Thermodynamics of weakly measured quantum systems

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We consider continuously monitored quantum systems and introduce definitions of work and heat along individual quantum trajectories that are valid for coherent superpositions of energy eigenstates. We use these quantities to extend the first and second laws of stochastic thermodynamics to the quantum domain. We illustrate our results with the case of a weakly measured driven two-level system and show how to distinguish between quantum work and heat contributions. We finally employ quantum feedback control to suppress detector backaction and determine the work statistics.

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Thermodynamics is, at its heart, a theory of work and heat. The first law is based on the realization that both quantities are two forms of energy and that their sum is conserved. At the same time, the fact that entropy, defined as the ratio of reversible heat and temperature, can only increase in an isolated system is an expression of the second law [1, 2]. In classical thermodynamics, work is defined as the change of internal energy in an isolated system, $W = \Delta U$, while heat is introduced as the difference, $Q = \Delta U - W$, in a nonisolated system. Thermal isolation is thus crucial to distinguish W from Q. In the last decades, stochastic thermodynamics has successfully extended the definitions of work and heat to the level of single trajectories of microscopic systems [3]. In this regime, thermal fluctuations are no longer negligible and the laws of thermodynamics have to be adapted to fully include them. The second law has, for instance, been generalized in the form of fluctuation theorems that quantify the occurrence of negative entropy production [4]. A particular example is the Jarzynski equality, $\langle \exp(-\beta W)\rangle = \exp(-\beta\Delta F)$, that allows the determination of equilibrium free energy differences ΔF from the nonequilibrium work statistics in systems at initial inverse temperature β [5]. The laws of stochastic thermodynamics have been verified in a large number of different experiments, see Refs. [6, 7] and the review [8].

The challenge of this day is to extend the principles of thermodynamics to include quantum effects which are expected to dominate at smaller scales and colder temperatures. Some of the unsolved key issues concern the correct definition of quantum work and heat, means to distinguish between the two quantities owing to the blurring effect of quantum fluctuations, and the proper clarification of the role of quantum coherence. A variety of approaches have been put forward to tackle these and related problems [9–20], and quantum work statistics has been measured in isolated systems in two pioneering experiments using NMR [21] and trapped ions [22]. However, it appears that each approach suffers from limitations and there still seems to be little consensus [20]. A new approach to the problem can emerge from the possibility of weakly monitoring quantum systems. Recently, individual quantum trajectories of a superconducting qubit in a microwave cavity have been observed using weak measurements [23, 24]. These measurements only slightly disturb quantum systems owing to the weak coupling to the measuring device [25]. They hence allow to gain information about quantum states without projecting them into eigenstates. They have been employed with great success to explain quantum paradoxes [26], detect and amplify weak signals [27, 28], determine a quantum virtual state [30], as well as directly measure a wave function [29]. Motivated by the two experiments [23, 24], we here investigate the first and second law for continuously monitored quantum systems and aim at developing a quantum stochastic thermodynamics based on quantum trajectories. Such an extension faces several technical difficulties. First, since a weakly measured system can be in a coherent superposition of energy eigenstates, energy is not a well-defined concept along a single quantum trajectory. Furthermore, even in the absence of an external environment, a continuously monitored quantum system is not isolated and the detector backaction, albeit small, will perturb its dynamics [25]. As a result, its time evolution will be nonunitary and energy, in the form of heat, will be exchanged with the detector.

In the following, we introduce suitable and consistent definitions of work and heat contributions to the quantum stochastic evolution of a weakly measured system that is externally driven. We use these definitions to determine the distributions of quantum work and heat for a driven two-level system, and demonstrate the general validity of the Jarzynski equality, hence of the second law. We finally use the tools of quantum feedback control [31] to suppress detector backaction and thus effectively achieve thermal isolation of the system. This provides a practical scheme to experimentally test our definitions of work and heat along individual quantum trajectories.

Quantum work and heat. We consider a system with time-dependent Hamiltonian H_t that is initially in a thermal state at inverse temperature β, $\rho_0 = \exp(-\beta H_0)/Z_0$, where Z_0 is the partition function. The system is driven by an external parameter λ_t during a time interval τ. This driving induces a change of energy in the form of work, while heat may be exchanged with the outside world. At the ensemble level, quantum work and heat are introduced by considering an infinitesimal variation of the mean energy, $U = \langle H \rangle = \mathrm{tr}\{\rho_t H_t\}$ [32, 33]:

$$dU = \mathrm{tr}\{\rho_t dH_t\} + \mathrm{tr}\{d\rho_t H_t\} = \delta W + \delta Q. \quad (1)$$

Heat is further related to entropy $S = -k\mathrm{tr}\{\rho_t \ln \rho_t\}$ via $\delta Q = TdS$ [32, 33]. For an isolated system with unitary dynamics heat vanishes, since $dS = 0$, and therefore $dU = \delta W$ in agreement with classical thermodynamics [1, 2]. Heat therefore appears to be fundamentally associated with the nonunitary part of the dynamics.

