Universal spectral statistics in Andreev graphs: The self-dual approximation

Sven Gnutzmann,[1,*] Burkhard Seif,[2,†] and Felix von Oppen[1,‡]

[1] *Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany*
[2] *Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln*

The symmetry classification of quantum systems has recently been extended beyond the Wigner-Dyson classes. In ergodic systems each class has universal spectral statistics as described by Gaussian random matrix ensembles. We show that Andreev graphs are simple quantum systems belonging to new symmetry classes and give a interpretation of some universal features of their spectra using trace formulae.

I. INTRODUCTION

More than 40 years ago Wigner [1] proposed to use random matrix theory to describe universal features of quantum spectra. Based on Wigner's idea Dyson [2] gave a symmetry classification of complex quantum systems according to their behavior under time reversal. In the ergodic limit each symmetry class defines a universality class which may be described by Gaussian random matrix ensembles. A quantum system is invariant under generalized time reversal if there is an anti-unitary operator $\hat{T}$ which commutes with the Hamilton operator $[\hat{H}, \hat{T}] = 0$. Dyson introduced three symmetry classes of irreducible systems: systems without any generalized time reversal invariance, systems with time reversal invariance and $\hat{T}^2 =$, systems with time reversal invariance and $\hat{T}^2 = -$. While the mean density of states remains system specific each class shows universal spectral fluctuations in the ergodic limit which are described by the well-known Wigner-Dyson ensembles of random matrix theory (GUE, GOE, GSE). Gaussian random matrix theory has since then been applied successfully to many different areas such as atomic nuclei, quantum chaos and disordered quantum systems.

However, quantum systems may also have a symmetry that relates the positive and the negative part of the spectrum: if E is an eigenvalue then, due to this symmetry so is $-E$. Such a symmetry is often realized in systems with a combined electron-hole or particle-antiparticle dynamics and has strong impact on both, the mean density of states and spectral correlations near $E = 0$. Dyson has not included this type of symmetry in his classification and the Wigner-Dyson ensembles of random matrix theory cannot be applied to their spectral statistics at $E = 0$. Extending Dyson's three-fold way an additional seven symmetry classes have been identified [4] which are naturally realized in part by Dirac fermions in random gauge fields (so called chiral classes) [5] and in part by quasi-particles in disordered mesoscopic superconductors [6] or superconducting-normalconducting (SN) hybrid systems [7]. In the ergodic limit this leads to seven new universality classes each described again by Gaussian random matrix ensembles. The ten symmetry classes of quantum systems correspond to Cartan's tenfold classification of symmetric spaces. As customary, we use the Cartan scheme to refer to the symmetry class (A, AI, AII for the Wigner-Dyson classes) which we distinguish from the corresponding universality class such as the Gaussian random matrix ensembles (GUE, GOE, GSE for the Wigner-Dyson classes).

In this paper we will give a semiclassical interpretation of the universal features of the density of states in terms of periodic orbits for Andreev graphs. A similar argument has been derived by us before for Andreev billiards [8] where we also touched graphs. In section II we will introduce two of the new symmetry classes and introduce the generalized form factor as the Fourier transform of the mean density of states. The universal mean density of states is known from the corresponding random matrix theory. In section III we will introduce to quantum graphs as simple ergodic quantum systems. An Andreev graph adds the concepts of an electron-hole symmetry and Andreev scattering. By construction they belong to the new symmetry classes. An exact semiclassical trace formula relates the density of states to a sum over periodic orbits on the graph. In section IV we will use this trace formula to partially reproduce the universal features of the generalized form factor as known from random matrix theory. Our self-dual approximation for the Fourier transform of the density of states (generalized form factor) is an analogue to Berry's diagonal approximation [9] for the Fourier transform of the spectral two-point correlation function (form factor) for the Wigner-Dyson ensembles. Berry's semiclassical derivation for chaotic quantum systems was based on Gutzwiller's trace formula, partially reproducing the results of random matrix theory and clarifying its limitations.

