Supplemental Material:
Hilbert-space geometry of random-matrix eigenstates

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I. DERIVATION

A. Quantum geometric tensor

Inserting a complete set of states into Eq. (2) of the main text, one obtains the expression

$$g^{(n)}_{\alpha\beta} = \sum_{m(\neq n)} \langle \partial_\alpha \tilde{n}|\tilde{m}\rangle\langle \tilde{m}|\partial_\beta \tilde{n}\rangle \tag{S1}$$

for the quantum geometric tensor. Differentiating $\langle \tilde{m}|H|\tilde{n}\rangle = 0$ and using $H_0|m\rangle = E_m|m\rangle$ gives

$$E_n\langle \partial_\alpha m|n\rangle + E_m\langle m|\partial_\alpha n\rangle + \langle m|\partial_\alpha H|n\rangle = 0, \tag{S2}$$

where we specialized to $x = y = 0$. Finally using that $\langle \tilde{m}|\tilde{n}\rangle = 0$ implies $\langle \partial_\alpha m|n\rangle + \langle m|\partial_\alpha n\rangle = 0$, one finds

$$\langle m|\partial_\alpha n\rangle = \frac{\langle m|\partial_\alpha H|n\rangle}{E_n - E_m}. \tag{S3}$$

Inserting this into Eq. (S1) gives Eq. (4) of the main text.

B. Integrable systems

For integrable systems, the eigenvalues E_n can be taken as statistically independent and the spacings $|E_n - E_m|$ in Eq. (4) obey a Poisson distribution. Thus, the distribution $p_s(s)$ of the spacings remains constant in the limit $s \to 0$. A small spacing implies a large term in the sum in Eq. (4). Due to the constant $p_s(s)$ in the limit $s \to 0$, the terms $x \sim 1/s^2$ in the sum in Eq. (4) have a probability distribution $p_x(x)$, which decays at large x as

$$p_x(x) = p_s(s)\left|\frac{ds}{dx}\right| \sim \frac{1}{|x|^{3/2}}. \tag{S4}$$

For this asymptotic decay of $p_x(x)$, both the average and the variance of x diverge. By consequence, in the limit of large N, the distribution function of the entire sum in Eq. (4) converges to an appropriate Levy stable distribution with the same asymptotic decay [1]. The stable distribution depends on whether the signs of the terms in the sum are random (off-diagonal element of the quantum geometric tensor) or not (diagonal element). The characteristic functions of the corresponding stable distributions are given in Eq. (5) in the main text.

We include a heuristic argument yielding Eq. (5) for the distribution of the diagonal elements of the quantum geometric tensor. Assuming the existence of a stable distribution, we can choose a convenient distribution $p_x(x)$ for the individual terms in the sum in Eq. (4), with the only requirement that the distribution fall off as $1/|x|^{3/2}$ at large $|x|$. Such a choice is a Gaussian distribution for the spacings s, with the numerators in Eq. (4) simply taken as fixed. As we saw above, the fact that $p_s(s) \sim \exp\{-\gamma_0 s^2/4N\}$ remains nonzero in the limit $s \to 0$ implies that $p_x(x) \sim 1/|x|^{3/2}$. With this choice, we find

$$p_x(x) \sim \int_0^\infty ds\, e^{-\frac{\gamma_0}{4N}s^2}\delta(x - \frac{1}{s^2}). \tag{S5}$$

Here, we focused on the diagonal element of the quantum geometric tensor, for which all terms in the sum in Eq. (4) are positive. We also made the dependence on the matrix size N explicit, choosing the same scalings as for the GUE. Using the Fourier representation of the δ-function, the corresponding characteristic function takes the form

$$\tilde{p}_x(\xi) \sim \int_0^\infty ds \exp\left(-\frac{\gamma_0}{4N}s^2 - \frac{i\xi}{s^2}\right). \tag{S6}$$

