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Parametric resonance and spin–charge separation in 1D fermionic systems

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Abstract. - We show that the periodic modulation of the Hamiltonian parameters for 1D correlated fermionic systems can be used to parametrically amplify its bosonic collective modes. Treating the problem within the Luttinger liquid picture, we show how charge and spin density waves with different momenta are simultaneously amplified. We discuss the implementation of our predictions for cold atoms in 1D modulated optical lattices, showing that the fermionic momentum distribution directly provides a clear signature of spin–charge separation.

Introduction. – The swing is the best known example of a classical system showing parametric resonances [1]. The periodic modulation of the effective swing length induced by the motion of legs leads to the exponential amplification of the oscillations if the modulation frequency is chosen commensurately with the natural frequency of the swing. Quantizing this classical problem as a harmonic oscillator with modulated parabolic confinement leads to the appearance of an exponential divergence in the time evolution of the bosonic rising and lowering operators. This effect is particularly strong if the modulation is around twice the natural oscillator frequency.

In a modulated system of many bosonic oscillators only those fulfilling the resonance condition will be amplified, making parametric resonance a spectroscopic tool in many-body quantum systems. These ideas acquired particular relevance since cold atoms in optical lattices have been realized [2]. As the intensity of the lattice can be fully controlled by the laser power one can study parametric modulations in correlated quantum systems with current experimental tools [3]. Parametric amplification of Bogoliubov quasiparticles for bosonic clouds in optical lattices have been already investigated in the past [4,5]. In these systems, the bosonic nature of quasiparticles appears already when interactions are treated at mean-field level, and allows for the amplification to occur.

In this paper we analyze parametric resonances in many-body *fermionic* systems, starting with the very question whether the amplification can occur at all. Indeed, in contrast to the bosonic case, in fermionic systems any mean-field treatment of interactions, including the presence of broken symmetries, preserves the fermionic nature of quasiparticles. The Pauli principle thus blocks their
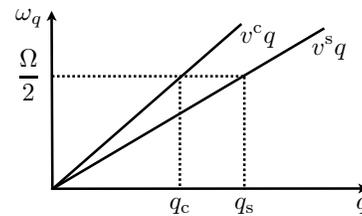

Fig. 1: Sketch of the dispersions ω_q of the charge and spin density waves as a function of momentum q, with group velocities v^c and v^s, respectively. The pumping frequency Ω amplifies collective spin and charge modes in the vicinity of the momenta q_c and q_s.

amplification, as can be easily checked by direct calculation [5]. Then the question rises if bosonic collective excitations of a fermionic many-body system can be subject to amplification by modulating a parameter in the microscopic Hamiltonian. In order to address this fundamental question we need to treat correlations in a fermionic system beyond mean field level. In this work we thus confine our investigation to one-dimensional (1D) correlated fermions within the Luttinger liquid picture, in which interactions are treated exactly and the system is naturally diagonalized in terms of collective bosonic spin and charge density waves [6,7]. According to the Luttinger liquid theory these modes disperse with two different group velocities, giving rise to the so-called spin–charge separation (see fig. 1). This fundamental issue in condensed matter physics has been detected in transport experiments on quantum wires [8] and by angle-resolved photoemission spectroscopy of 1D SrCuO$_2$ [9] only recently. Due to their great tunability, cold atomic gases in optical lattices are

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also promising candidates for the experimental detection of spin–charge separation in 1D systems, as shown by several theoretical proposals [10–16].

Here we show how a spatially homogeneous time-periodic modulation of the intensity of the optical lattice indeed leads to the amplification of charge and spin density waves of a 1D correlated cloud of ultracold fermionic atoms. If Ω is the modulation frequency, the charge and spin waves with energy $v^c q_c = v^s q_s = \Omega/2$ will be amplified (see fig. 1). Due to the different group velocities v^c and v^s for the charge and spin channels, respectively, the resonant condition amplifies different wavenumbers for the two branches, q_c and q_s. On top of showing the feasibility of parametric amplification in correlated fermionic systems, we also propose this technique as a tool to systematically investigate the spin–charge separation in experiments. Indeed, we discuss the effect of the amplification above on the fermionic momentum distribution and show how the latter exhibits well defined shoulders directly related to the wavenumbers q_c and q_s. These structures are particularly evident after not too long modulation times. As the momentum distribution is the standard quantity measured in time-of-flight experiments on cold atomic clouds [2], our analysis has direct implications on investigations of correlated fermionic systems with current experimental tools.