There have been several attempts to apply semiclassical theory to SN hybrid systems [10–12] – mainly Andreev billiards. Melsen *et al.* [10] pointed out that the gap induced by the proximity effect in a billiard coupled to a superconducting lead is sensitive to whether the classical dynamics of the (normal) billiard is integrable or chaotic. These authors showed that the proximity-induced hard gap in the chaotic case is *not* fully reproduced by semiclassical theory, the reasons for which have

*Electronic address: sven@gnutzmann.de
†Electronic address: bseif@thp.uni-koeln.de
‡Electronic address: vonoppen@physik.fu-berlin.de

been discussed further in [11].

Non-magnetic Andreev billiards, though belonging to the new symmetry classes, do *not* show the universal mean density of states as predicted by the corresponding Gaussian random matrix theory. The reason is that in this case, the combined electron-hole dynamics near the Fermi level is not chaotic in the classical limit (even if the billiard is chaotic when the superconductor is replaced by an insulator). This is also the reason for the problems in constructing a semiclassical theory for some Andreev billiards. Here, however, we deal with classically chaotic systems which exhibit the *universal* spectral statistics of the random matrix ensembles corresponding to the new symmetry classes. We identify the class of periodic orbits contributing to the form factor and show that in this case, semiclassics reproduces the universal spectral statistics.

II. THE NEW SYMMETRY CLASSES

A symmetry that relates positive and negative parts of the spectrum may either be represented by a unitary operator $\hat{P}$ or an anti-unitary operator $\hat{C}$ that anticommutes with the Hamilton operator

$$\hat{P}\hat{H} + \hat{H}\hat{P} = 0 \tag{1a}$$

$$\hat{C}\hat{H} + \hat{H}\hat{C} = 0. \tag{1b}$$

In the first case one also requires that $\hat{P}$ be chosen such that

$$\hat{P}^2 = \tag{2a}$$

while the anti-unitary operator $\hat{C}$ either has

$$\hat{C}^2 = \qquad \text{or} \qquad \hat{C}^2 = - \,. \tag{2b}$$

The seven new ensemble arise from possible combinations of these symmetries with time reversal. In time reversal invariant systems the anti-unitary operator $\hat{T}$ has to commute not only with the Hamilton operator but also with $\hat{P}$ or $\hat{C}$.

For the three chiral symmetry classes there is a unitary operator $\hat{P}$ anticommuting with the Hamilton operator. In class AIII there is no additional time reversal, class BDI is time reversal invariant with $\hat{T}^2 =$ and class CII is also time reversal invariant, but with $\hat{T}^2 = -$. The corresponding Gaussian random matrix ensembles have been called chGUE, chGOE and chGSE. Note that BDI and CII also have a anti-unitary operator $\hat{C} = \hat{P}\hat{T}$, thus $\hat{C}^2 =$ in BDI and $\hat{C}^2 = -$ in CII.

In the four remaining classes the symmetry in the spectrum is due to a anti-unitary operator ('charge conjugation') $\hat{C}$. If there is no time reversal symmetry there are two symmetry classes called C and D. In the first class $\hat{C}^2 = -$ while in the latter $\hat{C}^2 =$. We will call the corresponding Gaussian random matrix ensembles (universality classes) C-GE and D-GE. For time reversal invariant systems there are again two symmetry classes called DIII and CI. In the first class $\hat{C}^2 =$ and $\hat{T}^2 = -$ while in the latter $\hat{C}^2 = -$ and $\hat{T}^2 =$ and we call the corresponding Gaussian random matrix ensembles DIII-GE and CI-GE. The two other possibilities $\hat{C}^2 = \hat{T}^2 =$ and $\hat{C}^2 = \hat{T}^2 = -$ have already been accounted for in the chiral symmetry classes (BDI and CII). These four classes naturally appear in hybrid superconducting normalconducting heterostructures (Andreev systems) where quasi-electron and quasi-hole (we will omit the 'quasi' in the following) excitations are described by the Bogoliubov-de-Gennes equation. We will focus in the sequel on the classes C and CI – these are realized in Andreev systems that are invariant under spin-rotations. One then deals with effectively spinless electrons and holes and the Bogoliubov-de-Gennes equation is

$$\begin{pmatrix} \frac{(\hat{\mathbf{p}}+e\mathbf{A})^2}{2m} - \mu & \Delta \\ \Delta^* & -\frac{(\hat{\mathbf{p}}-e\mathbf{A})^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3}$$