Here, ξ should be taken to have an infinitesimal negative imaginary part. This integral can be performed and yields

$$\tilde{p}_(\xi) = e^{-\sqrt{\frac{\gamma_0}{2N}|\xi|}(1+i\mathrm{sgn}\xi)} \tag{S7}$$

Due to statistical independence, the characteristic function $\tilde{P}(\xi)$ of the entire sum in Eq. (4) is simply given by

$$\tilde{P}(\xi) = [\tilde{p}(\xi)]^N = e^{-\sqrt{\frac{N\gamma_0}{2}|\xi|}(1+i\mathrm{sgn}\xi)}. \tag{S8}$$

This is just a rescaled version of the characteristic function for the distribution of an individual term in Eq. (4) [whose distribution is thus already equal to the Levy stable distribution for our choice of $p_s(s)$] and coincides with Eq. (5) in the main text with the identification $\gamma = N\gamma_0$.

It is interesting to note that the characteristic function in Eq. (S8) can be readily Fourier tansformed to yield

$$P(g) = \frac{\sqrt{\gamma/\pi}}{2g^{3/2}} e^{-\gamma/4g} \tag{S9}$$

Similarly, the GUE distribution function of the trace of the quantum geometric tensor can actually be computed in fully analytical form, see Sec. II below.

C. GUE average

Following Refs. [2] (see also [3, 4]), we perform the average in Eq. (14) of the main text using the joint eigenvalue distribution for H_0,

$$p_N(E_1, \ldots, E_N) \propto \prod_{i<j} (E_i - E_j)^2 e^{-\frac{1}{2}N\sum_j E_j^2}. \tag{S10}$$

Writing the terms involving E_N separately and using the large N limit, this is

$$p_N(E_1, \ldots, E_N) \propto \prod_{i=1}^{N-1} (E_i - E_N)^2 e^{-\frac{1}{2}NE_N^2} p_{N-1}(E_1, \ldots, E_{N-1}), \tag{S11}$$

where p_{N-1} denotes the joint eigenvalue distribution of an $(N-1) \times (N-1)$-dimensional random matrix drawn from the GUE, denoted by $\tilde{H}$ in the following. Using that the δ-function in Eq. (14) allows us to set $E_N = 0$, we find

$$\tilde{P}(\xi_0, \boldsymbol{\xi}) \propto \mathbb{E}_{\text{GUE}} \left[\delta(E_N) \prod_{m=1}^{N-1} \frac{E_m^6}{(E_m^2 + \frac{i\xi_0}{2N})^2 + \frac{|\boldsymbol{\xi}|^2}{4N^2}} \right]. \tag{S12}$$

Here, we write the GUE average (denoted by $\langle \ldots \rangle_{\tilde{H}}$ in the main text) as $\mathbb{E}_{\text{GUE}}[\ldots]$.

Equation (S12) can be rewritten as a GUE average over determinants of $\tilde{H}$, as given in Eq. (15) in the main text. Factorizing the denominator gives

$$\tilde{P}(\xi_0, \boldsymbol{\xi}) \propto \lim_{b_j \to 0} \mathbb{E}_{\text{GUE}} \left[\frac{\prod_{j=1}^6 \det\left(\tilde{H} + \mathrm{i}b_j\right)}{\prod_{j=1}^4 \det\left(\tilde{H} + \mathrm{i}a_j\right)} \right]. \tag{S13}$$

Here, the a_j with $j = 1, \ldots, 4$ solve $a_j^2 = i(\xi_0 \pm |\boldsymbol{\xi}|)/2N$. There are two roots with $\mathrm{Re}\, a_j > 0$, which we denote as a_1 and a_3, and two roots with $\mathrm{Re}\, a_j < 0$, which we denote as a_2 and a_4. We also introduced parameters b_j with $j = 1, \ldots, 6$. The b_j need to be set to zero at the end, but it turns out to be convenient to retain them at intermediate steps of the calculation.