Model. – We consider interacting fermionic atoms with (pseudo-)spin $1/2$ confined into 1D cigars (as, *e.g.*, realized in optical lattices [2]), with a further periodic potential along the 1D axis. Despite this additional potential, we assume the atoms to be in their metallic phase, as opposed to the recently experimentally-investigated Mott-insulator phase [17,18]. The parametric excitation of the system is achieved by periodically modulating the intensity of the optical lattice along the 1D system, thereby shrinking the Wannier wavefunctions associated to each lattice site. In the usual microscopic tight-binding picture this has the two-fold effect of modulating the hopping rate between neighboring sites (*i.e.*, the kinetic energy) as well as the on-site repulsion for multiple occupancy. As the former is exponentially sensitive to the wavefunction overlap it is the dominant parametric modulation term in the problem and we will focus on this for simplicity. It should be however noticed that the inclusion of the smaller parametric modulation of the interactions would lead to the very same consequences discussed in the following.

Once translated into the Bloch-band language, and if we focus on the low-energy physics of the problem, the modulation in the kinetic term yields an effectively time-dependent Fermi velocity for the atoms close to the Fermi level, $v_F(t) = v_F + \delta v_F(t)$. Here $\delta v_F(t) = \gamma v_F \sin(\Omega t)$, where v_F is the Fermi velocity at equilibrium (*i.e.*, for times $t < 0$ before the modulation starts), and Ω and γ are the frequency and intensity of the parametric modulation, respectively. Our focus on the low-energy sector of the many-body problem naturally suggests a Luttinger liquid approach [6,7] to the correlated system, which allows for

an exact treatment of interactions. We thus linearize the single-particle spectrum in the vicinity of the Fermi level for momenta $||k| - k_F| < \Lambda$, with k_F the Fermi wavevector and Λ an ultraviolet cutoff. The time-dependent Hamiltonian describing the system of size L therefore reads[1]

$$H(t) = v_F(t) \sum_{k\sigma\tau} (\tau k - k_F) c_{k\sigma\tau}^\dagger c_{k\sigma\tau}$$

$$+ \frac{1}{2L} \sum_{\substack{q \neq 0 \\ \tau\tau'\sigma\sigma'}} V_{\tau\tau',q}^{\sigma\sigma'} \rho_{\tau\sigma,-q}\rho_{\tau'\sigma',q}. \quad (1)$$

Here, $c_{k\sigma\tau}^\dagger$ ($c_{k\sigma\tau}$) creates (annihilates) a τ-moving fermion with momentum k and spin $\sigma = \uparrow,\downarrow$ ($\tau = 1$ (-1) corresponds to right (left) movers) while $\rho_{\tau\sigma,q} = \sum_k c_{k-q,\sigma,\tau}^\dagger c_{k,\sigma,\tau}$ is the corresponding fermionic density operator. In (1) we keep the general form of the interaction between fermionic densities with generic spin and branch indices. Within the standard Luttinger liquid theory we have $V_{\tau\tau,q}^{\sigma\sigma} = V_0^\parallel$, $V_{\tau\tau,q}^{\sigma,-\sigma} = V_0^\perp$, $V_{\tau,-\tau,q}^{\sigma\sigma} = V_0^\parallel - V_{2k_F}^\parallel$, $V_{\tau,-\tau,q}^{\sigma,-\sigma} = V_0^\perp$, where $V_q^{\parallel/\perp}$ is the Fourier transform of the microscopic interaction between fermions with parallel/antiparallel spins. This allows to treat short range interactions (like contact s-wave scattering for neutral fermions) where Pauli principle imposes $V_q^\parallel = 0$, as well as finite range spin-invariant ones (*e.g.*, between charged fermions) with $V_q^\parallel = V_q^\perp$.