At the level of individual realizations, energy is a stochastic quantity owing to thermal and quantum fluctuations. The distribution $p(u)$ of the total energy change u may be determined by performing projective measurements Π_n and Π_m, with outcomes E_n^0 and E_m^τ, at the beginning and at the end of the driving protocol [9, 34],

$$p(u) = \sum_{m,n} P^\tau_{m,n} P^0_n \delta(u - \Delta E_{m,n}). \quad (2)$$

Here $P_n^0 = \mathrm{tr}\{\Pi_n \rho_0\}$ denotes the probability of the eigenvalue E_n^0, $P^\tau_{m,n} = \mathrm{tr}\{\Pi_m \rho_{n,\tau}\}$ the transition probability from state n to m, with $\rho_{n,\tau}$ the time evolved projected density operator $\rho_{n,0} = \Pi_n \rho_0 \Pi_n / P_n^0$, and $\Delta E_{m,n} = E_m^\tau - E_n^0$ the energy difference. For unitary dynamics, $p(u)$ reduces to the work distribution $p(W)$, but, in general, Eq. (2) does not allow to distinguish work from heat. In the following, we generalize Eq. (2) and identify work and heat for a weakly measured quantum system.

A quantum system continuously monitored by a quantum limited detector may be assigned, for each individual trajectory, a conditional density operator $\tilde{\rho}_t$ that reduces to the usual density operator ρ_t when averaged over all the trajectories, $\rho_t = \langle\langle \tilde{\rho}_t \rangle\rangle$ [31, 35]. The evolution of $\tilde{\rho}_t$ is commonly described by a stochastic master equation that contains a random parameter $\xi(t)$ that accounts for the detector shot noise, see Eqs. (9)-(10) below for an example. An important observation is that such master equation has a unitary component, corresponding to the dynamics generated by the system's Hamiltonian, and a nonunitary part that stems from the continuous coupling to the detector. For an infinitesimal time step, these two contributions are additive and may be written as,

$$\delta\tilde{\rho}_t = \delta\mathbb{W}[\tilde{\rho}_t]dt + \delta\mathbb{Q}[\tilde{\rho}_t]dt, \quad (3)$$

where $\delta\mathbb{W}[\tilde{\rho}_t]$ and $\delta\mathbb{Q}[\tilde{\rho}_t]$ are the operators associated with the respective unitary and nonunitary parts of the dynamics. We identify them as corresponding to work and heat at the level of an infinitesimal quantum trajectory. We note that this separation cannot be directly extended to the entire (time integrated) trajectory, since the stochastic master equation is generally a nonlinear function of the operator $\tilde{\rho}_t$. However, when averaged over quantum fluctuations, Eq. (3) allows to extend the first law (1) to single realizations of the stochastic measurement outcome,

$$\begin{aligned} d\tilde{U}_t =& \mathrm{tr}\{H_t\tilde{\rho}_t\} - \mathrm{tr}\{H_{t-dt}\tilde{\rho}_{t-dt}\} \\ =& \mathrm{tr}\{\tilde{\rho}_t dH_t\} + \mathrm{tr}\{H_{t-dt}\delta\mathbb{W}dt\} + \mathrm{tr}\{H_{t-dt}\delta\mathbb{Q}dt\} \\ =& \delta\tilde{W}_t + \delta\tilde{Q}_t, \end{aligned} \quad (4)$$

where $dH_t = H_t - H_{t-dt}$ and $\mathrm{tr}\{H_t \delta\mathbb{W}dt\} = 0$ since $\delta\mathbb{W}$ is unitary [36]. Equation (4) is a direct extension of stochastic thermodynamics to the quantum domain. The familiar first law (1) is recovered when Eq. (3) is averaged over both stochastic and quantum fluctuations. Furthermore, the integrated work and heat contributions to the changes in transition probabilities, $d\tilde{P}_{m,n} = \tilde{P}^\tau_{m,n} - \tilde{P}^0_{m,n}$, with $P^0_{m,n} = \delta_{n,m}$ the initial transition probability, can be obtained from Eq. (3) by carefully adding all the different terms (see Supplemental Material [37]). We find, for each individual quantum trajectory,

$$d\tilde{P}_{m,n} = \delta\tilde{P}^W_{m,n} + \delta\tilde{P}^Q_{m,n}, \quad (5)$$

with the two quantities,

$$\delta\tilde{P}^W_{m,n} = \mathrm{tr}\left\{\Pi_m \int_0^\tau dt\, \delta\mathbb{W}[\tilde{\rho}_t]\right\}, \quad (6)$$

$$\delta\tilde{P}^Q_{m,n} = \mathrm{tr}\left\{\Pi_m \int_0^\tau dt\, \delta\mathbb{Q}[\tilde{\rho}_t]\right\}. \quad (7)$$

