Musterlösung – Blatt 6

(1) (a)

$$\int x^2 e^x \, dx = x^2 e^x - \int (2x) \cdot e^x \, dx$$
$$= x^2 e^x - 2\left(x e^x - \int e^x \, dx \right)$$
$$= x^2 e^x - 2 x e^x + 2 e^x$$
$$= e^x \left(x^2 - 2x + 2 \right)$$

(b)

$$\int x^3 e^{-\frac{x^2}{a}} \, dx = a \int \frac{x^2}{a} \cdot x \cdot e^{-\frac{x^2}{a}} \, dx$$

Substituiere: $\frac{x^2}{a} = u$

$$\frac{2x}{a} = \frac{du}{dx} \Rightarrow dx = \frac{a\,du}{2x}$$

$$= a \int u \cdot x \cdot e^{-u} \, \frac{a\,du}{2x}$$
$$= \frac{a^2}{2} \int u \cdot e^{-u} \, du$$
$$= \frac{a^2}{2} \left[u \cdot (-e^{-u}) - \int 1 \cdot (-e^{-u}) \, du \right]$$
$$= \frac{a^2}{2} e^{-u} (-u - 1)$$
$$= \frac{-a^2}{2} e^{-\frac{x^2}{a}} \left(\frac{x^2}{a} + 1 \right)$$

(c) $$\int e^x \cos x \, dx = e^x \cos x - \int e^x \cdot (-\sin x)\, dx$$

$$= e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, dx \right)$$

$$= e^x (\cos x + \sin x) - \int e^x \cos x \, dx$$

$$\Rightarrow \quad 2 \int e^x \cos x \, dx = e^x (\cos x + \sin x)$$

$$\int e^x \cos x \, dx = \frac{e^x (\cos x + \sin x)}{2}$$

(d) $$\int \frac{\ln x}{x} \, dx = \int \frac{1}{x} \cdot \ln x \, dx$$

$$= \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x} \, dx$$

$$\Rightarrow \quad \int \frac{\ln x}{x} \, dx = \frac{\ln^2 x}{2}$$

(e) $$\int e^x \sin^2 x \, dx = e^x \sin^2 x - \int e^x \cdot (2 \sin x \cos x)\, dx$$

$$= e^x \sin^2 x - \int e^x \sin(2x)\, dx$$

Nebenrechnung: $$\int e^x \sin(2x)\, dx = e^x \sin(2x) - \int e^x (2\cos(2x))\, dx$$

$$= e^x \sin(2x) - 2\left[e^x \cos(2x) - \int e^x (-2\sin(2x)) \right]$$

$$= e^x \sin(2x) - 2 e^x \cos(2x) - 4 \int e^x \sin(2x)\, dx$$

$$\Rightarrow \quad \int e^x \sin(2x)\, dx = \frac{e^x \sin(2x) - 2 e^x \cos(2x)}{5}$$

$$\int e^x \sin^2 x \, dx = e^x \sin^2 x - \frac{e^x \sin(2x) - 2 e^x \cos(2x)}{5}$$

$$= \frac{e^x}{5} \left[5 \sin^2 x - \sin(2x) + 2 \cos(2x) \right]$$

Fortsetzung 1 (e)

$$\int e^x \sin^2 x \, dx = \frac{e^x}{5}\left[5 \sin^2 x - \sin(2x) + 2\cos(2x)\right]$$

$$= \frac{e^x}{10}\left[10 \sin^2 x - 2\sin(2x) + 4\cos(2x)\right]$$

Nutze: $\cos(2x) = \cos^2 x - \sin^2 x$

$$= 1 - 2\sin^2 x$$

$$\Leftrightarrow 2\sin^2 x = 1 - \cos(2x)$$

$$= \frac{e^x}{10}\left[5(1 - \cos(2x)) - 2\sin(2x) + 4\cos(2x)\right]$$

$$= \frac{e^x}{10}\left[5 - \cos(2x) - 2\sin(2x)\right]$$

(f) $$\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx$$

$$= x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$$

Nebenrechnung: $\int x \cdot \frac{1}{\sqrt{1-x^2}} = \int \sin\varphi \frac{1}{\sqrt{1-\sin^2\varphi}} \cos\varphi \, d\varphi$