Introducing the bosonic operators for charge and spin density fluctuations $b_q^c = (\pi/L|q|)^{1/2} \sum_\tau \Theta(\tau q)(\rho_{q\uparrow\tau} + \rho_{q\downarrow\tau})$ and $b_q^s = (\pi/L|q|)^{1/2} \sum_\tau \Theta(\tau q)(\rho_{q\uparrow\tau} - \rho_{q\downarrow\tau})$, respectively, eq. (1) transforms into the separable Hamiltonian [6]

$$H(t) = \sum_{\substack{a=c,s \\ q \neq 0}} |q| \left[A_q^a(t) b_q^{a\dagger} b_q^a + B_q^a \left(b_q^{a\dagger} b_{-q}^{a\dagger} + b_{-q}^a b_q^a \right) \right],$$
$$(2)$$

with $A_q^c(t) = v_F(t) + (V_0^\parallel + V_0^\perp)/2\pi$, $A_q^s(t) = v_F(t) + (V_0^\parallel - V_0^\perp)/2\pi$, $B_q^c = (V_0^\parallel + V_0^\perp - V_{2k_F}^\parallel)/4\pi$, and $B_q^s = (V_0^\parallel - V_0^\perp - V_{2k_F}^\parallel)/4\pi$.

The time-independent part in (2) can be diagonalized by means of the Bogoliubov transformation $b_q^a = \cosh\varphi_q^a\, \beta_q^a + \sinh\varphi_q^a\, \beta_{-q}^{a\dagger}$ in terms of the bosonic fields β_q^a, such that

$$H(t) = \sum_{a,q\neq 0} \left[\left(\omega_q^a + \delta v_F(t)|q| \cosh(2\varphi_q^a)\right) \beta_q^{a\dagger} \beta_q^a \right.$$

$$\left. + \frac{\delta v_F(t)}{2}|q| \sinh(2\varphi_q^a) \left(\beta_q^{a\dagger} \beta_{-q}^{a\dagger} + \beta_{-q}^a \beta_q^a \right) \right] \quad (3)$$

with $\omega_q^a = v_q^a|q|$. The different charge and spin group velocities are $v_q^a = [A_q^a(0)^2 - 4B_q^{a2}]^{1/2}$. For short range interactions, one can neglect the momentum dependence of the group velocities such that $v_q^a \simeq v^a$, and the dispersion of the spin and charge density waves is linear

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[1]In the remaining of the paper, we set $\hbar = k_B = 1$.

(see fig. 1). The coefficients of the Bogoliubov transformation read $\sinh \varphi_q^c = -[A_q^c(0)/2v_q^c - 1/2]^{1/2}$, $\cosh \varphi_q^c = [A_q^c(0)/2v_q^c + 1/2]^{1/2}$, $\sinh \varphi_q^s = [A_q^s(0)/2v_q^s - 1/2]^{1/2}$, and $\cosh \varphi_q^s = [A_q^s(0)/2v_q^s + 1/2]^{1/2}$.

The crucial point to notice at this level is that the parametric modulation introduces a time-dependent anomalous term in the Hamiltonian (3), creating and annihilating pairs of bosonic charge and spin density waves. As the modulation is homogeneous in space (*i.e.*, at zero wavenumber), the new terms create or annihilate pairs of excitations with opposite wavenumber, as requested by momentum conservation [5]. In addition, the induced anomalous terms are proportional to $\sinh(2\varphi_q^a)$ and thus correctly vanish in the non-interacting limit $V_q^{\|/\perp} = 0$, where fermionicity forbids parametric amplifications.

Parametric amplification. – From the Hamiltonian (3) above we can now determine the time evolution of the operators β_q^a, showing that indeed parametric amplification of collective bosonic modes in a fermionic system is possible. With eq. (3), the Heisenberg equation of motion for the operator β_q^a reads

$$\dot{\beta}_q^a(t) = -\,\mathrm{i}\left[\omega_q^a + \delta v_F(t)|q|\cosh(2\varphi_q^a)\right]\beta_q^a(t)$$
$$- \,\mathrm{i}\,\delta v_F(t)|q|\sinh(2\varphi_q^a)\beta_{-q}^{a\dagger}(t). \quad (4)$$

Defining $\beta_q^a(t) = \mathrm{e}^{-\mathrm{i}\int_0^t \mathrm{d}s[\omega_q^a + \delta v_F(s)|q|\cosh(2\varphi_q^a)]}\tilde{\beta}_q^a(t)$, assuming a weak parametric modulation ($\gamma \ll 1$) and retaining only slow terms near the resonance (rotating wave approximation, *i.e.*, for Ω in the vicinity of $2\omega_q^a$), the equation of motion (4) and its adjoint transform into [4,5] $\ddot{\tilde{\beta}}_q^a(t) + \mathrm{i}(\Omega - 2\omega_q^a)\dot{\tilde{\beta}}_q^a(t) - \xi_q^{a\,2}\tilde{\beta}_q^a(t) = 0$, where $\xi_q^a = \gamma v_F|q|\sinh(2\varphi_q^a)/2$. Solving this equation with the appropriate initial conditions $\tilde{\beta}_q^a(0) = \beta_q^a(0)$ and $\dot{\tilde{\beta}}_q^a(0) = \xi_q^a\beta_{-q}^{a\dagger}(0)$, we obtain