The required averages in Eq. (S13) can be found from general results obtained in Refs. [5, 6]. Here we include a dedicated calculation for completeness. We represent the determinants as Gaussian integrals. The determinants in the denominator are written as integrals over complex variables $z, \bar{z}$ (with Einstein's summation convention in force)

$$\det{}^{-1}(\tilde{H} + \mathrm{i}a) = \int_{z,\bar{z}} \mathrm{e}^{\pm \mathrm{i}\bar{z}_k (\tilde{H}+\mathrm{i}a)^k{}_l z^l}. \tag{S14}$$

For convergence, we choose the upper sign when $\mathrm{Re}\, a > 0$ and thus for the determinants involving a_1 and a_3, and the lower sign when $\mathrm{Re}\, a < 0$ and thus for a_2 and a_4. The determinants in the numerator are written as integrals over Grassmann variables $\zeta, \bar{\zeta}$,

$$\det[\mathrm{i}(\tilde{H} + \mathrm{i}b)] = \int_{\zeta,\bar{\zeta}} \mathrm{e}^{-\mathrm{i}\bar{\zeta}_k (\tilde{H}+\mathrm{i}b)^k{}_l \zeta^l}, \tag{S15}$$

where we note that $\prod_{j=1}^6 \det(\tilde{H} + \mathrm{i}b_j) = (-1)^N \prod_{j=1}^6 \det[\mathrm{i}(\tilde{H} + \mathrm{i}b_j)]$.

We now collect the random factors into

$$X \equiv \exp\left\{ \mathrm{i}\tilde{H}^k{}_l \left(z^l{}_1 \bar{z}^1{}_k - z^l{}_2 \bar{z}^2{}_k + z^l{}_3 \bar{z}^3{}_k - z^l{}_4 \bar{z}^4{}_k + \zeta^l{}_f \bar{\zeta}^f{}_k \right) \right\}, \tag{S16}$$

where $f = 1, \ldots, 6$, and introduce supervectors

$$\{\Psi^l{}_\mu\} = \left(z^l{}_1, z^l{}_2, z^l{}_3, z^l{}_4, \zeta^l{}_1, \zeta^l{}_2, \zeta^l{}_3, \zeta^l{}_4, \zeta^l{}_5, \zeta^l{}_6 \right) \tag{S17}$$

to abbreviate the notation. Then, we have

$$\Psi^l{}_\mu (s\bar{\Psi})^\mu_k = z^l{}_1 \bar{z}^1{}_k - z^l{}_2 \bar{z}^2{}_k + z^l{}_3 \bar{z}^3{}_k - z^l{}_4 \bar{z}^4{}_k + \zeta^l{}_f \bar{\zeta}^f{}_k \tag{S18}$$

with

$$s = \mathrm{diag}(1, -1, 1, -1, 1, 1, 1, 1, 1, 1). \tag{S19}$$

Taking the GUE expectation value has now been reduced to a Gaussian integral, which yields

$$\mathbb{E}_{\mathrm{GUE}}(X) = \mathbb{E}_{\mathrm{GUE}} \left(\mathrm{e}^{\mathrm{i}\tilde{H}^k{}_l (\Psi s \bar{\Psi})^l{}_k} \right) = \mathrm{e}^{(-\lambda^2/2N)(\Psi s\bar{\Psi})^l{}_k (\Psi s \bar{\Psi})^k{}_l}. \tag{S20}$$

Using the cyclicity of trace and supertrace, the exponent on the right hand side can be written as a supertrace,

$$\mathbb{E}_{\mathrm{GUE}}(X) = \mathrm{e}^{-(\lambda^2/2N)\,\mathrm{tr}(\Psi s \bar{\Psi})^2} = \mathrm{e}^{-(\lambda^2/2N)\,\mathrm{STr}(\bar{\Psi}\Psi s)^2}. \tag{S21}$$

Here, λ denotes the disorder strength parameter of the GUE, which was set to $\lambda = 1$ in the main text.