Substituiere: $x = \sin\varphi$

$dx = \cos\varphi \, d\varphi$

$$= \int \sin\varphi \, d\varphi$$

$$= -\cos\varphi$$

$$= -\sqrt{1-\sin^2\varphi}$$

$$= -\sqrt{1-x^2}$$

$$= x \arcsin x + \sqrt{1-x^2}$$

② (a) $\int \frac{x^2}{x^2-a^2} = \int \frac{\cancel{x}^2}{a^2}\cdot\frac{1}{\frac{x^2}{a^2}-1}\,dx = a\int\frac{u^2}{u^2-1}\,du$

Substituiere: $\frac{x}{a} = u$

$dx = a\,du$

$$a\int\frac{u^2}{u^2-1}\,du = a\int\frac{u^2}{(u-1)(u+1)}\,du = \frac{a}{2}\int\left(\frac{u}{u+1}+\frac{u}{u-1}\right)du$$

Nebenrechnung: $\frac{u^2}{(u-1)(u+1)} \overset{!}{=} \frac{Au}{(u+1)} + \frac{Bu}{(u-1)}$

$$= \frac{Au(u-1)+Bu(u+1)}{(u+1)(u-1)}$$

$$= \frac{Au^2 - A + Bu^2 + Bu}{(u+1)(u-1)}$$

$$\Rightarrow \begin{array}{l} A+B=1 \\ B-A=0 \end{array} \Rightarrow B = A = \frac{1}{2}$$

$$\frac{a}{2}\int\left(\frac{u}{u+1}+\frac{u}{u-1}\right)du = \frac{a}{2}\int\frac{u}{u+1}du + \frac{a}{2}\int\frac{u}{u-1}du$$

Nebenrechnung: $\int\frac{u}{u\pm 1}\,du = \int\frac{z\mp 1}{z}\,dz = \int\left(1\mp\frac{1}{z}\right)dz$

Substituiere: $z = u\pm 1 \Rightarrow u = z\mp 1$

$dz = du$

$$= z \mp \ln z$$
$$= (u\pm 1) \mp \ln(u\pm 1)$$

$$\frac{a}{2}\int\frac{u}{u+1}du + \frac{a}{2}\int\frac{u}{u-1}du = \frac{a}{2}(u+1) - \frac{a}{2}\ln(u+1) + \frac{a}{2}(u-1) + \frac{a}{2}\ln(u-1)$$

$$= au + \frac{a}{2}\ln\left(\frac{u-1}{u+1}\right) = x + \frac{a}{2}\ln\left(\frac{x-a}{x+a}\right)$$

(b) Nebenrechnung:

$$(x^2-3x+2) = (x-2)(x-1) \qquad \text{da} \quad 0 \overset{!}{=} x^2-3x+2$$

$$x_{1/2} = \frac{3}{2} \pm \sqrt{\frac{9}{4}-2}$$

$$= \frac{3}{2} \pm \frac{1}{2}$$

$$\frac{x+7}{(x-2)(x-1)} \overset{!}{=} \frac{A}{(x-2)} + \frac{B}{(x-1)}$$

$$= \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$$

$$= \frac{(A+B)x+(-A-2B)}{(x-2)(x-1)}$$

$$\Rightarrow \begin{array}{l} A+B=1 \\ -A-2B=7 \end{array} \Rightarrow \begin{array}{l} A=1-B \\ -1+B-2B=7 \end{array} \Rightarrow \begin{array}{l} A=9 \\ B=-8 \end{array}$$

$$\int \frac{x+7}{x^2-3x+2}\,dx = \int\left(\frac{9}{x-2} - \frac{8}{x-1}\right)dx$$

$$= 9\ln(x-2) - 8\ln(x-1)$$

(c) $$\int \frac{1}{e^x-1}\,dx = \int \frac{1}{(u-1)}\,\frac{du}{u} = \int\left(\frac{1}{u-1} - \frac{1}{u}\right)du$$