$$\beta_q^a(t) = \sum_{\eta=\pm} \eta\, \mathrm{e}^{\mathrm{i}(\omega_{q\eta}^a - \omega_q^a)t}\left[\bar{\omega}_{q,-\eta}^a\beta_q^a(0) + \mathrm{i}\,\bar{\xi}_q^a\beta_{-q}^{a\dagger}(0)\right], \quad (5)$$

where $\omega_{q\pm}^a = \omega_q^a - \Omega/2 \pm \sqrt{\left(\omega_q^a - \Omega/2\right)^2 - \xi_q^{a\,2}}$. In (5), we defined $\bar{\omega}_{q\pm}^a = \omega_{q\pm}^a/(\omega_{q-}^a - \omega_{q+}^a)$ and $\bar{\xi}_q^a = \xi_q^a/(\omega_{q-}^a - \omega_{q+}^a)$. Thus, in a narrow "resonant window" of energy $|\omega_q^a - \Omega/2| < |\xi_q^a|$, the frequencies $\omega_{q\pm}$ acquire an imaginary part, leading to the exponential amplification of the corresponding bosonic modes. Outside this window, the modes evolve according to their coherent dynamics and are therefore not amplified. Indeed, out of eq. (5), it is easy to verify that, on- and off-resonance, the evolution of the Bogoliubov operators is

$$\beta_q^a(t) \simeq \mathrm{e}^{-\mathrm{i}\Omega t/2}\Big[\cosh(|\xi_q^a|t)\beta_q^a(0)$$
$$+ \,\mathrm{sign}(\xi_q^a)\sinh(|\xi_q^a|t)\beta_{-q}^{a\dagger}(0)\Big] \quad (6a)$$

for $|\omega_q^a - \Omega/2| \ll |\xi_q^a|$, and

$$\beta_q^a(t) \simeq \mathrm{e}^{-\mathrm{i}\omega_q^a t}\beta_q^a(0) \quad (6b)$$

for $|\omega_q^a - \Omega/2| \gg |\xi_q^a|$.

Fermionic momentum distribution. – The results above prove the possibility of amplifying bosonic collective modes (with different wavenumbers for charge and spin modes) while parametrically modulating the underlying 1D fermionic many-body Hamiltonian. However, the detection of this amplification is not necessarily easy from the experimental point of view. In the case of cold atoms in optical lattices this would require a (spin-resolved) measurement of the cloud density during the parametric modulation, without opening the trap. In view of this experimental difficulty we now discuss the consequences of our analysis on the fermionic momentum distribution, which is the standard quantity measured in time-of-flight experiments [2]. Due to the different amplified momenta for charge and spin modes, we show that the fermionic momentum distribution shows clear signatures of spin–charge separation.

The momentum distribution function (per spin channel) for τ-movers is defined as

$$n_{k\tau}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx}\left\langle \psi_{\tau\sigma}^\dagger(x,t)\psi_{\tau\sigma}(0,t)\right\rangle \quad (7)$$

where, within our perturbative treatment in $\gamma \ll 1$, $\langle\cdots\rangle$ represents a thermal average with respect to the *time-independent* part of the Hamiltonian (1), *i.e.*, for $\gamma = 0$. The fermionic field operators creating a τ-mover with spin σ at position x and time t can be expressed in terms of the bosonic operators as [6]

$$\psi_{\tau\sigma}^\dagger(x,t) = \frac{\mathrm{e}^{-\mathrm{i}\tau k_F x - \theta_{\tau\sigma}(x,t)}}{\sqrt{L(1 - \mathrm{e}^{-2\pi/L\Lambda})}}U_{\tau\sigma}(t), \quad (8)$$

with $\theta_{\tau\sigma}(x,t) = [\theta_\tau^c(x,t) + \sigma\theta_\tau^s(x,t)]/\sqrt{2}$ ($\sigma = 1$ and -1 correspond to spin up and down, respectively) and $\theta_\tau^a(x,t) = \sum_{\tau q>0}(2\pi/L|q|)^{1/2}\left[\mathrm{e}^{\mathrm{i}qx}b_q^a(t) - \mathrm{e}^{-\mathrm{i}qx}b_q^{a\dagger}(t)\right]$. In (8), the unitary Klein operator $U_{\tau\sigma}$ increases the number of τ-movers with spin σ by one.

