$. Express the solution in terms of the error function

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \tag{5}$$

Suggestion: It is instructive to plot the solution $c(x,t)$ vs. the position x for various times t. (This is not mandatory.)

Problem 2 – **Diffusion-degradation equation**

Suppose we would like to calculate the concentration of some pollutant inside the ground as a function of depth x from the surface. Assuming lateral translational invariance and homogeneous composition of the soil, the problem is effectively one dimensional and thus the diffusion equation (1) might be a reasonable model. Due to bacterial activity, the pollutant degrades at a constant rate $a > 0$. To take this effect into account, we consider the modified diffusion equation that includes a degradation that is proportional to concentration:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2} + a\right)\rho(x,t) = f(x,t) \tag{6}$$

a) (4P) Suppose that the pollutant is released into the ground at a constant rate at depth x_0. We set the source term on the r.h.s. of (6) to $f(x,t) = Q\delta(x - x_0)$ and neglect the presence of the surface at $x = 0$. Find the stationary solution $c_{\text{stat}}(x)$ of (6) for this case!

b) (3P) Interpret the solution: What is the characteristic length scale λ of the stationary density profile? What happens in the limit of zero degradation rate $a \to 0$ and what does this suggest for the stationary solution of the standard diffusion equation (1), for $f(x,t) = Q\delta(x - x_0)$, i.e. a constant source?