Substituiere: $e^x = u$

$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x} = \frac{du}{u}$$

Nebenrechnung: $$\frac{1}{u(u-1)} \overset{!}{=} \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1)+Bu}{u(u-1)}$$

$$= \frac{(A+B)u-A}{u(u-1)}$$

$$\Rightarrow \begin{array}{l} A+B=0 \\ -A=1 \end{array} \Rightarrow \begin{array}{l} A=-1 \\ B=1 \end{array}$$

Fortsetzung 2 (e):

$$\int \frac{1}{e^x-1} = \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$
$$= \ln(u-1) - \ln u$$
$$= \ln\left(\frac{e^x-1}{e^x}\right)$$

3 (a) $\int \frac{1}{1+\sqrt{x}}\, dx = 2 \int \frac{u}{1+u}\, du$

Substituiere: $\sqrt{x} = u$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u} \iff dx = 2u\,du$$

Nebenrechnung: $\frac{u}{1\cdot(1+u)} \stackrel{!}{=} \frac{A}{1} + \frac{B}{1+u}$

$$= \frac{A(1+u)+B}{1+u} = \frac{Au + (A+B)}{1+u}$$

$$\Rightarrow \begin{array}{c} A=1 \\ A+B=0 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \end{array}$$

$$2\int \frac{u}{1+u} = 2\int \left(1 - \frac{1}{1+u}\right) du$$
$$= 2\left(u - \ln(1+u)\right)$$
$$= 2\sqrt{x} - 2\ln(1+\sqrt{x})$$

(l.) $\int \frac{1}{\sinh x} = \int \frac{2}{e^x - e^{-x}} = \int \frac{2}{u - u^{-1}} \; \frac{du}{u} = \int \frac{2}{u^2 - 1}$

Substituiere: $u = e^x$

$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{u}$$

Nebenrechnung: $\frac{2}{u^2-1} \overset{!}{=} \frac{A}{u+1} + \frac{B}{u-1}$

$$= \frac{A(u-1) + B(u+1)}{(u+1)(u-1)}$$

$$= \frac{(A+B)u + (B-A)}{(u+1)u-1)}$$

$$\Rightarrow A + B = 0 \quad \Rightarrow \quad A = -1$$
$$B - A = 2 \qquad\qquad B = 1$$

$$\int \frac{2}{u^2-1} = \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du$$

$$= \ln(u-1) - \ln(u+1)$$

$$= \ln\left(\frac{u-1}{u+1}\right) = \ln\left(\frac{2e^{-x/2}}{2e^{-x/2}} \cdot \frac{e^x - 1}{e^{\cancel{x}} + 1}\right)$$

$$= \ln\left(\frac{\sinh \frac{x}{2}}{\cosh \frac{x}{2}}\right)$$

$$= \ln\left(\tanh\left(\frac{x}{2}\right)\right)$$

(4)

$$F_n = \int dx \, \frac{1}{(x^2+1)^n}$$

(a) $F_1 = \int dx \, \frac{1}{(x^2+1)^1} = \int \frac{du}{\cos^2 u} \, \frac{1}{1+\tan^2 u} = \int \frac{du}{\cancel{\cos^2 u}} \, \frac{1}{\frac{1}{\cancel{\cos^2 u}}} = u = \arctan x$

Substituiere: $x = \tan u$

$$\frac{dx}{du} = \frac{1}{\cos^2 u} = 1 + \tan^2 u$$

(b) Schreibe F_n um:

$$F_n = \int dx \, \frac{1}{(1+x^2)^n} = \int \frac{du}{\cos^2 u} \, \frac{1}{(1+\tan^2 u)^n} = \int du \, \frac{\cos^{2n} u}{\cos^2 u}$$

Benutze Substitution aus (a)