This is a two-component eigenequation for the excitation energy E. The electron and hole component interact via the superconducting order parameter Δ, μ is the Fermi-energy and $\hat{\mathbf{p}} = -i\hbar\nabla$. The spectrum is symmetric: if $\Psi = \begin{pmatrix} u \\ v \end{pmatrix}$ is an eigenfunction with energy E then $\Psi^C = \hat{C}\Psi = \begin{pmatrix} -v^* \\ u* \end{pmatrix}$ is also an eigenfunction but now with energy $-E$. The equation (3) is generally not time reversal invariant and belongs to class C. For time reversal invariance (class CI) one has to restrict (3) to $\mathbf{A} = 0$ (no magnetic field) and $\Delta = \Delta^*$ (real superconducting order parameter). Then $\Psi^T = \hat{T}\Psi = \begin{pmatrix} u^* \\ v^* \end{pmatrix}$ differs only by a global phase from the eigenfunction Ψ – thus one may choose Ψ real from the start.

In contrast to the Wigner-Dyson universality classes the spectral statistics of the Gaussian random matrix ensembles for the new symmetry classes, while still universal, is no longer stationary under shifts of the energy. The additional spectral correlations are most prominent near $E = 0$ while for energies much larger than the mean level spacing $E \gg \Delta E$ Wigner-Dyson statistics is recovered.

One significant difference is in the mean density of states. In the Wigner-Dyson ensembles the mean density of states is given by Wigner's semi-circle law. This cannot be used to predict the density of states of a any quantum system in the same class as the latter is given by Weyl's law

$$\rho_{\text{Weyl}}(E) = \int \frac{d^f\mathbf{p}\, d^f\mathbf{q}}{(2\pi\hbar)^f}\, \delta(E - H(\mathbf{p},\mathbf{q})) \tag{4}$$

where $H(\mathbf{p},\mathbf{q})$ is the classical Hamilton function and f the number of freedoms. Weyl's law shows that the mean density is system specific and universality can only be found in the fluctuations

$$\delta\rho(E) = \rho(E) - \rho_{\text{Weyl}}(E) \tag{5}$$

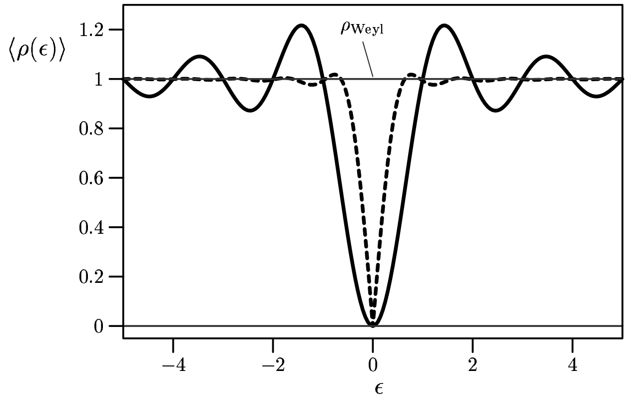

FIG. 1: Mean density of states for the random matrix ensembles C-GE (full line) and CI-GE (dashed line). Weyl's law (thin full line) predicts a constant mean density of states near $\epsilon = 0$.

where $\rho(E) = \sum_i \delta(E - E_i)$ is the full density of states. Upon averaging over some system parameter $\delta\rho(E)$ vanishes in the Wigner-Dyson case. The latter is not true for the new symmetry classes with symmetric spectra. Even after averaging over a system parameter $\delta\rho(E)$ does not vanish near $E = 0$ – thus the mean density of states differs from Weyl's law near $E = 0$. The difference between Weyl's law and the mean density of states vanishes however for energies much larger than the mean level spacing $E \gg \Delta E = \frac{1}{\rho_{\rm Weyl}}$. The corresponding Gaussian random matrix theories predict a universal function for each symmetry class that describes the difference of Weyl's law and the mean density of states near $E = 0$ on the scale of the mean level spacing. For C-GE and CI-GE the mean density of states is

$$C\text{-GE:} \quad \langle\rho(\epsilon)\rangle = 1 - \frac{\sin \pi\epsilon}{\pi\epsilon} \tag{6a}$$

$$C\text{I-GE:} \quad \langle\rho(\epsilon)\rangle = \frac{\pi^2|\epsilon|}{2}\left(J_0(\pi\epsilon)^2 + J_1(\pi\epsilon)^2\right) - \frac{\pi}{2}J_0(\pi\epsilon)J_1(\pi|\epsilon|) \tag{6b}$$

where the energy ϵ is measured in units of the mean level spacing and $J_{0,1}(\xi)$ are Bessel functions (see figure 1).