D. Superbosonization step

Consider the composite object (with $k = 1, 2, \ldots, N$ for $N \times N$ GUE matrices)

$$M^\mu{}_\nu = N^{-1} \bar{\Psi}^\mu{}_k \Psi^k{}_\nu. \tag{S22}$$

This is a supermatrix of dimension $(4|6) \times (4|6)$. The superbosonization method [7, 8] allows us to switch from the original variables $z, \bar{z}$ and $\zeta, \bar{\zeta}$ of integration to supermatrices M as new integration variables. In the fermion-boson block decomposition,

$$M = \begin{pmatrix} M_{\mathrm{BB}} & M_{\mathrm{BF}} \\ M_{\mathrm{FB}} & M_{\mathrm{FF}} \end{pmatrix}, \tag{S23}$$

the block M_{BB} is a positive Hermitian 4×4 matrix,

$$M_{\mathrm{BB}} = N^{-1} \begin{pmatrix} \bar{z}^1{}_k z^k_1 & \ldots & \bar{z}^1{}_k z^k_4 \\ \vdots & \ddots & \vdots \\ \bar{z}^4{}_k z^k_1 & \ldots & \bar{z}^4{}_k z^k_4 \end{pmatrix}, \tag{S24}$$

while M_{FF},

$$M_{\mathrm{FF}} = N^{-1} \begin{pmatrix} \bar{\zeta}^1{}_k \zeta^k_1 & \ldots & \bar{\zeta}^1{}_k \zeta^k_6 \\ \vdots & \ddots & \vdots \\ \bar{\zeta}^6{}_k \zeta^k_1 & \ldots & \bar{\zeta}^6{}_k \zeta^k_6 \end{pmatrix}, \tag{S25}$$

turns into a unitary 6×6 matrix, and the entries of M_{BF} and M_{FB} are Grassmann variables. The change of variables is carried out by using the superbosonization identity

$$\int_{z,\bar{z}} \int_{\zeta,\bar{\zeta}} F\left(M(\bar{z}, z, \bar{\zeta}, \zeta) \right) = \int \mathcal{D}M \, \mathrm{SDet}^N(M) F(M), \tag{S26}$$

where a normalization constant is absorbed into the new integration measure, $\mathcal{D}M$. The new measure is scale invariant and, up to a constant, uniquely determined by the symmetries of the problem.

E. Saddle-point approximation

After superbosonization, we have

$$\tilde{P}(\xi_0, \boldsymbol{\xi}) = \int \mathcal{D}M \, \mathrm{SDet}^N(M) \, \mathrm{e}^{-(N\lambda^2/2)\,\mathrm{STr}(Ms)^2 - N\mathrm{STr}(smM)} \tag{S27}$$

with $m = \mathrm{diag}(a_1, a_2, a_3, a_4, b_1, \ldots, b_6)$ as defined in the main text. In the limit of large random matrices, $N \to \infty$, the integral can now be performed by saddle-point integration. Since $m \sim N^{-1}$, the corresponding term can be neglected in determining the saddle-point manifold, and the saddle-point equation becomes

$$M^{-1} - \lambda^2 sMs = 0\,. \tag{S28}$$

This has the supermanifold of dominant (for $N \to \infty$) solutions

$$Ms = \lambda^{-1}Q, \quad Q = T\Sigma_3 T^{-1}, \tag{S29}$$

where

$$\Sigma_3 = \mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1, 1, -1), \quad T \in \mathrm{U}(2, 2|6). \tag{S30}$$

Thus, saddle-point integration yields

$$\tilde{P}(\xi_0, \boldsymbol{\xi}) = \int DQ \, \mathrm{e}^{-(N/\lambda)\,\mathrm{STr}(Qm)}, \tag{S31}$$

where DQ is the invariant measure on $\mathrm{U}(2, 2|6)/\mathrm{U}(2|3) \times \mathrm{U}(2|3)$. Up to a multiplicative constant, this measure is again determined uniquely by symmetries.