Folglich ist

$$F_n = \int du \, \cos^{2n-2} u$$

und

$$F_{n+1} = \int du \, \cos^{2n} u$$

Berechne F_{n+1}

$$F_{n+1} = \int du \, \cos^{2n} u = \int du \, \cos u \cdot \cos^{2n-1} u \, du$$

$$= \sin u \cdot \cos^{2n-1} u - \int \sin u \cdot ((2n-1) \cdot \cos^{2n-2} u \cdot (-\sin u)) du$$

$$= \sin u \cdot \cos^{2n-1} u + \int \sin^2 u \cdot (2n-1) \cos^{2n-2} u \, du$$

$$= \sin u \cdot \cos^{2n-1} u + (2n-1) \int (1-\cos^2 u) \cos^{2n-2} u \, d$$

$$= \sin u \cdot \cos^{2n-1} u + (2n-1) \int \cos^{2n-2} u \, du + (2n-1) \int \cos^{2n} u \, d$$

Fortsetzung ④ (l.)

$$F_{n+1} = \int du \cos^{2n} u \, du$$

$$= \sin u \cos^{2n-1} u + (2n-1) \int \cos^{2n-2} u \, du + (2n-1) \int \cos^{2n} u \, du$$

$$= \sin u \cos^{2n-1} u + (2n-1) F_n + (2n-1) F_{n+1}$$

$$\Rightarrow \quad 2n F_{n+1} = \sin u \cos^{2n-1} u + (2n-1) F_n$$

Nebenrechnung:

$$\sin u \cos^{2n-1} u = \frac{\sin u}{\cos u} \cos^{2n} u = \tan u \frac{1}{(1+\tan^2 u)^n}$$

Rücksubstitution:

$$2n F_{n+1} = \frac{x}{(1+x^2)^n} + (2n-1) F_n$$

$$F_{n+1} = \frac{1}{2n} \frac{x}{(1+x^2)^n} + \frac{(2n-1)}{2n} F_n$$

$$F_2 = \frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} F_1 = \frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} \arctan x$$

$$F_3 = \frac{1}{4} \frac{x}{(1+x^2)^2} + \frac{3}{4} F_2$$

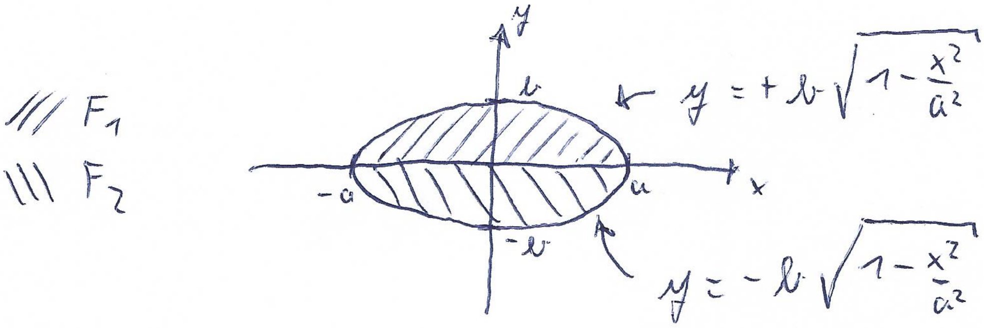
$$= \frac{1}{4} \frac{x}{(1+x^2)^2} + \frac{3}{8} \frac{x}{1+x^2} + \frac{3}{8} \arctan x$$

$$= \frac{2}{8} \frac{x}{(1+x^2)^2} + \frac{3}{8} \frac{x(1+x^2)}{(1+x^2)^2} + \frac{3}{8} \arctan x$$

$$= \frac{5x + 3x^2}{8(1+x^2)^2} + \frac{3}{8} \arctan x$$

(5) (a)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$



$$F = F_1 + F_2$$

$$= \int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}} + \left(-\int_{-a}^{a} (-1)\, b \sqrt{1 - \frac{x^2}{a^2}}\right)$$