With (8), we find after a lengthy but straightforward calculation

$$\left\langle \psi_{\tau\sigma}^\dagger(x,t)\psi_{\tau\sigma}(0,t)\right\rangle = \frac{\mathrm{e}^{-\mathrm{i}\tau k_F x + \phi_\tau(x,t)}}{L(1 - \mathrm{e}^{-2\pi/L\Lambda})}. \quad (9)$$

Here, within the rotating wave approximation, we have

$$\phi_\tau(x,t) = \sum_{a,q>0} \frac{\pi}{Lq}\Big\{\mathrm{i}\tau\sin(qx)\left[\beta_q^a(t), \beta_q^{a\dagger}(t)\right]$$
$$+ [\cos(qx) - 1]\cosh(2\varphi_q^a)\left\langle\left\{\beta_q^a(t), \beta_q^{a\dagger}(t)\right\}\right\rangle\Big\}, \quad (10)$$

with $\beta_q^a(t)$ given in eq. (5). The summation over q in (10) is simplified by taking the evolution of the Bogoliubov operators in eq. (6) for wavenumbers belonging to the off-resonance and the narrow on-resonance windows. The latter are centered around the two resonant wavenumbers $q_{c/s} = \Omega/2v^{c/s}$ for the charge and spin modes, respectively.

Our procedure allows for a fully analytical treatment of the problem, leading to the fermionic correlator

$$\langle \psi_{\tau\sigma}^\dagger(x,t)\psi_{\tau\sigma}(0,t)\rangle = \langle \psi_{\tau\sigma}^\dagger(x)\psi_{\tau\sigma}(0)\rangle_0 \, \mathcal{A}(x,t), \quad (11)$$

where the correlator without parametric amplification (*i.e.*, for $\gamma = 0$) in the zero temperature limit is [6]

$$\langle \psi_{\tau\sigma}^\dagger(x)\psi_{\tau\sigma}(0)\rangle_0 = \frac{\mathrm{i}e^{-\mathrm{i}\tau k_\mathrm{F} x}}{2\pi(\tau x + \mathrm{i}0^+)}\left(\frac{\lambda^2}{x^2 + \lambda^2}\right)^\alpha \quad (12)$$

with $\alpha = (\sinh^2\varphi^\mathrm{c} + \sinh^2\varphi^\mathrm{s})/2$ and $\varphi^a = \varphi_{q\simeq0}^a$. Here, the ultraviolet cutoff Λ has been identified with the inverse of the screening length λ. This leads to the fermionic momentum distribution in the absence of the parametric amplification $n_{k,\tau}^0$. As the problem is fully symmetric between right and left movers, we focus on the former without loss of generality. For $k > k_\mathrm{F}$, the momentum distribution can be expressed as

$$n_{k,\mathrm{R}}^0 = \frac{2^{1/2-\alpha}}{\sqrt{\pi}\Gamma(\alpha)}\int_0^\infty \mathrm{d}Q'(Q+Q')^{\alpha-1/2}\mathrm{K}_{\alpha-1/2}(Q+Q'), \quad (13)$$

with $Q = \lambda(k-k_\mathrm{F})$, $\Gamma(z)$ and $\mathrm{K}_\nu(z)$ being the gamma and the modified Bessel functions, respectively. In particular, $n_{k-k_\mathrm{F},\mathrm{R}}^0 = 1 - n_{-k+k_\mathrm{F},\mathrm{R}}^0$. The function $n_{k,\mathrm{R}}^0$ is presented in fig. 2 at time $t = 0$ for contact and finite-range interactions. The latter may be of relevance for the treatment of trapped cold ions.

