Universal spectral statistics of Andreev billiards: semiclassical approach

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The classification of universality classes of random-matrix theory has recently been extended beyond the Wigner-Dyson ensembles. Several of the novel ensembles can be discussed naturally in the context of superconducting-normal hybrid systems. In this paper, we give a semiclassical interpretation of their spectral form factors for both quantum graphs and Andreev billiards.

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Based on early work of Wigner [1], Dyson [2] proposed a classification of complex quantum systems according to their behavior under time reversal and spin rotations. While initially motivated by atomic nuclei, the proposed universality classes – called Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, and GSE) – have since been applied successfully to a large variety of systems, most notably chaotic and disordered quantum systems [3]. More recently, an additional seven ensembles have been identified [4] which are naturally realized in part by Dirac fermions in random gauge fields (Gaussian chiral ensembles) [5] and in part by quasi-particles in disordered mesoscopic superconductors [6] and in superconducting-normalconducting (SN) hybrid systems [7]. These novel universality classes differ from the Wigner-Dyson ensembles in that their spectral statistics, while still universal, is no longer stationary under shifts of the energy due to additional discrete symmetries.

Much insight into the range of validity of the Wigner-Dyson random-matrix ensembles has been gained from the semiclassical approach to the spectral statistics of chaotic quantum systems, based on Gutzwiller's trace formula. In a seminal paper [8], Berry gave a semiclassical derivation of the spectral form factor of chaotic quantum systems for the Wigner-Dyson ensembles, partially reproducing the results of random-matrix theory (RMT) and clarifying its limitations. It is the purpose of the present letter to provide such a semiclassical interpretation for the spectral statistics of the novel random-matrix ensembles associated with disordered superconductors, termed classes C, CI, D, and DIII.

There have been earlier attempts to apply semiclassical theory to SN hybrid systems [9, 10, 11]. Melsen *et al.* [9] pointed out that the gap induced by the proximity effect in a billiard coupled to a superconducting lead (Andreev billiard – cf. Fig. 1a) is sensitive to whether the classical dynamics of the billiard is integrable or chaotic. These authors showed that the proximity-induced hard gap in the chaotic case is *not* fully reproduced by semiclassical theory, the reasons for which have been discussed further in Ref. [10]. By contrast, we focus here on a semiclassical calculation of the *universal* spectral statistics of SN hybrid systems corresponding to the novel ensembles. We show that in this case, semiclassics reproduces the spectral statistics as captured by the form factor.

We start with a summary of pertinent random-matrix results for the new universality classes. For the Wigner-Dyson ensembles, the average density of states (DOS) is nonuniversal and random-matrix theory makes universal predictions only about spectral fluctuations such as the correlation function $C(\epsilon) = \langle \delta\rho(E)\delta\rho(E+\epsilon)\rangle$ of the deviations $\delta\rho$ of the density of states $\rho(E)$ from its mean value $\langle\rho(E)\rangle$. A central quantity is the spectral form factor $K_{\rm WD}(t) = \langle\rho\rangle^{-1}\int_{-\infty}^{\infty} d\epsilon\, e^{-{\rm i}\epsilon t/\hbar} C(\epsilon)$. The novel random-matrix ensembles differ from the Wigner-Dyson case by the fact that even the average density of states has universal features close to the Fermi energy. Thus, we define the spectral form factor $K(t) = 2\int_{-\infty}^{\infty} dE \langle\rho(E)\rangle e^{-{\rm i}Et/\hbar}$, where E is the energy measured from the Fermi energy. For class C (invariant under spin rotations, while time reversal is broken), this form factor is [7]

$$K^{\rm C}(t) = -\theta(1 - |t|/t_H). \tag{1}$$

Here, $t_H = 2\pi\hbar\rho_{av}$ is the Heisenberg time defined in terms of the mean DOS ρ_{av} sufficiently far from the Fermi energy. For the ensemble CI (which differs from C by invariance under time reversal), the short time expansion is [7] $K^{C{\rm I}}(t) = -1 + |t|/2t_H + \mathcal{O}(|t|^2)$.