We emphasize that these expressions depend on the quantum trajectory $\tilde{\rho}_t$ and, hence, on the specific stochastic detector outcome. They provide an unambiguous way to distinguish between work and heat at the level of a single trajectory. Remarkably, they are valid even if the system remains in a coherent superposition of energy eigenstates, that is, when its energy is ill-defined. Equation (5) holds for the trajectory averaged quantities $dP_{m,n} = \delta P^W_{m,n} + \delta P^Q_{m,n}$, with $\delta P^\alpha_{m,n} = \langle\langle \delta\tilde{P}^\alpha_{m,n} \rangle\rangle$, $\alpha = W, Q$. It is important to note that this averaged distinction between work and heat contributions to transition probabilities requires access to single trajectories, thus cannot be established directly at the ensemble level.

The second law in the form of the Jarzynski equality, $\int dW p(W) \exp(-\beta W) = \exp(-\beta \Delta F)$, immediately follows from Eq. (2) for an isolated system [9]. However, the equality is not satisfied for an open system with nonunitary dynamics owing to the heat term [34]. The second law may be restored by replacing $P^\tau_{m,n}$ by $P^W_{m,n} = P^0_{m,n} + \delta P^W_{m,n}$, that is, by setting $\delta P^Q_{m,n}$ to zero at each time step, see Fig. 3 below. We shall next show how the identification of quantum work and heat may be achieved theoretically, by numerically analyzing a weakly measured driven two-level system, and experimentally, by means of quantum feedback control.

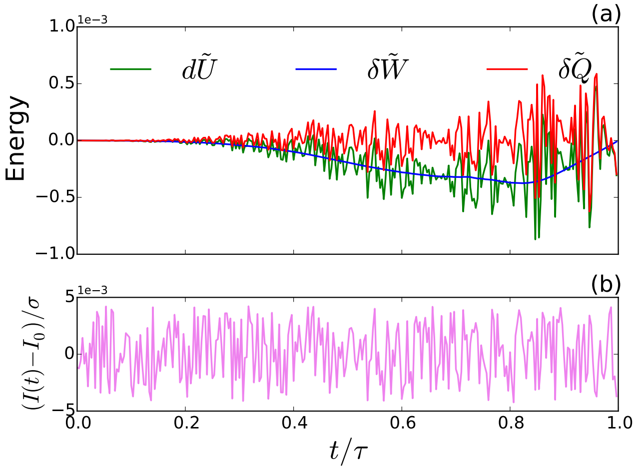

FIG. 1. (color online) First law for a weakly measured qubit. a) Infinitesimal change of work, heat and energy along a single quantum trajectory $\tilde{\rho}_t$; for each realization $d\tilde{U}_t = \delta\tilde{W}_t + \delta\tilde{Q}_t$, Eq. (4). b) Corresponding signal $I(t)$ in the detector. Parameters are $S_0/\Delta I^2 = 2.5\cdot 10^5 dt$, $\hbar/g = 1.6\cdot 10^2 dt$, $\hbar/\epsilon = 10^3 dt$, $\nu = 8$ and $\tau = 3\cdot 10^3 dt$ (see main text).

Application to a monitored qubit. In order to illustrate our approach, we consider a driven two-level system S with Hamiltonian, $H_S = \epsilon\sigma_z + \lambda_t\sigma_x$, where λ_t is the external driving and σ_i the usual Pauli matrices. The system is continuously coupled to a quantum limited detector D via the interaction Hamiltonian H_I:

$$H_t = H_S + H_D + \sigma_z H_I, \qquad (8)$$

where, without loss of generality, we identify σ_z as the system's observable that is monitored by the detector. The effect of the detector is fully characterized by the averaged signals, (I_1, I_2), and Gaussian noises, (S_1, S_2), measured when the qubit is in the two eigenstates, $(|1\rangle, |2\rangle)$, of the measured observable. We assume that the detector is weakly coupled to the system, i.e. the signal that distinguishes state $|2\rangle$ from $|1\rangle$, $\Delta_I = I_2 - I_1$, is much weaker than the typical signal, $I_0 = (I_2+I_1)/2$. The measurement time $\tau_M = (S_1+S_2)/(I_1-I_2)^2$ sets the time scale after which the detector clearly distinguishes state $|1\rangle$ from $|2\rangle$, and thus performs a projective measurement. In the following, we focus on the weak measurement regime, $\tau \ll \tau_M$, and assume that τ_M is the largest time scale of the problem. Our analysis is solely based on this assumption and is, therefore, rather general. For concreteness and simplicity, we will interpret the qubit in Hamiltonian (8) as describing a double quantum dot sharing a single electron and interacting with a quantum point contact (QPC), but it can also be applied to a qubit coupled to a microwave resonator [38] in a circuit QED set-up as in the experiments [23, 24]. We accordingly identify the configurations where the electron occupies only one dot by $\langle\sigma_z\rangle = \pm 1$. Coherent superpositions of the two are possible. The detector monitoring the occupation of the dots is a QPC with Hamiltonian [39–42] $H_D = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r}\Omega(a_r^\dagger a_l + a_l^\dagger a_r)$ and the interaction term reads $H_I = \sum_{l,r}\delta\Omega/2(a_r^\dagger a_l + a_l^\dagger a_r)$. The signal in the detector is the current $I(t)$ across the QPC. It is characterized by $I_{1(2)} 2\pi\Omega_{1(2)}^2 \rho_l\rho_r e^2 V/\hbar = e^2 T_{1(2)} V/h$ and $S_{1(2)} = e(1-T_{1(2)})I_{1(2)}$, where $\rho_{l,r}$ are the densities of states in the left and right electrodes, and $T_{1(2)}$ are the dimensionless transmission probabilities across the QPC. The voltage bias V maintained across the QPC makes the detector an out-of-equilibrium environment for the qubit system.