F. Semiclassical exactness

Our integral representation for $\tilde{P}$ is semiclassically exact, c.f., [9], which significantly simplifies the calculation. The principle of semiclassical exactness is easiest to apply if the critical points of the integrand are isolated. In the present case, that is not the case once we set $b_j \to 0$. It is for this reason that we introduced the b_j at all intermediate stages of the calculation and take the limit $b_j \to 0$ only at the very end.

Now all critical points are isolated and using the semiclassical exactness, we can evaluate the integral (S31) semiclassically. The isolated critical points are given by

$$Q_{\mathrm{crit}} = \mathrm{diag}(+1, -1, +1, -1, s_1, s_2, s_3, s_4, s_5, s_6) \tag{S32}$$

where $s_f \in \{\pm 1\}$ and $\sum_f s_f = 0$. There exist $6!/(3!3!) = 20$ critical points, namely $Q_{\mathrm{crit}} = \Sigma_3$ and 19 more.

Then, the value of the integral (S31) is a sum of 20 terms (one for each critical point) and each term contributes by the value of the integral at the critical point times a factor originating from the corresponding fluctuation integral in Gaussian approximation. The contribution from the critical point $Q_{\mathrm{crit}} = \Sigma_3$ takes the form

$$\tilde{P}(\xi_0, \boldsymbol{\xi})_{\Sigma_3} = \frac{\lambda}{N} \, \Delta(\xi_0, \boldsymbol{\xi}) \, \mathrm{e}^{-(N/\lambda)\,\mathrm{STr}(\Sigma_3 m)}, \tag{S33}$$

where $\Delta(\xi_0, \boldsymbol{\xi})$ is given by

$$\Delta(\xi_0, \boldsymbol{\xi}) = \frac{\prod_{i=1}^{2} \prod_{j=2,4,6} (a_i - b_j) \prod_{i=3}^{4} \prod_{j=1,3,5} (a_i - b_j)}{\prod_{i=1}^{2} \prod_{j=3}^{4} (a_i - a_j) \prod_{i=1,3,5} \prod_{j=2,4,6} (b_i - b_j)}. \tag{S34}$$

The contributions from the other 19 critical points Q_{crit} are obtained by applying to $[b_1, b_2, b_3, b_4, b_5, b_6]$ the same permutation that turns Σ_3 into the given Q_{crit}, and $\tilde{P}(\xi_0, \boldsymbol{\xi})$ follows by summing over the contributions of all critical points.

The denominator of Eq. (S34) is singular in the limit $b_j \to 0$. However, after summing over all critical points one finds that there is a compensating factor in the numerator and the limit becomes well defined. Performing this calculation [10] gives Eqs. (9) and (10) of the main text.

