Blatt 5

1 a) $f(x) = ax + b$

$F(x) = \frac{1}{2}ax^2 + bx + c$

b) $f(x) = (ax+b)^{-1}$

$F(x) = \frac{1}{a}\ln(ax+b)$

c) $f(x) = x^{-5}$

$F(x) = -\frac{1}{4}x^{-4}$

d) $f(x) = \sin(3x)$

$F(x) = -\frac{1}{3}\cos(3x)$

e) $f(x) = (x+1)^{1/2}$

$F(x) = \frac{2}{3}(x+1)^{3/2}$

f) $f(x) = (a+x)^{-1/2}$

$F(x) = 2(a+x)^{1/2}$

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a) $F(x) = \frac{1}{3}x^3$

$$\int_a^b dx\, x^2 = \left[\frac{1}{3}x^3\right]_a^b = \underline{\underline{\frac{1}{3}\left(b^3 - a^3\right)}}$$

b) $F(x) = \ln x$

$$\int_a^b dx\, \frac{1}{x} = \ln b - \ln a = \underline{\underline{\ln \frac{b}{a}}}$$

c) $F(x) = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x$

$$\int_0^3 dx\,(x^2+1)^2 = 3\left(\frac{81}{5} + 6 + 1\right)$$

d) $F(x) = -e^{-x}$

$$\int_0^a dx\, e^{-x} = e^{-a} - e^{0} = \underline{\underline{e^{-a} - 1}}$$

e) $$\int_a^b dx\, \frac{1}{\sqrt{(x-a)(b-x)}}$$

Substitution

$$z = x - \underbrace{\frac{(a+b)}{2}}_{=c/2}$$

$$\int_{-c/2}^{c/2} dz\, \frac{1}{\sqrt{-z^2 + \frac{c^2}{4}}}$$

Substitution

$$\xi = \frac{2}{c}z$$

$$\int_{-1}^{1} d\xi \underbrace{\frac{1}{\sqrt{-\xi^2+1}}}_{f(\xi)} \Rightarrow F(\xi) = \arcsin(\xi)$$

$$\int_a^b dx \frac{1}{\sqrt{(x-a)(b-x)}} = \Big(\underbrace{\arcsin(1)}_{= \pi/2} - \underbrace{\arcsin(-1)}_{= -\pi/2}\Big)$$

$$= \pi$$

f) $$\int_0^{2\pi} dx \cos^2 x \underset{\uparrow \text{ Symmetrie}}{=} \int_0^{2\pi} dx \sin^2 x$$

$$\Rightarrow 2\int_0^{2\pi} dx \cos^2 x = \int_0^{2\pi} (\underbrace{\cos^2 x + \sin^2 x}_{=1})\, dx = \int_0^{2\pi} dx$$

$$\Rightarrow 2\int_0^{2\pi} dx \cos^2 x = 2\pi \; ; \; \int_0^{2\pi} dx \cos^2 x = \underline{\underline{\pi}}$$

A3

a) $\int \frac{x^2}{x^3+1}\,dx$

$$\boxed{\begin{aligned} &\text{Subst.} \\ &u = x^3+1 \\ &\frac{du}{dx} = 3x^2 \end{aligned}}$$

$$\int \frac{1}{3}\,\frac{1}{u}\,du = \frac{1}{3}\ln u + C$$

$$= \frac{1}{3}\ln(x^3+1) + C$$

b) $\int \frac{\cos x}{\sin x}\,dx$

$$\boxed{\begin{aligned} &u = \sin x \\ &\frac{du}{dx} = \cos x \end{aligned}}$$

$$\int \frac{1}{u}\,du = \ln u + C$$

$$= \ln(\sin x) + C$$

c) $\int e^{\cos x}\sin x\,dx$

$$\boxed{\begin{aligned} &u = \cos x \\ &\frac{du}{dx} = -\sin x \end{aligned}}$$

$$-\int e^u\,du = -e^u + C$$

$$= -e^{\cos x} + C$$

d) $\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$

$$\boxed{\begin{aligned} u &= \cosh x \\ \frac{du}{dx} &= \sinh x \end{aligned}}$$

$= \int \frac{1}{u} \, du = \ln u + C$

$= \ln(\cosh x) + C$

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e) $\int \frac{1}{x \ln^2 x} \, dx$

$$\boxed{\begin{aligned} \ln x = u \,&,\quad x = e^u \\ \frac{du}{dx} &= \frac{1}{x} \end{aligned}}$$

$\int \frac{1}{u^2} \, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$

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f) $\int x \, e^{-ax^2} dx$

$$\boxed{\begin{aligned} u &= ax^2 \\ \frac{du}{dx} &= 2ax \end{aligned}}$$

$= \int \frac{1}{2a} e^{-u} \, du = -\frac{1}{2a} e^{-u} + C$

$= -\frac{1}{2a} e^{-ax^2} + C$

4 a) $\displaystyle\int \frac{1}{\sin x}\,dx$

$$\boxed{\begin{aligned} &t = \tan\frac{x}{2} \\ &\frac{dt}{dx} = \frac{1}{\cos^2\frac{x}{2}} \end{aligned}}$$

$$\underset{\substack{\uparrow \\ \sin x \\ = \\ 2\cdot\sin\frac{x}{2}\cos\frac{x}{2}}}{=} \int \frac{\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dt = \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}\ln(t) + C = \frac{1}{2}\ln\left(\tan\frac{x}{2}\right) + C$$

b) $\displaystyle\int \frac{1}{1+\sin^2 x}\,dx$

$$\boxed{\begin{aligned} &t = \sqrt{2}\tan x \\ &\frac{dt}{dx} = \sqrt{2}\,\frac{1}{\cos^2 x} \end{aligned}}$$

$$\underset{\substack{\uparrow \\ 1=\sin^2 x+\cos^2 x}}{=} \frac{1}{\sqrt{2}}\int dt\,\frac{\cos^2 x}{2\sin^2 x + \cos^2 x}$$

$$= \frac{1}{\sqrt{2}}\int dt\,\frac{1}{\underbrace{2\frac{\sin^2 x}{\cos^2 x}}_{t^2} + 1} = \sqrt{2}\int dt\,\frac{1}{t^2+1}$$

$$= \frac{1}{\sqrt{2}}\arctan(t) + C$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right) + C$$