The amplification factor in eq. (11) reads

$$\mathcal{A}(x,t) = \exp\left(\sum_{a=\mathrm{c,s}} h^a(t)\left[\cos\left(q_a x\right) - 1\right]\right), \quad (14)$$

where

$$h^a(t) = [1 + 2n_\mathrm{B}(\Omega/2)]\kappa^a\cosh\left(2\varphi^a\right)\left[\cosh\left(\kappa^a\Omega t\right) - 1\right], \quad (15)$$

and $\kappa^a = \gamma\,|\sinh\left(2\varphi^a\right)|\,v_\mathrm{F}/2v^a$. Here, $n_\mathrm{B}(\omega) = (e^{\omega/T} - 1)^{-1}$ is the Bose distribution at temperature T. It is important to notice that the amplification factor equals 1 in the non-interacting limit (where $\sinh(\varphi^a) = 0$). Once again, this highlights the importance of fermionic interactions as a necessary ingredient for the parametric amplification to occur. Out of eq. (11) the momentum distribution at finite times thus results in the convolution

$$n_{k,\mathrm{R}}(t) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}q}{2\pi}n_{k-q,\mathrm{R}}^0\tilde{\mathcal{A}}(q,t) \quad (16)$$

with $\tilde{\mathcal{A}}(q,t)$ the Fourier transform of (14).

Two qualitatively different regimes occur in the small- or large-time regimes, *i.e.*, if $h^a(t) \ll 1$ or $h^a(t) \gg 1$ in eq. (14). In the first case, valid up to times of order $t_0 \simeq -\ln\left([1 + 2n_\mathrm{B}(\Omega/2)]\kappa^a\cosh\left(2\varphi^a\right)\right)/\kappa^a\Omega$, the expansion of $\mathcal{A}(x,t) \simeq 1 + \sum_a h^a(t)\left[\cos\left(q_a x\right) - 1\right]$ yields

$$\tilde{\mathcal{A}}(q,t) = 2\pi\delta(q)$$
$$+ \pi\sum_a h^a(t)\left[\delta(q - q_a) + \delta(q + q_a) - 2\delta(q)\right], \quad (17)$$
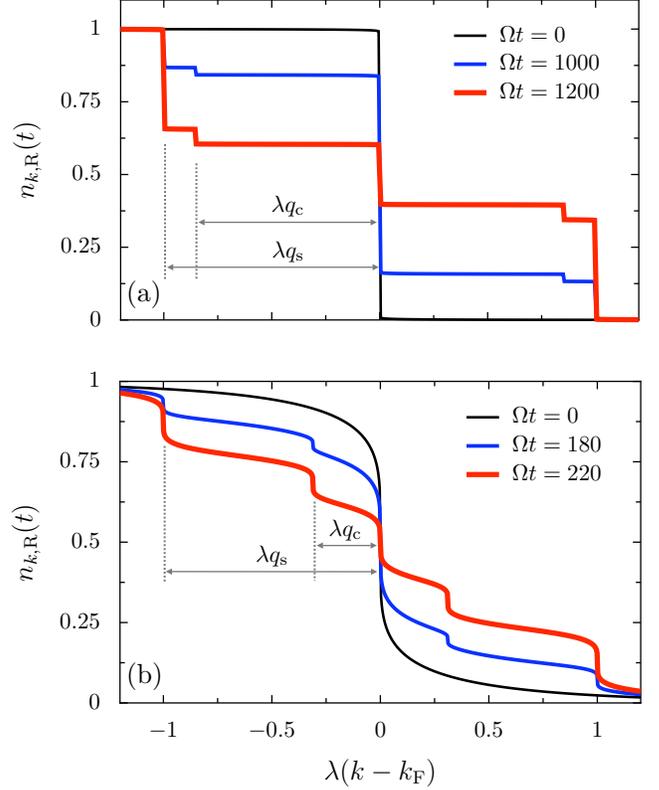

Fig. 2: Fermionic momentum distribution for right movers $n_{k,\mathrm{R}}(t)$ in the short-time regime (see eq. 18) as a function of momentum k, scaled by the screening length λ. In the figure, $T = 0$, $\gamma = 0.1$ and $\lambda q_\mathrm{s} = 1$. (a) Case of contact s-wave scattering interaction, with $V_0^\perp/v_\mathrm{F} = 0.5$ and $V_0^\parallel = V_{2k_\mathrm{F}}^\parallel = 0$, such that $\lambda q_\mathrm{c} = \lambda q_\mathrm{s}v_\mathrm{s}/v_\mathrm{c} \simeq 0.85$ (b) Case of finite-range interactions, with $V_0^\parallel/v_\mathrm{F} = V_0^\perp/v_\mathrm{F} = 10$ and $V_{2k_\mathrm{F}}^\parallel/v_\mathrm{F} = 2$, such that $\lambda q_\mathrm{c} = \lambda q_\mathrm{s}v_\mathrm{s}/v_\mathrm{c} \simeq 0.31$. Thicker lines correspond to larger times t.