Much insight can be gained from a semiclassical argument for the short-time behavior of the spectral form factor of class C. We first review the semiclassical derivation of the spectral form factor of the GUE. There one starts from the Gutzwiller trace formula, which relates the oscillatory contribution $\delta\rho(E)$ to the density of states to a sum over periodic orbits p, $\delta\rho(E) = \frac{1}{\pi\hbar}{\rm Re}\sum_p t_p A_p e^{{\rm i}S_p/\hbar}$. Here, S_p denotes the classical action of the orbit, A_p denotes its stability amplitude, and t_p is the primitive orbit traversal time. The explicit factor t_p arises because the traversal of the periodic orbit can start anywhere along the orbit. Inserting this expression into the definition of the spectral form factor and making the diagonal approximation, one obtains for broken time-reversal symmetry $K_{\rm WD}(t) = \sum_p (t^2/t_H)|A_p|^2\delta(t - t_p)$. Finally averaging over some time interval Δt and using the Hannay-Ozorio-de-Almeida sum rule [12] $\sum_p^{t_p\in[t,t+\Delta t]} |A_p|^2 = \Delta t/t$ one obtains the result $K_{\rm WD}(t) = t/t_H$ valid for $t \ll t_H$.

The basic process in SN hybrid systems is Andreev reflection converting electrons into holes (and vice versa) at the interface to the superconductor (see Fig. 1). In this process, the incoming electron (hole) acquires a phase $-\mathrm{i}e^{-\mathrm{i}\alpha}$ ($-\mathrm{i}e^{\mathrm{i}\alpha}$), where α is the phase of the superconducting order parameter [13]. In the absence of a magnetic field, electrons (holes) sufficiently close to the Fermi energy are reflected as holes (electrons) which then retrace the electron (hole) trajectory backwards (retroflection). In chaotic (ergodic) billiards, essentially all trajectories eventually hit the superconducting interface, thus leading to a periodic orbit bouncing back and forth between two points on the superconducting interface. The resulting trajectories are all periodic, leading to nonuniversal behavior such as the proximity-induced hard gap for time-reversal invariant systems.

One expects to recover universal spectral statistics *only* if the periodic orbits are isolated as in conventional chaotic (hyperbolic) systems. In Andreev billiards this occurs naturally when time-reversal symmetry is broken by a perpendicular magnetic field (class C). In this case the retroflected hole (electron) does not retrace the trajectory of the incoming electron (hole) since both electron and hole trajectories are curved in the same direction (cf. Fig. 1a). When the cyclotron radius is larger than, but of the order of, the size of the billiard one may expect chaotic dynamics. This allows one to express the semiclassical DOS by a Gutzwiller-type trace formula as a sum over the isolated periodic orbits of the Andreev billiard, $\delta\rho(E) \sim \sum_p t_p A_p e^{\mathrm{i}S_p(E)/\hbar+\mathrm{i}\chi}$. The orbit amplitudes A_p are products of electron and hole contributions, $A_p = A_p^{(e)} A_p^{(h)}$, while the orbit actions are sums of electron and hole actions, $S_p(E) = S_p^{(e)}(E) + S_p^{(h)}(E)$. The prefactor t_p again reflects the fact that the traversal of the orbit can start anywhere along the orbit and χ denotes the phase accumulated due to Andreev reflections.