Under the assumption of a weakly coupled detector with Gaussian noise, the detector signal is a random variable, and the evolution of the system depends on the specific realization of the stochastic process. This is captured by a well-established Bayesian formalism [41, 42] which describes the evolution of the system conditional to the detector's outcome in terms of a nonlinear stochastic differential equation for the system's density matrix $\tilde{\rho}(t)$. In the Ito formulation, we have [41, 42],

$$\dot{\tilde{\rho}}_{11} = -2\frac{\lambda(t)}{\hbar}\,\mathrm{Im}\,(\tilde{\rho}_{12}) + \tilde{\rho}_{11}(1-\tilde{\rho}_{11})\frac{2\Delta I}{S_0}\xi(t), \qquad (9)$$

$$\dot{\tilde{\rho}}_{12} = 2i\frac{\epsilon}{\hbar}\tilde{\rho}_{12} - i\frac{\lambda(t)}{\hbar}(1-2\tilde{\rho}_{11}) - \tilde{\rho}_{12}\frac{(\Delta I)^2}{4S_0} + (1-2\tilde{\rho}_{11})\tilde{\rho}_{12}\frac{\Delta I}{S_0}\xi(t), \qquad (10)$$

where $\xi(t)$ is the white noise of the detector's signal with $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = \sigma^2\delta(t-t')$ and $\sigma = \sqrt{S_0/2\Delta t}$. The detector current $I(t)$ is further given by,

$$I(t) = I_0 + \frac{\Delta I}{2}(2\tilde{\rho}_{11} - 1) + \xi(t). \qquad (11)$$

For each realization of the measurement outcome, Eqs. (9) and (10) allow us to identify the unitary and nonunitary contributions to the time evolution, since the nonunitary part is proportional to the current I. We rewrite Eq. (3) as $\delta\tilde{\rho}_t = \delta\mathbb{W}[\tilde{\rho}_t]dt + (\Delta I/S_0)\delta\mathbb{M}[\tilde{\rho}_t]dt$, and identify $\delta\mathbb{W}$ with the work done by the external driving λ_t along an infinitesimal trajectory and $\delta\mathbb{Q} = (\Delta I/S_0)\delta\mathbb{M}$ as the heat associated with the fluctuating backaction of the detector. Due to the nonlinearity of the stochastic master equation, we can only determine the distributions of work and heat numerically. To this goal, we specify the driving as $\lambda_t = g(1/\cosh(\nu(1-t/\tau)))$, where time is measured in units of the duration of the experiment τ, and reformulate the stochastic differential equations (9) and (10) in the Stratonovich form [41, 42]. We solve them numerically by the Monte-Carlo method for an ensemble of realizations of the random signal $I(t)$ in the interval $t/\tau \in [0,1]$ (we use a time step $dt = 0.01$ and compute averages over 300 trajectories). The results obtained for work, heat and energy along a given quantum trajectory, Eq. (4), are shown in Fig. 1, while those for the work and