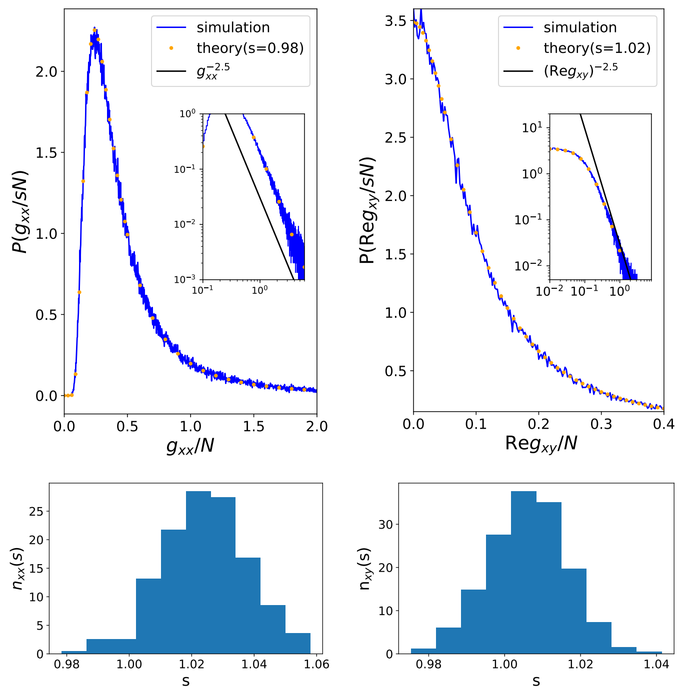

Figure S1. Top panels: Distribution functions of matrix elements of the quantum geometric tensor (left: g_{xx}; right: $\mathrm{Re}g_{xy}$), obtained by sampling 10^6 realizations of H_0 in Eq. (3) with H_0 drawn from the Gaussian Unitary Ensemble with $N = 100$. The sampling is performed for fixed perturbation matrices H_x and H_y (chosen as matrices drawn independently from the Gaussian Unitary Ensemble). The insets show a corresponding log-log plot, emphasizing the asymptotic $1/|g|^{5/2}$ decay. A plot of $f(g) \propto 1/|g|^{5/2}$ is shown for comparison. Numerical data (blue) are compared to the analytical prediction (orange dots) given in Eq. (9) in the main text with $\gamma = s\gamma^{\mathrm{GUE}}$ and $s = 0.978$ (left) and $s = 1.018$ (right). Bottom panels: Distribution of scaling factors as defined in Eq. (S43). The scale factors describe the fits of the distributions of the quantum geometric tensor to our analytical result in Eq. (9) and are obtained by sampling and fitting the distributions of g_{xx} and $\mathrm{Re}g_{xy}$ for 600 sets of random, but fixed perturbation matrices drawn from the GUE.

II. TRACE OF QUANTUM GEOMETRIC TENSOR

Berry and Shukla [11] give approximate results for the distribution of the trace G of the quantum geometric tensor. This section collects corresponding analytical results based on the exact solution.

For one and two parameters, the characteristic function of the distribution of the trace of the quantum geometric tensor can be directly obtained from our general result in Eq. (9). Similar to Eq. (S9) for the diagonal elements of the quantum geometric tensor of integrable systems, the characteristic functions can be explicitly Fourier transformed to obtain the corresponding distributions. For just one parameter, the quantum geometric tensor reduces to a number (also known as fidelity susceptibility). In this case, the distribution function of the "trace" is equal to the distribution function of a diagonal element of the quantum geometric tensor, which follows by setting $\xi_0 = \xi_3 = \xi$ and $\xi_1 = \xi_2 = 0$. This yields $\xi_+ = 2\xi$ and $\xi_- = 0$, and thus $X_+ = \frac{1}{2}(1 + \mathrm{sgn}\xi)\sqrt{2\gamma\xi}$ and $X_- = 0$, so that

$$\tilde{P}(\xi) = r(X_+, 0)e^{-X_+}. \tag{S35}$$

We can write this as

$$\tilde{P}(\xi) = (1 - \partial_\beta + \frac{1}{3}\partial_\beta^2 - \frac{1}{24}\partial_\beta^3)e^{-\beta X_+}\Big|_{\beta=1}. \tag{S36}$$

The Fourier transform involves now just the same integral as in the integrable case. This yields (rescaling G such that $\gamma = 4$) [12]

$$P(G) = \frac{1 + G + \frac{3}{4}G^2}{3\sqrt{\pi}G^{9/2}}e^{-\frac{1}{G}}. \tag{S37}$$