Coherent contributions to the form factor can be expected from those periodic orbits that retrace the same trajectory – *in the same direction* – both as electron and hole, cf. Fig. 1b. (Here, we restrict ourselves to systems in which the ergodic time $t_{\rm erg}$ is small compared to the escape time $t_{\rm esc}$ into the leads, and consider times $t \lesssim t_{\rm esc}$ in the form factor. The more general case will be discussed below.) For these trajectories, the dynamical contributions to the action largely cancel between the electron and hole actions, due to the relation $S_p^{(e)}(E) = -S_p^{(h)}(-E)$, and we obtain $S_p(E) \simeq Et_p$ for small E. Moreover, in this case the amplitudes of electron and hole are just complex conjugates of one another so that $A_p = |A_p^{(e)}|^2$. The accumulated Andreev phase is $(-\mathrm{i})^2 = -1$. Keeping only the periodic orbits giving coherent contributions – the diagonal approximation – we therefore find $\delta\rho(E) \sim -\sum_p t_p |A_p^{(e)}|^2 e^{\mathrm{i}Et_p/\hbar}$. It is important to note that there is only *one* factor t_p arising from the arbitrary starting point of the periodic orbit. Taking the Fourier transform, we obtain for the form factor $K(t) \sim -\sum_p t_p |A_p^{(e)}|^2 \delta(t-t_p)$.

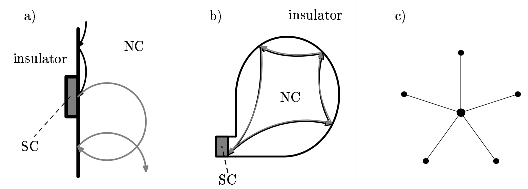

FIG. 1: (a) Andreev scattering with a perpendicular magnetic field. (b) Andreev billiard with a lead connected to a superconductor where Andreev reflection takes place. When traversed both as electron and hole, the orbit contributes in the diagonal approximation. The singly traversed orbit is a periodic orbit of a billiard without electron-hole conversion but unusual boundary conditions. (c) Star graph with five peripheral vertices connected to superconductors.

Finally, we observe that there is a complete correspondence between the *diagonal* periodic orbits of the Andreev billiard and those of an ordinary (magnetic) billiard with unusual boundary conditions along the interface (retroflection). Thus, the Hannay-Ozorio-de-Almeida sum rule is expected to hold for the amplitudes $A_p^{(e)}$ as well, and we find that the form factor

$$K(t) \sim -\sum_p t_p |A_p^{(e)}|^2 \delta(t-t_p) \sim -\mathrm{const} \qquad (2)$$

is indeed constant for short times.

In the remainder of this letter, we will make this argument rigorous for two types of models: We first consider quantum graphs [14] which were recently introduced as particularly simple quantum chaotic systems. Introducing Andreev reflection as a new ingredient, we show *semiclassically* that the form factor of the resulting *Andreev graph* takes on the universal result. Subsequently, we come back to Andreev billiards.

Quantum graphs – A quantum graph consists of vertices connected by bonds. A particle (electron or hole) propagates freely on a bond and is scattered at a vertex according to a prescribed vertex scattering matrix. For definiteness, we discuss star graphs with N bonds of equal length L. These have one *central* vertex and N *peripheral* vertices. Each bond connects the central vertex to one peripheral vertex (cf. Fig. 1c).

Andreev (star) graphs are obtained by introducing (complete) electron-hole conversions at the peripheral vertices while the central vertex preserves the particle type. The quantization condition is $\det(\mathcal{S}(k) - \mathbb{1}) = 0$ with the unitary $N\times N$ matrix $\mathcal{S}(k) = S_C \mathcal{L} D_- \mathcal{L} S_C^* \mathcal{L} D_+ \mathcal{L}$. Here S_C (S_C^*) is the central scattering matrix for an electron (hole). We choose $S_{C,kl} = e^{2\pi\mathrm{i}kl/N}/\sqrt{N}$ [15] (S_C by itself does not break time reversal symmetry). The matrix $\mathcal{L} = e^{\mathrm{i}kL}\mathbb{1}$ contains the