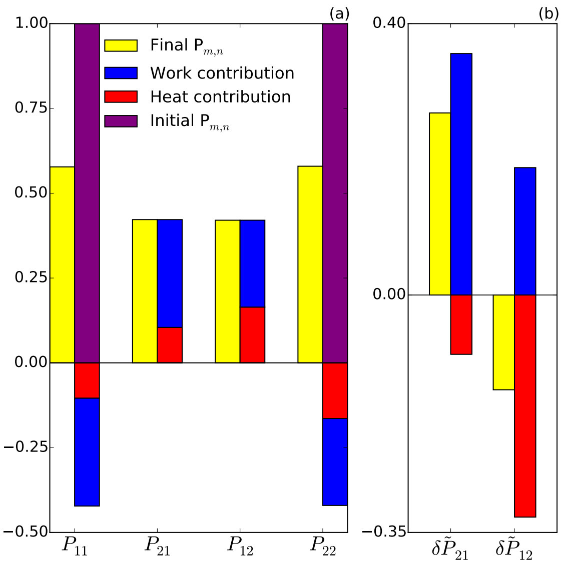

FIG. 2. (color online) a) Averaged final transition probabilities $P^{\tau}_{m,n}$ (yellow) for a continuously monitored qubit with their work and heat contributions, $\delta P^{W}_{m,n}$ (blue) and $\delta P^{Q}_{m,n}$ (red), and the initial transition probability, $P^{0}_{m,n} = \delta_{mn}$ (purple). The first law like equation $dP_{m,n} = \delta P^{W}_{m,n} + \delta P^{Q}_{m,n}$ is verified. b) Work and heat contributions, $\delta\tilde{P}^{W}_{m,n}$ and $\delta\tilde{P}^{Q}_{m,n}$ at the single trajectory level. Same parameters as in Fig. 1.

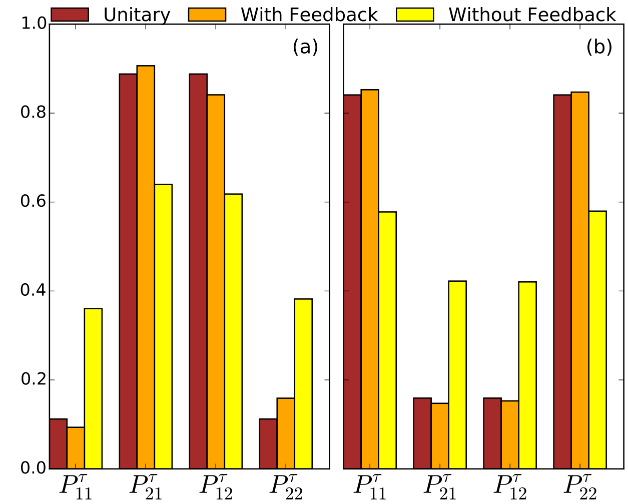

FIG. 3. (color online) Transition probabilities P^{τ}_{mn} for the weakly measured qubit with (orange) and without (yelllow) feedback control. The isolated, unitary, case is shown in red as a reference. The feedback loop effectively suppresses the detector backaction and the associated heat exchange, achieving thermal isolation. a) $\tau_1 = 1.4 \cdot 10^3 dt$ and b) $\tau_2 = 2.5 \cdot 10^3 dt$.

heat contributions to the transition amplitudes, Eqs. (6)-(7), are presented in Fig. 2.

Figure 1a) demonstrates the reconstruction of the quantum averaged work and heat changes, $\delta\tilde{W}_t$ and $\delta\tilde{Q}_t$, along a single quantum trajectory, from the signal $I(t)$ measured in the detector, displayed in Fig. 1b). Contrary to the case of an isolated system for which $d\tilde{U} = \delta\tilde{W}$, the heat contribution $\delta\tilde{Q}_t$ is here clearly visible. Equation (4) holds for each individual realization and thus extends the first law of stochastic thermodynamics to the quantum regime. Figure 2a) shows the unambiguous distinction of work and heat contributions, $\delta P^{W}_{m,n}$ and $\delta P^{Q}_{m,n}$, to the final transition probability $P^{\tau}_{m,n}$. It is important to note that, although $P^{\tau}_{m,n}$ is always positive, as a proper probability should be, the work and heat contributions need not be: the probability to go from state n to m at time τ can, for instance, be smaller than the initial transition probability. An example at the level of single trajectories, $\delta\tilde{P}^{W}_{m,n}$ and $\delta\tilde{P}^{Q}_{m,n}$, can be seen in Fig. 2b). An insightful discussion of the role and interpretation of negative probabilities in physics has been provided by Feynman [43]. It is finally worth mentioning that a quantity, $dP^{\alpha}_{m,n} = P^{\alpha,\tau}_{m,n} - P^{\alpha,0}_{m,n}, \alpha = W, Q$, cannot be defined, reflecting the fact that there no heat or work operator.

Quantum feedback control. In classical thermodynamics work is associated with the variation of the internal energy of the isolated system [1, 2]. We have shown above how heat can be theoretically identified, and thus substracted, in order to achieve isolation numerically. We next take advantage of a feedback loop protocol to suppress the detector backaction [31], offering a scheme to reach isolation experimentally. Quantum feedback has recently been demonstrated experimentally for a superconducting qubit [44]. To control the parameters of the system, depending on the continuous outcome of the detector, we make the amplitude g of the external driving λ_t time dependent, $g(t) = g(1 - f\Delta\phi_t)$, where f is the feedback strength and $\Delta\phi_t$ the phase difference between the actual vector (with backaction) and desired vector (without backaction) in the Bloch sphere of the qubit (see Ref. [41] for details). This allows us to operationally counter the effects induced by the continuous monitoring. From a thermodynamic point of view, the feedback adds an extra amount of work that exactly cancels the heat contribution to the transition probabilities.