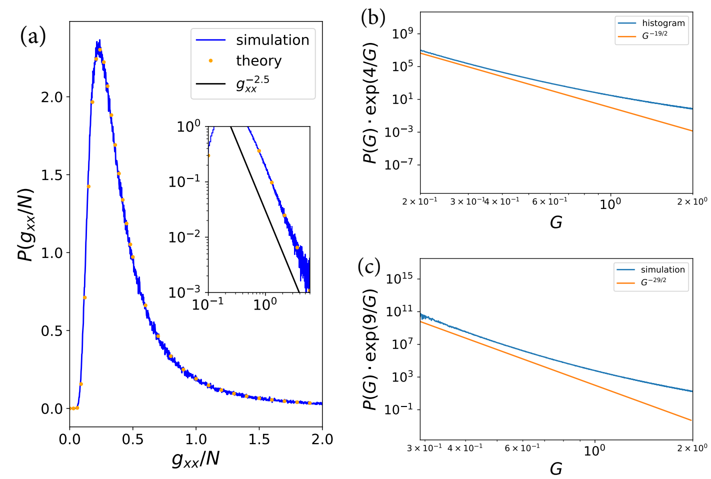

Figure S2. Distribution functions of the trace of the quantum metric tensor. (a) Quantum metric tensor for one parameter (also known as fidelity susceptibility; previously calculated in [12] by an orthogonal-polynomial method). The exact analytic expression in Eq. (S37) exhibits nonanalytic small-G asymptotics $\propto G^{-3/2}e^{-1/G}$. Simulations for GUE matrices with $N = 100$ using 10^6 realizations. (b) Comparison of the nonanalytic small-G asymptotics $\propto G^{-19/2}e^{-4/G}$ for two parameters (see the main text for full distribution function of the trace and Eq. (S38) for the exact analytical result). Simulations for GUE matrices with $N = 100$ using 10^6 realizations. (c) Comparison of the nonanalytic small-G asymptotics $\propto G^{-29/2}e^{-9/G}$ for three parameters (see Eq. (S39) for the general expression for the asymptotics for any number of parameters). Simulations for GUE matrices with $N = 100$ using 5×10^5 realizations.

The distribution function exhibits an asymptotic $1/G^{5/2}$ dependence at large G. At small G, it falls off faster than any power $\propto G^{-9/2}e^{-1/G}$. Notice that this result falls off more slowly than the result for integrable systems, $\propto G^{-3/2}e^{-1/G}$. A comparison to numerical results is shown in Fig. S2(a).

For two parameters, we obtain the distribution of the trace by setting $\xi_0 = \xi$ and $\xi_1 = \xi_2 = \xi_3 = 0$. This yields

$$P(G) = \frac{4096 + 18432G + 42624G^2 + 61728G^3 + 58257G^4 + 35505G^5 + 13140G^6 + 2700G^7}{8640G^{19/2}} \frac{2}{\sqrt{\pi}} e^{-4/G}. \tag{S38}$$

The distribution function again exhibits an asymptotic $1/G^{5/2}$ dependence at large G. At small G, it falls off $\propto G^{-19/2}e^{-4/G}$.

For k parameters, we can obtain the asymptotic behavior by power counting. Generalizing the derivation to include k parameters, we can readily determine the powers of determinants in the numerator and denominator in the expression analogous to Eq. (15). By extracting the power of the polynomial resulting from the prefactor in the analog of Eq. (9), we then find

$$P(G) \propto G^{-(k^2+2k+3/2)} e^{-k^2/G}. \tag{S39}$$

at small G, which complements the $G^{-5/2}$ dependence at large G. These asymptotics of the distribution of the trace were previously considered for $k = 3$ within an approximate approach by Berry and Shukla [11]. While their result at small G has the same overall structure, their preexponential factor has a different power. Numerical results comparing the asymptotics in Eq. (S39) to numerical results are shown in Fig. S2(b) and (c) for $k = 2$ and $k = 3$.

III. AVERAGING OVER H_0 ONLY

In the main text, we assume that the two parameters x and y couple to independent random matrices, i.e., we average over both the unperturbed Hamiltonian H_0 and the perturbations H_x and H_y. This assumption can be relaxed. Averaging only over the unpertubed Hamiltonian H_0, the matrix elements in the numerator of Eq. (4) are still random variables as they involve the eigenvectors of the GUE matrix H_0. In the limit $N \to \infty$, the matrix elements of the perturbation matrices in the eigenbasis of H_0 become Gaussian random variables with zero mean and covariance

$$\mathbb{E}_{\mathrm{GUE}}\{\langle n|H_\alpha|m\rangle\langle m|H_\beta|n\rangle\} = \frac{1}{N^2}\mathrm{tr}H_\alpha H_\beta \tag{S40}$$

$$\mathbb{E}_{\mathrm{GUE}}\{\langle n|H_\alpha|m\rangle\langle n|H_\beta|m\rangle\} = 0. \qquad (m \neq n) \tag{S41}$$