A central quantity in the semiclassical analysis of chaotic systems in the Wigner-Dyson classes is the form factor

$$K_{\rm WD}(\tau) = \int_{-\infty}^{\infty} d\epsilon\, e^{-2\pi i\epsilon\tau} C(\epsilon) \tag{7}$$

where $C(\epsilon) = \langle\delta\rho(\epsilon')\delta\rho(\epsilon' + \epsilon)\rangle$ is the two-point correlation function. For the Gaussian unitary ensemble (GUE) the form factor is given by

$$K_{\rm WD}(\tau) = \begin{cases} |\tau| & \text{for} |\tau| < 1 \\ 1 & \text{for} |\tau| \geq 1 \end{cases} \tag{8}$$

Much insight into the range of validity of the Wigner-Dyson random-matrix ensembles has been gained from the semiclassical approach to the spectral statistics of chaotic quantum systems, based on Gutzwiller's trace formula. In a seminal paper [9], Berry gave a semiclassical derivation of the spectral form factor of chaotic quantum systems for the Wigner-Dyson ensembles, partially reproducing the results of RMT and clarifying its limitations. We will now briefly review the semiclassical derivation of the spectral form factor of the GUE. There one starts from the Gutzwiller trace formula [13], which relates the oscillatory contribution $\delta\rho(E)$ to the density of states to a sum over periodic orbits p,

$$\delta\rho(E) = \frac{1}{\pi\hbar}{\rm Re}\sum_p t_p A_p e^{iS_p/\hbar}. \tag{9}$$

Here, S_p denotes the classical action of the orbit, A_p denotes its stability amplitude, and t_p is the primitive orbit traversal time. The explicit factor t_p arises because the traversal of the periodic orbit can start anywhere along the orbit. Inserting this expression into the definition of the spectral form factor, and employing the diagonal approximation, one finds

$$K_{\rm WD,diag}(\tau) = \sum_p \tau^2 |A_p|^2 \delta(\tau - t_p/t_H). \tag{10}$$

Here $t_H = \frac{2\pi\hbar}{\Delta E}$ is the Heisenberg time. Finally averaging over some time interval $\Delta\tau$ and using the Hannay-Ozorio-de-Almeida (HOA) sum rule [14]

$$\sum_{p: t_p/t_H \in [\tau, \tau+\Delta\tau]} |A_p|^2 = \frac{\Delta\tau}{\tau} \tag{11}$$

one obtains the result

$$K_{\rm WD,diag}(\tau) = \tau \tag{12}$$

valid for $\tau \ll 1$.

Now, for the new symmetry classes the central quantity of our study is the generalized spectral form factor which we define to be the Fourier transform of the *density of states*

$$\begin{aligned} K(\tau) &= 2\int_{-\infty}^{\infty} d\epsilon \langle\delta\rho(\epsilon)\rangle\, e^{-i2\pi\epsilon\tau} \\ &= 2\int_{-\infty}^{\infty} dE \langle\delta\rho(E)\rangle\, e^{-i\frac{E t_H \tau}{\hbar}}. \end{aligned} \tag{13}$$

Here $E = \epsilon\Delta E$ and $\delta\rho(E) = \Delta E \delta\rho(\epsilon)$ (Note that $\Delta E = \frac{1}{\rho_{\rm Weyl}}$ is taken from Weyl's law). In ergodic systems in the symmetry class C random matrix theory (C-GE) predicts [7]