Figure 3 shows the numerically simulated final transition probabilities $P^{\tau}_{m,n}$ for the weakly measured qubit with (orange) and without (yellow) quantum feedback. The unitary case (red) without coupling to the detector is displayed as a reference. We have considered two driving times: a) $\tau_1 = 1.4 \cdot 10^3 dt$ and b) $\tau_2 = 2.5 \cdot 10^3 dt$ (with feedback strength $f = 3$). We observe in both cases that the feedback process effectively suppresses the heat contributions (identified in Fig. 2) and that the transition probabilities agree with those of the isolated system with unitary dynamics. Quantum feedback control thus appears as a powerful tool to determine the statistics of the work done by the external driving in a continuously monitored system. We mention that the heat statistics can be easily obtained by measuring the undriven system, that is, when no work is performed and $d\tilde{U}_t = \delta\tilde{Q}_t$.

The above findings can be directly used to verify the quantum Jarzynski equality for the driven qubit. We find $\Delta F_1 = -0.488$ and $\Delta F_2 = -0.496$ with feedback control for $\tau_1 = 1.4 \cdot 10^3 dt$ and $\tau_2 = 2.8 \cdot 10^3 dt$ and $\Delta F = -0.495$ in the unitary case (for an inverse temperature $\beta = 10$). The excellent agreement establishes the second law for a weakly measured quantum system.

Conclusions. We have extended the laws of stochastic thermodynamics along individual quantum trajectories of a weakly measured system. We have shown how to distinguish work and heat contributions to both the energy changes and the transition probabilities. We have further demonstrated the usefulness of our approach with the analysis of a driven qubit and introduced methods to identify work from heat numerically as well as experimentally with the help of quantum feedback control.

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Note added. While completing this manuscript, we came aware of a preprint by Elouard *et al.* [45] that also discusses quantum stochastic thermodynamics.

[1] A. B. Pippard, *Elements of Classical Thermodynamics* (Cambridge, Cambridge, 1966).

[2] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, (Wiley, New York, 1985).

[3] U. Seifert, *Thermodynamics, Fluctuation Theorems and Molecular Machines*, **75** 126001 (2012).

[4] C. Jarzynski, *Equalities and Inequalities: Irreversibility and the Second Law of Thermodynamics at the Nanoscale*, C. Annu. Rev. Condens. Matter Phys. **2**, 329 (2011).

[5] C. Jarzynski, *Nonequilibrium Equality for Free Energy Differences*, Phys. Rev. Lett. **78**, 2690 (1997).

[6] J. Liphardt, S. Dumont, S. B. Smith, I. Tinoco, and C. Bustamante, *Equilibrium Information from Nonequilibrium Measurements in an Experimental Test of the Jarzynski Equality*, Science **296**, 1832 (2002).

[7] V. Blickle, T. Speck, L. Helden, U. Seifert, and C. Bechinger, *Thermodynamics of a Colloidal Particle in a Time-Dependent Nonharmonic Potential*, Phys. Rev. Lett. **96**, 070603 (2006).

[8] S. Ciliberto, R. Gomez-Solano, and A. Petrosyan, *Fluctuations, Linear Response, and Currents in Out-of-Equilibrium Systems*, Annu. Rev. Condens. Matter Phys. **4** 235 (2013).

[9] P. Talkner, E. Lutz, P. Hänggi, *Fluctuation Theorems: Work is not an Observable*, Phys. Rev. E **75**, 050102 (2007).

[10] G. Huber, F. Schmidt-Kaler, S. Deffner, and E. Lutz, *Employing Trapped Cold Ions to Verify the Quantum Jarzynski Equality*, Phys. Rev. Lett. **101**, 070403 (2008).

[11] M. Esposito, U. Harbola and S. Mukamel, *Nonequilibrium Fluctuations, Fluctuation Theorems, and Counting Statistics in Quantum Systems*, Rev. Mod. Phys. **81**, 1665 (2009).

[12] M. Campisi, P. Hänggi, and P. Talkner, *Quantum Fluctuation Relations: Foundations and Applications*, Rev. Mod. Phys. **83**, 771 (2011).

[13] J.M. Horowitz, *Quantum-Trajectory Approach to the Stochastic Thermodynamics of a Forced Harmonic Oscillator*, Phys. Rev. E **85**, 031110 (2012).

[14] F. W. J. Hekking and J. P. Pekola, *Quantum Jump Approach for Work and Dissipation in a Two-Level System* Phys. Rev. Lett. **111**, 093602 (2013).

[15] R. Dorner, S. R. Clark, L. Heaney, R. Fazio, J. Goold, and V. Vedral, *Extracting Quantum Work Statistics and Fluctuation Theorems by Single-Qubit Interferometry*, Phys. Rev. Lett. **110**, 230601 (2013).