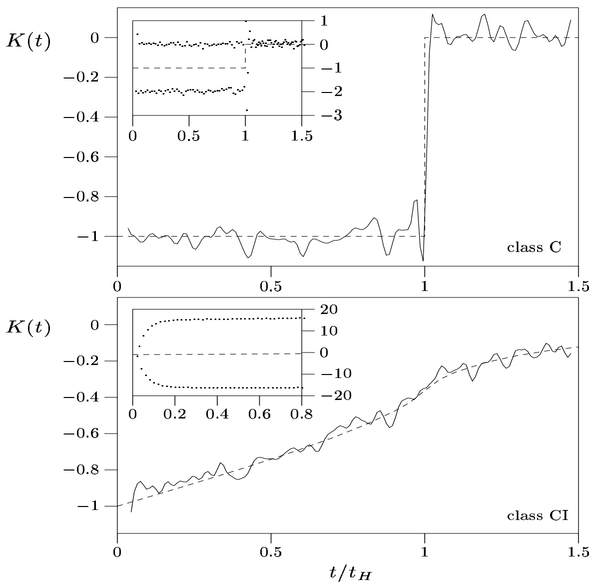

FIG. 2: Form factors for class C (top) and CI (bottom) calculated numerically for a star graph (full lines) with $N = 100$ bonds (averaged over 50000 realizations) and as obtained from RMT (dashed lines). The inserts give the coefficients K_m as a function of "time" m/N.

phases accumulated when the quasi-particle propagates along the bonds (k is the wavenumber measured from the Fermi wavenumber). Finally, $D_\pm = -\mathrm{i}\,\mathrm{diag}(e^{\mp \mathrm{i}\alpha_i})$ contains the *Andreev* phases accumulated at the vertices, where α_i denotes the order-parameter phase at peripheral vertex i. Time-reversal symmetry is obeyed if all Andreev phases are either $\alpha_i = 0$ or $\alpha_i = \pi$ but is broken otherwise. Accordingly, we build ensembles corresponding to the symmetry classes C (uncorrelated Andreev phases α_i with uniform distributions in the interval $[0, 2\pi)$) and CI (uncorrelated Andreev phases taking values $\alpha_i = 0$ or $\alpha_i = \pi$ with equal probability). Numerically computed ensemble averages are in excellent agreement with RMT results as shown in Fig. 2.

Following previous work on quantum graphs [14], we write the DOS in k-space as $\rho(k) = \rho_{av} + \delta\rho(k)$ with $\rho_{av} = 2NL/\pi$ and obtain the exact trace formula

$$\delta\rho(k) = \frac{1}{\pi}\mathrm{Re}\sum_p t_p A_p e^{\mathrm{i}S_p + \mathrm{i}\chi} \tag{3}$$

as a semiclassical sum over periodic orbits p of the graph. Here, periodic orbits are defined as a sequence $i_1, i_2, \ldots, i_l$ of peripheral vertices, with cyclic permutations identified. Since the particle type changes at the peripheral vertices, the sequences must have even length $l = 2m$. The primitive traversal 'time' [16] of a periodic orbit is $t_p = 4mL/r$ (where r is the repetition number), the stability amplitude is $A_p = 1/N^m$ and the action is $S_p = 4mkL + \sum_{j=1}^{2m}(-1)^{j+1} 2\pi i_j i_{j+1}/N$. The accumulated Andreev phase is $\chi = -m\pi - \sum_{j=1}^{2m}(-1)^{j+1}\alpha_{i_j}$. Then, the form factor becomes $K(t) = 2\int_{-\infty}^{\infty} dk\, e^{-\mathrm{i}kt}\langle\delta\rho(k)\rangle = (t_H/N)\sum_{m=1}^{\infty} K_m\, \delta(t - m t_H/N)$ with the Heisenberg time $t_H = 4LN$.($\langle\cdot\rangle$ denotes the average over Andreev phases.) The coefficients can be written as a sum over periodic orbits p_m with $2m$ Andreev reflections,

$$K_m = 2\sum_{p_m}\frac{m}{r}\left\langle A_p e^{\mathrm{i}S_p(k=0)+\mathrm{i}\chi}\right\rangle. \tag{4}$$

K_m can be viewed as a form factor in discrete time m/N.