leading to

$$n_{k,\mathrm{R}}(t) = n_{k,\mathrm{R}}^0 + \sum_a h^a(t)\left(\frac{n_{k+q_a,\mathrm{R}}^0 + n_{k-q_a,\mathrm{R}}^0}{2} - n_{k,\mathrm{R}}^0\right). \quad (18)$$

Despite parametric amplification, the fermionic momentum distribution fulfills $n_{k-k_\mathrm{F},\mathrm{R}} = 1 - n_{-k+k_\mathrm{F},\mathrm{R}}$ as in the unperturbed case, guaranteeing particle-number conservation. For $k > k_\mathrm{F}$ it shows two steps of size $h^{\mathrm{c/s}}(t)/2$ involving fermionic states with momenta up to $q_{\mathrm{c/s}}$ away from the Fermi level, as exemplified in fig. 2.

These are direct signatures of the parametric amplification of the charge and spin density waves and their observation can thus be used to detect spin–charge separation in interacting 1D Fermi systems. By spanning the external modulation frequency Ω, the whole dispersion of the collective modes can be mapped. From the point of view of the measurement, our result is best visible in the "short-time regime" where the expansion above holds, leading to two well resolved steps.
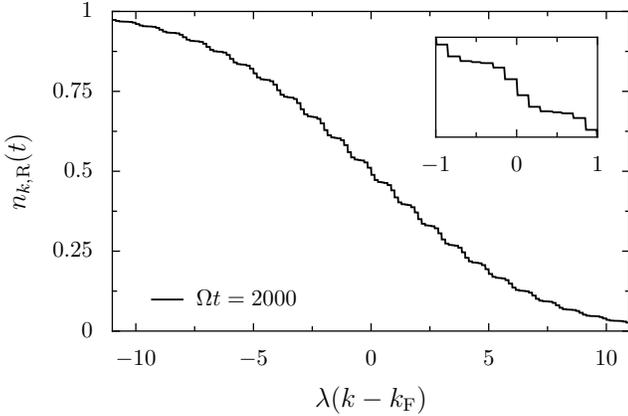
Fig. 3: Momentum distribution for right movers $n_{k,\mathrm{R}}(t)$ in the long-time regime (see eq. 20) as a function of momentum k. The parameters are the same as in fig. 2a. The inset presents the momentum distribution at a finer wavenumber scale close to the Fermi level.

Our analytical treatment allows formally the analysis of the "large-time regime" as well, where $h^a(t) \gg 1$. In this case the amplification factor (14) can be approximated as $\mathcal{A}(x,t) = \mathcal{A}^c(x,t)\mathcal{A}^s(x,t)$, with $\mathcal{A}^a(x,t) = \sum_{n=-\infty}^{+\infty} \exp\big(-(q_a x - 2\pi n)^2 h^a(t)/2\big)$. As a consequence, the Fourier transform results in

$$\tilde{\mathcal{A}}(q,t) = \sum_{m,n=-\infty}^{\infty} f_m^c(t) f_n^s(t) \delta(q - mq_c - nq_s) \qquad (19)$$

with $f_m^a(t) = 1/\sqrt{h^a(t)} \exp\big(-m^2/2h^a(t)\big)$ leading to

$$n_{k,\mathrm{R}}(t) = \sum_{m,n=-\infty}^{\infty} \frac{f_m^c(t) f_n^s(t)}{2\pi} n_{k-mq_c-nq_s,\mathrm{R}}^0. \qquad (20)$$

Thus, in the large-time limit the fermionic momentum distribution shows many small-size steps stemming from both the charge and the spin sectors (see fig. 3). Indeed, the form of $f_m^a(t)$ shows how for large $h^a(t)$ more and more peaks of $\mathcal{A}(q,t)$ in (19) become relevant. This will limit the experimental resolution of the structures in $n_{k,\mathrm{R}}(t)$ in contrast to the short timescales. Moreover, at large times the exponential amplification of bosonic modes requires a treatment of their residual interaction beyond the Luttinger liquid model, associated to parabolic corrections to the linearized spectrum around the Fermi level [16].