[16] L. Mazzola, G. De Chiara, and M. Paternostro, *Measuring the Characteristic Function of the Work Distribution*, Phys. Rev. Lett. **110**, 230602 (2013).

[17] J. Aberg, *Truly Work-Like Work Extraction via a Single-Shot Analysis*, Nat. Commun. **4**, 1925 (2013).

[18] N. Y. Halpern, A. J. P. Garner, O. C. O. Dahlsten, and V. Vedral, *Unification of Fluctuation Theorems and One-Shot Statistical Mechanics*, arXiv:1409.3878.

[19] A. J. Roncaglia, F. Cerisola, and J. P. Paz, *Work Measurement as a Generalized Quantum Measurement*, Phys. Rev. Lett. **113**, 250601 (2014).

[20] R. Gallego, J. Eisert, and H. Wilming, *Defining Work from Operational Principles*, arXiv:1504.05056.

[21] T. Batalhao, A. M. Souza, L. Mazzola, R. Auccaise, R. S. Sarthour, I. S. Oliveira, J. Goold, G. De Chiara, M. Paternostro, and R. M. Serra, *Experimental Reconstruction of Work Distribution and Study of Fluctuation Relations in a Closed Quantum System*, Phys. Rev. Lett. **113**, 140601 (2014).

[22] S. An, J.-N. Zhang, M. Um, D. Lv, Y. Lu, J. Zhang, Z.-Q. Yin, H. T. Quan, and K. Kim, *Experimental test of the quantum Jarzynski equality with a trapped-ion system*, Nature Phys. *11*, 193 (2014).

[23] K. W. Murch, S. J. Weber, C. Macklin, and I. Siddiqi, *Observing Single Quantum Trajectories of a Superconducting Quantum Bit*, Nature **502**, 211 (2013).

[24] S. J. Weber, A. Chantasri, J. Dressel, A. N. Jordan, K. W. Murch, and I. Siddiqi, *Mapping the optimal route between two quantum states*, Nature **511**, 570 (2014).

[25] A. A. Clerk, M. H. Devoret, S. M. Girvin, Florian Marquardt, and R. J. Schoelkopf, *Introduction to Quantum Noise, Measurement, and Amplification,* Rev. Mod. Phys. **82**, 1155 (2010).

[26] Y. Aharonov and D. Rohrlich, *Quantum Paradoxes* (Wiley-VCH, New York, 2005).

[27] O. Hosten and P. Kwiat, *Observation of the Spin Hall Effect of Light via Weak Measurements,* Science **319**, 787 (2008).

[28] P. B. Dixon, D. J. Starling, A. N. Jordan, and J. C. Howell, *Ultrasensitive Beam Deflection Measurement via Interferometric Weak Value Amplification,* Phys. Rev. Lett. **102**, 173601 (2009).

[29] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, *Direct Measurement of the Quantum Wavefunction,* Nature **474**, 188 (2011).

[30] A. Romito, Y. Gefen, *Weak Measurement of Cotunneling Time*, Phys. Rev. B, **90**, 085417 (2014).

[31] H. W Wiseman and G. Milburn, *Quantum Measurement and Control*, (Cambridge, Cambridge, 2010).

[32] F. Reif. *Fundamentals of Statistical and Thermal Physics*, (McGraw-Hill, New York, 1965).
[33] A. Peres, *Quantum Theory, Concepts and Methods*, (Kluwer, New York, 1993).
[34] D. Kafri and S. Deffner, *Holevo's Bound from a General Quantum Fluctuation Theorem*, Phys. Rev. A **86**, 044302 (2012).
[35] K. Jacobs, *Quantum Measurement Theory and its Applications*, (Cambridge, Cambridge, 2014).
[36] For a unitary process $d\tilde{\rho}_t = \delta\mathbb{W}dt = i[\tilde{\rho}_t, H_t]dt/\hbar$ and $\mathrm{tr}\{H_t[\tilde{\rho}_t, H_t]\} = 0$.
[37] See Supplemental Material.
[38] A. Korotkov, *Quantum Bayesian Approach to Circuit QED Measurement*, arXiv:1111.4016.
[39] S. A. Gurvitz, *Measurements with a Noninvasive Detector and Dephasing Mechanism,* Phys. Rev. B **56**, 15 215 (1997).
[40] A. Korotkov, *Continuous Quantum Measurement of a Double Dot,* Phys. Rev. B **60**, 5737 (1999).
[41] A. Korotkov, *Selective Quantum Evolution of a Qubit State Due to Continuous Measurement*, Phys. Rev. B **63**, 115403 (2001).
[42] A. Korotkov, *Noisy Quantum Measurement of Solid-State Qubits: Bayesian Approach*, NATO Science Series **97**, 205 (2003).
[43] R. P. Feynman, *Negative Probability*, in F. D. Peat and B. Hiley, *Quantum Implications: Essays in Honour of David Bohm* (Routledge, New York, 1987) pp. 235-248.
[44] K. W. Murch, U. Vool, D. Zhou, S. J. Weber, S. M. Girvin, and I. Siddiqi, *Cavity-Assisted Quantum Bath Engineering*, Phys. Rev. Lett. **109**, 183602 (2012).
[45] C. Elouard, A. Auffèves, and M. Clusel, *Stochastic Thermodynamics in the Quantum Regime,* arXiv:1507.00312.