For graphs in class C, only those periodic orbits survive the average over Andreev phases that visit each peripheral vertex an even number of times – half as incoming electron and half as incoming hole. In the diagonal approximation, only those orbits contribute whose total phase due to the scattering matrix of the central vertex vanishes. As the phase factors due to scattering between bonds i and j for electrons and for holes are complex conjugates of one another, this requires that the periodic orbits contain equal numbers of scatterings from i to j as electron and hole. This leads to the orbits sketched in Fig. 3: An *odd* number of peripheral vertices are visited twice, once as an electron and once as a hole. First, the peripheral vertices are visited once, alternating between electrons and holes, and subsequently the vertices are visited again in the same order but with the roles of electrons and holes interchanged. For these orbits, we have $A_p = 1/N^m$, $S_p = 4mkL$, and $\chi = m\pi$. The number of such orbits of length $2m$ is N^m/m, where the denominator m reflects the identification of cyclic permutations of peripheral vertices. With these ingredients, we find the short-time result

$$K^{\mathrm{C}}_{m,\mathrm{diag}} = -1 + (-1)^m \quad \Rightarrow \quad \overline{K}^{C}_{\mathrm{diag}}(t) = -1, \tag{5}$$

where $\overline{K}(t)$ is the time averaged form factor. This reproduces the RMT result.

For class CI, the average over Andreev phases requires only an even number of visits to each vertex. In the diagonal approximation, this leads to additional orbits (see Fig. 3) and to the result $K^{\mathrm{CI}}_{m,\mathrm{diag}} = -1 + (-1)^m(2m+1)$ so that $\overline{K}^{\mathrm{CI}}_{\mathrm{diag}}(t) = -1$, again as in RMT.

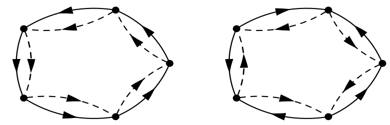

FIG. 3: Periodic orbits contributing in the diagonal approximation (at $m = 5$). The vertices in the diagram correspond to peripheral vertices of the original star graph, full and dashed lines represent electron and hole propagation. In class C, only the left diagram contributes. In class CI, the right diagram gives m additional contributions as the turning point can be any of the m vertices.

The results can be extended to the symmetry classes D and DIII. We also note that our results remain valid for a rather large class of central scattering matrices S_C. Finally, by going beyond the diagonal approximation, it is possible to extract the orbits contributing to the form factor to linear order in t (weak localization corrections). These extensions will be discussed elsewhere [17].

Andreev billiards – We first discuss Andreev billiards with N leads containing one channel each for which the problem can be mapped to star graphs. The quantization condition for Andreev billiards with N leads is [11] $\det(\mathcal{S}(E)-\mathbb{1})=0$. Here $\mathcal{S}(E)$ is the $N\times N$ Andreev billiard scattering matrix $\mathcal{S}(E)=S_{NC}(E)D_-S^*_{NC}(-E)D_+$ with $S_{NC}(E)$ the scattering matrix describing the coupling of the N channels by the normal region. The matrices $D_\pm$ describing the Andreev scattering in the leads are diagonal, $D_\pm=-\mathrm{i}\,\mathrm{diag}(e^{\mp\mathrm{i}\alpha_i})$, with a specific Andreev phase α_i for each lead. Then a detailed correspondence between billiard and star graph is obtained by substituting $\mathcal{L}S_C\mathcal{L}\to S_{NC}(E)$ and $\mathcal{L}S^*_C\mathcal{L}\to S^*_{NC}(-E)$ (with a more general central scattering matrix). Thus, the form factor of the billiard can be obtained in the diagonal approximation in complete analogy with the star graph.