As long as we consider perturbations H_x and H_y such that, to leading order in the large-N limit, the covariance matrix

$$C_{\alpha\beta} = \frac{1}{N^2}\mathrm{tr}H_\alpha H_\beta \tag{S42}$$

for H_α is proportional to the unit matrix, the calculations can now proceed exactly as in the case discussed in the bulk of this paper, in which one averages over the perturbations H_x and H_y.

This situation occurs when the perturbations are drawn independently from a GUE, but then held fixed while averaging over H_0. The resulting distributions are in excellent agreement with our analytical result. A comparison between the numerical results and the exact distribution of the quantum geometric tensor in Eq. (9) is shown in Fig. S1 (top panels). The random fluctuations of the strength of the perturbation matrices across the GUE can be accounted for by introducing a scale factor s through

$$\gamma = s\gamma^{\mathrm{GUE}}, \tag{S43}$$

relative to the GUE result $\gamma^{\mathrm{GUE}} = 4N$. By fitting the numerical results to Eq. (9) for different GUE matrices H_x and H_y, we can numerically obtain the corresponding distributions of scaling factors as shown in Fig. S1 (bottom panels). In accordance with random-matrix estimates, the deviation of the scale factor from unity is of order $1/N$.

We note that our approach to computing the joint distribution function for the quantum geometric tensor can also be extended to the case of a general covariance matrix. Then, we first define new perturbations $\overline{H}_\alpha$ and parameters $\overline{\mathbf{r}} = (\overline{x}, \overline{y})$ through

$$\overline{\boldsymbol{r}} = D\boldsymbol{r} \tag{S44}$$

$$\overline{H}_\alpha = \sum_\beta D_{\alpha\beta}H_\beta, \tag{S45}$$

where we choose the orthogonal matrix D such that the covariance matrix becomes diagonal. We then have to extend the calculation to situations in which the effective averages over $\overline{H}_x$ and $\overline{H}_y$ are still GUE-like, albeit with different disorder parameters λ_x and λ_y. Performing the average over the eigenvectors of the unperturbed Hamiltonian will then result in Eq. (S13) with

$$a_j^2 = \frac{i}{4N}\left[\xi_0(\lambda_x+\lambda_y)+\xi_3(\lambda_x-\lambda_y)\right] \pm \frac{i}{2N}\sqrt{\frac{1}{4}[\xi_0(\lambda_x-\lambda_y)+\xi_3(\lambda_x+\lambda_y)]^2+\lambda_x\lambda_y(\xi_1^2+\xi_2^2)}. \tag{S46}$$

We first consider the distributions of the diagonal and off-diagonal elements of the quantum geometric tensor. To obtain the distribution of the off-diagonal elements, we set $\xi_0 = \xi_3 = 0$. In this case, the product $\lambda_x\lambda_y$ simply rescales the otherwise unchanged distribution function. To obtain the distribution functions of the diagonal elements, we set $\xi_0 = \pm\xi_3 = \xi$ and $\xi_1 = \xi_2 = 0$. Again, the distribution functions are merely rescaled, though differently for g_{xx} and g_{yy}. Finally, the joint distribution function follows by setting

$$\xi_\pm = \frac{1}{2}\left[\xi_0(\lambda_x+\lambda_y)+\xi_3(\lambda_x-\lambda_y)\right] \pm \sqrt{\frac{1}{4}[\xi_0(\lambda_x-\lambda_y)+\xi_3(\lambda_x+\lambda_y)]^2+\lambda_x\lambda_y(\xi_1^2+\xi_2^2)} \tag{S47}$$

in the characteristic function in Eq. (9), Fourier transforming, and reverting to the quantum geometric tensor with respect to the original parameters x and y.

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