SUPPLEMENTAL MATERIAL

We here derive the first law like equation (5) for the transition probabilities. The evolution of a system in the time interval $[0, \tau]$ along an individual quantum trajectory $\tilde{\rho}_t$ can be divided into $N = \tau/dt$ successive infinitesimal unitary steps $\mathcal{U}_i$ (generated by the system's Hamiltonian) and N infinitesimal nonunitary steps $\mathcal{N}_i$ (induced by the coupling to the detector), ($i = 1$ to N):

$$\begin{aligned} &\tilde{\rho}_0 \xrightarrow{\mathcal{U}_1} \tilde{\rho}_1 \xrightarrow{\mathcal{N}_1} \tilde{\rho}_2 \dots \\ \dots\, &\tilde{\rho}_{2i-2} \xrightarrow{\mathcal{U}_i} \tilde{\rho}_{2i-1} \xrightarrow{\mathcal{N}_i} \tilde{\rho}_{2i} \dots \\ \dots\, &\tilde{\rho}_{2N-2} \xrightarrow{\mathcal{U}_N} \tilde{\rho}_{2N-1} \xrightarrow{\mathcal{N}_n} \tilde{\rho}_{2N}, \end{aligned} \tag{12}$$

where, without loss of generality, the unitary (nonunitary) processes were chosen to correspond to the odd (even) steps. Each transformation induces a change in the state of the system defined as,

$$\delta\mathbb{W}[\tilde{\rho}_i]dt = \tilde{\rho}_{2i-1} - \tilde{\rho}_{2i-2}, \tag{13}$$

$$\delta\mathbb{Q}[\tilde{\rho}_i]dt = \tilde{\rho}_{2i} - \tilde{\rho}_{2i-1}, \tag{14}$$

where $\delta\mathbb{W}[\tilde{\rho}_t]$ and $\delta\mathbb{Q}[\tilde{\rho}_t]$ are the operators associated with the respective unitary and nonunitary parts of the dynamics. The total change of the density operator can then be written as,

$$\begin{aligned} d\tilde{\rho} =& \tilde{\rho}_{2N} - \tilde{\rho}_0 = \tilde{\rho}_{2N} - \tilde{\rho}_1 + \tilde{\rho}_1 - \tilde{\rho}_0 \\ =& \tilde{\rho}_{2N} - \tilde{\rho}_2 + \tilde{\rho}_2 - \tilde{\rho}_1 + \delta\mathbb{W}[\tilde{\rho}_1]dt \\ =& \tilde{\rho}_{2N} - \tilde{\rho}_3 + \tilde{\rho}_3 - \tilde{\rho}_2 + \delta\mathbb{Q}[\tilde{\rho}_1]dt + \delta\mathbb{W}[\tilde{\rho}_1]dt \\ =& \sum_{i=1}^{N} \delta\mathbb{W}[\tilde{\rho}_i]dt + \sum_{i=1}^{N} \delta\mathbb{Q}[\tilde{\rho}_i]dt. \end{aligned} \tag{15}$$

The different notation d and δ in the above equations is a reminder that $d\tilde{\rho}$ only depends on the initial and final states, while $\delta\mathbb{W}$ and $\delta\mathbb{Q}$ are path-dependent quantities.

Taking the initial state $\tilde{\rho}_0$ to be in the eigenstate n of the Hamiltonian at time 0, and projecting Eq. (15) onto the eigenstate m of the Hamiltonian at time τ, we can write the transition probability $\tilde{P}_{m,n} = \mathrm{tr}\{\tilde{\rho}_{2N}\Pi_m\}$ as,

$$\begin{aligned} \tilde{P}_{m,n} =& \tilde{P}^0_{m,n} + \mathrm{tr}\left\{\sum_{i=1}^{N} \Pi_m \delta\mathbb{W}[\tilde{\rho}_i]dt\right\} \\ &+ \mathrm{tr}\left\{\sum_{i=1}^{N} \Pi_m\ \delta\mathbb{Q}[\tilde{\rho}_i]dt\right\} \\ =& \tilde{P}^0_{m,n} + \delta\tilde{P}^Q_{m,n} + \delta\tilde{P}^W_{m,n}. \end{aligned} \tag{16}$$

Equation (16) readily leads to Eqs. (5), (6) and (7) with $d\tilde{P}_{m,n} = \tilde{P}^\tau_{m,n} - \tilde{P}^0_{m,n}$ and in the limit dt to zero.