For class C, we may alternatively consider a *magnetic* billiard with one N-channel lead and an arbitrary order-parameter phase along the superconducting interface. In this case, we can write the trace formula as a sum over isolated periodic orbits in the limit of large N. Replacing sums by integrals in the trace formula and applying stationary phase, $\delta\rho$ can be expressed as a sum over periodic orbits of the billiard. Assuming chaotic dynamics, the periodic orbits are isolated and we have a Gutzwiller-type trace formula. A general periodic orbit will hit the superconducting interface m times as an incoming electron and m times as an incoming hole. Contributions to the diagonal approximation have a structure that closely resembles the periodic orbits in the case of Andreev graphs: The trajectory hits the interface at m locations, alternating between electron and hole, and then repeats itself with the roles of electrons and holes interchanged. Such orbits exist for odd m only. The calculation of the form factor is again completely analogous to the calculation for quantum graphs. In the semiclassical argument given at the beginning of the paper, we restricted ourselves to the case with $t_{\rm esc}\gg t_{\rm erg}$ and times $t\lesssim t_{\rm esc}$ in the form factor. In this case, only orbits with $m=1$ contribute significantly. One therefore obtains contributions from orbits in which the same trajectory is traversed first as an electron and subsequently as a hole (or vice versa). We note that in the general situation, there is no need for the condition $t_{\rm esc}\gg t_{\rm erg}$.

A similar construction for class CI is difficult since time-reversal symmetry implies that the holes necessarily retrace the electron trajectory, thus leading to non-isolated periodic orbits and nonuniversal spectral statistics. Universal spectral statistics of CI can however be found in Andreev billiards with N one-channel leads.

In conclusion, we considered the universal spectral statistics for the new random-matrix universality classes associated with SN hybrid systems, in the semiclassical approximation. While it was known that semiclassics has problems in some types of Andreev systems [9, 10], we showed both for billiards and for quantum graphs that the universal spectral statistics of the new random-matrix ensembles C and CI as reflected by the form factor is correctly reproduced by semiclassical theory. We find that an important condition for finding the universal spectral statistics is that the Andreev-reflected hole do not retrace the incoming electron trajectory so that the periodic orbits are isolated. In class C, this is naturally the case in magnetic billiards. Our results clarify under which conditions to expect spectral statistics described by the novel random-matrix ensembles.

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[1] E.P. Wigner, Ann. Math. **67**, 325 (1958).
[2] F.J. Dyson, J. Math. Phys. **3**, 140, 1199 (1962).
[3] For a review, see T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[4] M.R. Zirnbauer, J. Math. Phys. **37**, 4989 (1996).
[5] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3853 (1993); J.J.M. Verbaarschot, *ibid* **72**, 2531 (1994).
[6] A. Altland, B.D. Simons, and M.R. Zirnbauer, Phys. Rep. **359**, 283 (2002).
[7] A. Altland and M.R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[8] M.V. Berry, Proc. Roy. Soc. London Ser. A **400**, 229 (1985).
[9] J. Melsen, P. Brouwer, K. Frahm, and C. Beenakker, Europhys. Lett. **35**, 7 (1996).
[10] D. Taras-Semchuk and A. Altland, Phys. Rev. B **64**, 014512 (2001).
[11] W. Ihra, M. Leadbeater, J.L. Vega, and K. Richter, Eur. Phys. J. B **21**, 425 (2001).
[12] J.H. Hannay and A.M. Ozorio de Almeida, J. Phys. A **17**, 3429 (1984).
[13] Strictly speaking, the phase of the order parameter is not a gauge invariant quantity. For class CI, we assume $B=0$ with the gauge $\mathbf{A}=0$ throughout this paper. The discussion of class C is valid in any fixed gauge. All physical properties such as the form factor are easily shown to be gauge independent.
[14] T. Kottos and U. Smilansky, Phys. Rev. Lett **79**, 4794 (1997); Ann. Phys. **274**, 76 (1999).
[15] G. Tanner, J. Phys. A **34**, 8485 (2001).
[16] Although the bond length is a dimensionless quantity, we nevertheless refer to it loosely as time.
[17] S. Gnutzmann, B. Seif, F. von Oppen and M.R. Zirnbauer, unpublished.