A final issue to be addressed in view of the experimental realization of our proposal is the role of finite temperatures, where our analysis above applies as well. The only differences are: (i) The presence of a non-vanishing Bose distribution in eq. (15). This yields a thermal seed for the amplification on top of the pure quantum fluctuations at $T = 0$, and decreases the time needed for the formation of well-resolved steps in $n_{k,\tau}(t)$. (ii) The unperturbed momentum distribution $n_{k,\mathrm{R}}^0$ in eq. (16) has to be replaced with the finite temperature one, involving a thermal smearing of order T around the Fermi level

(for $\alpha \ll 1$) on top of that purely induced by interactions. As thermal smearing involves wavenumbers up to order T/v_F around k_F, the shoulders in the final momentum distribution at short times are thus better visible if $v_\mathrm{F} q_a \simeq \Omega/2 \gtrsim T$ ($a = \mathrm{c, s}$), which can be guaranteed by choosing a sufficiently large Ω at a given temperature.

Experimental realization. – Our proposal could be experimentally realized with an equal mixture of quantum degenerate fermionic $^{40}\mathrm{K}$ atoms confined into 1D cigars in the two hyperfine states $|F, m_F\rangle = |9/2, -9/2\rangle = |\downarrow\rangle$ and $|F, m_F\rangle = |9/2, -7/2\rangle = |\uparrow\rangle$. Here, F is the total angular momentum and m_F its projection along the quantization axis. It is important to realize that the atomic density corresponds to the charge channel, while the two hyperfine states above correspond to the (pseudo-)spin channel. Assuming the cigar of length $L = 0.1\,\mathrm{mm}$ to be homogeneous and containing $N = 10^2$ atoms, we have $k_\mathrm{F} = 3 \times 10^6\,\mathrm{m}^{-1}$, which corresponds to a Fermi energy and temperature of order $E_\mathrm{F}/\hbar = 7\,\mathrm{kHz}$ and $T_\mathrm{F} = 60\,\mathrm{nK}$, respectively. For neutral atoms, we assume contact s-wave scattering for which $V_0^\| = V_{2k_\mathrm{F}}^\| = 0$ as required by Pauli principle, and $V_0^\perp/v_\mathrm{F} = \pi k_\mathrm{F} a \omega_\perp/E_\mathrm{F}$ [19]. Here, a is the 3D s-wave scattering length and $\omega_\perp$ the frequency of the transverse confining lasers creating the cigars. Notice that $E_\mathrm{F} \ll \omega_\perp$ justifies the effective 1D treatment of the cigars. Assuming a scattering length of the order of $a = 10\,\mathrm{nm}$ and $\omega_\perp = 40\,\mathrm{kHz}$, we obtain $V_0^\perp/v_\mathrm{F} = 0.5$, which corresponds to the parameters in fig. 2a.

As our proposal is optimized in the regime $T \lesssim \Omega/2 \lesssim E_\mathrm{F}$, and assuming $T/T_\mathrm{F} \simeq 0.2$, this corresponds to pumping frequencies in the range $3\,\mathrm{kHz} \lesssim \Omega \lesssim 14\,\mathrm{kHz}$. Measuring the fermionic momentum distribution by a time-of-flight experiment [2], one should thus obtain a clear signature of spin–charge separation for pumping times of the order of $100\,\mathrm{ms}$ (for $\Omega = 10\,\mathrm{kHz}$ and $\gamma = 0.1$), as exemplified in fig. 2a. Notice that pumping for larger times would lead to a situation similar to the one depicted in fig. 3 where spin–charge separation is much less clearcut and where temperature effects are likely to smear out most signatures of shoulders.

Conclusion. – In this work we have shown the possibility of parametrically amplifying collective modes in a modulated 1D fermionic many-body system. The amplification is crucially affected by fermionic interactions which are here exactly treated within the Luttinger liquid picture. This opens the perspective of similar observations in systems of higher dimensionality as well.

Our analysis shows that the amplification of charge and spin density waves of the Luttinger liquid results in clear steps in the fermionic momentum distribution. The wavenumber extension of the steps directly reveals the different momenta of the excited charge and spin modes and thus offers a tool for the detection of spin–charge separation. In parallel, we show that the best resolution of the steps is achieved by modulations of relatively short times

and that they survive thermal effects for large enough modulation frequencies.

Our proposal of detection of spin–charge separation is particularly suitable for systems of cold fermionic atoms in 1D optical lattices with modulated intensity. The fermionic momentum distribution is indeed the standard quantity measured in time-of-flight experiments. We stress that for our proposal no additional experimental setup is required on top of the already present tunable lasers creating the optical lattice.[2]

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[2]During the completion of this work, we became aware of ref. [16] where similar effects have been investigated.