$$= 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}$$

Substituiere $\quad \frac{x}{a} = \sin\varphi \qquad x = a \to \varphi = 0$

$\qquad x = -a \to \varphi = -\pi$

$$dx = a\cos\varphi\, d\varphi$$

$$= 2b \int_{-\pi}^{0} d\varphi\, a\cos\varphi \sqrt{1 - \sin^2\varphi}$$

$$= 2ba \int_{-\pi}^{0} \cos^2\varphi\, d\varphi$$

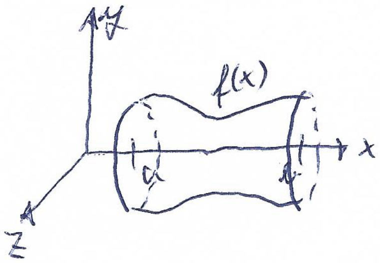
$$= 2ab \int_{-\pi}^{0} \frac{1}{2}(1 + \cos(2x))\, d\varphi$$

$$= ab\left[\varphi + \frac{1}{2}\sin(2x)\right]_{-\pi}^{0} = ab\left[0 - (-\pi) + 0 - 0\right]$$

$$= \pi ab$$

(b)

$$V = \int_a^b dx\, \pi\, [f(x)]^2 \quad \text{für}$$



Für Ellipse

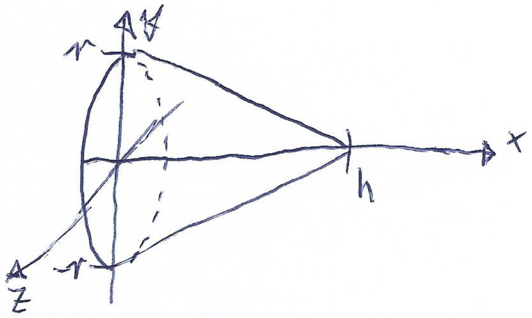
$$V = \int_{-a}^{a} dx\, \pi b^2 \left(1 - \frac{x^2}{a^2}\right) = 2\pi a b - \int_{-a}^{a} dx\, \pi \frac{b^2}{a^2} x^2$$

$$= 2\pi a b^2 - \frac{\pi}{3} \frac{b^2}{a^2} \left[x^3\right]_{-a}^{a}$$

$$= 2\pi a b^2 - \frac{2\pi}{3} a b^2$$

$$= \frac{4\pi}{3} a b^2$$

(c)



$$\Rightarrow f(x) = r - \frac{r}{h} x$$

$$V = \int_0^h dx\, \pi \left(r - \frac{r}{h} x\right)^2 = \int_0^h dx\, \pi \left(r^2 - 2\frac{r^2}{h} x + \frac{r^2}{h^2} x^2\right)$$

$$= 2\pi r^2 \int_0^h \left(\frac{1}{2} - \frac{x}{h} + \frac{x^2}{2h^2}\right) dx$$

$$= 2\pi r^2 \left[\frac{x}{2} - \frac{x^2}{2h} + \frac{x^3}{6h^2}\right]_0^h$$

$$= 2\pi r^2 \left[\frac{h}{2} - \frac{h}{2} + \frac{h}{6}\right]$$

$$= \frac{\pi}{3} r^2 h$$

⑥

$$F(x,s) = \frac{x^s - 1}{\ln x}$$

(a) $$g(x,s) = \frac{\partial}{\partial s} F(x,s) = \frac{\partial}{\partial s} \frac{x^s - 1}{\ln x}$$

$$= \frac{\partial}{\partial s} \frac{x^s}{\ln x} - \frac{\partial}{\partial s} \frac{1}{\ln x} = \frac{\partial}{\partial s} \frac{x^s}{\ln x}$$

$$= \frac{\partial}{\partial s} \frac{e^{s \ln x}}{\ln x}$$

$$= e^{s \ln x}$$

$$= x^s$$

(b) $$G(x,s) = \int_0^1 g(x,s)\, dx = \int_0^1 x^s\, dx$$

$$= \left[\frac{1}{1+s} x^{s+1}\right]_0^1$$

$$= \frac{1}{1+s}$$

(c) $$H(s) = \int ds\, G(x,s)$$

$$= \int ds\, \frac{1}{1+s}$$

$$= \ln(1+s)$$

(d) $H(\lambda) = \int_0^1 dx\ \frac{x^\lambda - 1}{\ln x} + C$

Betrachte

$$\int_0^1 dx\ \frac{x^\lambda - 1}{\ln x} = \int_{-\infty}^0 dy\ e^y\ \frac{e^{y\lambda} - 1}{y} = \int_{-\infty}^0 \frac{e^{y(1+\lambda)} - e^y}{y}\ dy$$

Substituiere $x = e^y \iff \ln x = y$

$\frac{dx}{dy} = e^y$ $\qquad \ln 0 = -\infty$

$\ln 1 = 0$

Nebenrechnung:

$$e^{y(1+\lambda)} - e^y = e^{y(1+\frac{\lambda}{2})}\left(e^{y\frac{\lambda}{2}} - e^{-y\frac{\lambda}{2}}\right)$$

$$= 2e^{y(1+\frac{\lambda}{2})} \sinh\left(y\frac{\lambda}{2}\right)$$

$$\int_0^1 dx\ \frac{x^\lambda - 1}{\ln x} = \int_{-\infty}^0 2\, e^{y(1+\frac{\lambda}{2})}\ \frac{\sinh(y\frac{\lambda}{2})}{y}\ dy$$

$$= -2\int_0^{-\infty} e^{y(1+\frac{\lambda}{2})}\ \frac{\sinh(y\frac{\lambda}{2})}{y}\ dy$$

Substituiere: $z = -y$

$dz = -dy$

$$= +2\int_0^\infty e^{-\cancel{y}z(1+\frac{\lambda}{2})}\ \frac{\sinh(\cancel{y}z\frac{\lambda}{2})}{-z}\ dz$$

$$= 2\int_0^\infty e^{-z(+1+\frac{\lambda}{2})}\ \frac{\sinh(z\frac{\lambda}{2})}{z}\ dz$$

Integrale der Form

$$\int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}\{f(t)\} = F(s)$$

nennt man Laplace-Transformation

Die Laplace-Transformation von sinh ist

$$\int_0^\infty e^{-st} \sinh(at)\, dt = \frac{a}{s^2-a^2}$$

Weiterhin gilt, dass falls sowohl die Laplace-Transformation einer Funktion $f(t)$ als auch der Grenzwert

$$\lim_{t\to 0} \frac{f(t)}{t}$$

existieren, dann gilt

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\, du$$

Folglich gilt in unserem Fall

$$\int_0^1 dx\, \frac{x^t-1}{\ln x} = 2\int_0^\infty e^{-z\left(1+\frac{t}{2}\right)} \frac{\sinh\left(z\,\frac{t}{2}\right)}{z}\, dz$$

$$= 2\int_{\left(1+\frac{t}{2}\right)}^\infty \frac{\frac{t}{2}}{u^2-\left(\frac{t}{2}\right)^2}\, du$$

Fortsetzung von ⑥ (e)

mit $a = \frac{1}{2}$

$$2\int_{(1+a)}^{\infty} \frac{a}{u^2-a^2}\,du = \int_{(1+a)}^{\infty}\left(\frac{1}{u-a} - \frac{1}{u+a}\right)du$$

$$= \left[\ln\left(\frac{u-a}{u+a}\right)\right]_{1+a}^{\infty}$$

$$= -\ln\left(\frac{1+a-a}{1+a+a}\right)$$

$$= -\ln\left(\frac{1}{1+2a}\right)$$

$$= \ln(1+2a)$$

$$= \ln(1+1)$$

Folglich ist $C = 0$

und

$$H(1) = \int_0^1 dx\,\frac{x-1}{\ln x} = \ln(1+1) = \ln(2)$$

$$H(2) = \int_0^1 dx\,\frac{x^2-1}{\ln x} = \ln(3)$$

Alternativ kann man $H(1)$ aus Tabellen von Exponential- und Logarithmen-integral nachschlagen