$$K^{C-{\rm GE}}(\tau) = -\theta(1 - |\tau|), \tag{14a}$$

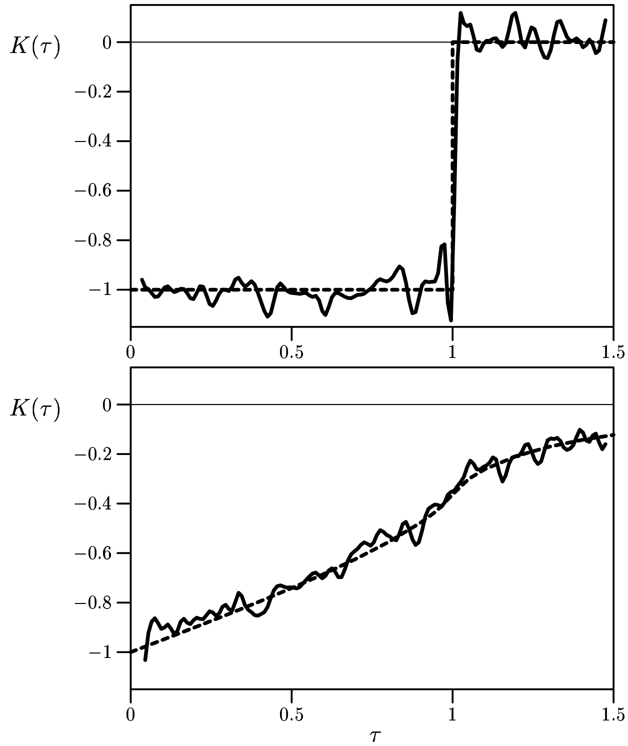

FIG. 2: The dashed lines give the generalized form factors for the random matrix ensembles C-GE (top) and CI-GE (bottom). The full lines are calculated numerically for a star graph with $N = 100$ bonds (averaged over 50000 realizations) and have been averaged over a small time interval

while the prediction (CI-GE) for class CI is [7]

$$\begin{aligned} K^{C\mathrm{I-GE}}(\tau) =& -1 + \frac{\tau+1}{\pi\tau}\mathcal{E}\left(\frac{4\tau}{(\tau+1)^2}\right) + \\ &+ \frac{\tau-1}{\pi\tau}\mathcal{K}\left(\frac{4\tau}{(\tau+1)^2}\right) \\ =& -1 + \frac{|\tau|}{2} + \mathcal{O}(|\tau|^2). \end{aligned} \tag{14b}$$

The result is given in terms of the complete elliptic integrals of first kind $\mathcal{K}(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1-x\sin^2\theta}}\, d\theta$ and of second kind $\mathcal{E}(x) = \int_0^{\pi/2} \sqrt{1-x\sin^2\theta}\, d\theta$. The generalized form factors are shown together with numerical results obtained for Andreev graphs (see section III) in figure 2.

III. ANDREEV GRAPHS

Quantum graphs have recently been introduced [15] as simple quantum chaotic systems. Due to an exact semiclassical trace formula they are extremely powerful and transparent in the semiclassical analysis of universal spectral statistics. The main new ingredient to quantum graphs which allows for the construction of very simple

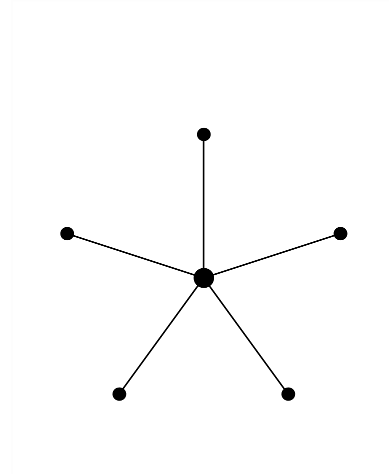

FIG. 3: Andreev star graph with five peripheral vertices connected to superconductors.

graphs belonging to new symmetry classes is Andreev reflexion. We show in section IV *semiclassically* that the form factor of the resulting *Andreev graph* takes on the universal result.

A quantum graph consists of vertices connected by bonds. A particle (electron/hole) propagates freely on a bond and is scattered at a vertex according to a prescribed scattering matrix. For definiteness, we discuss star graphs with N bonds of equal length L. These have one *central* vertex and N *peripheral* vertices. Each bond connects the central vertex to one peripheral vertex (see figure 3).

Andreev (star) graphs are obtained by introducing (complete) electron-hole conversions at the peripheral vertices with the vertex scattering matrix at the ith vertex

$$\sigma_i = \begin{pmatrix} 0 & -\mathrm{i}e^{\mathrm{i}\alpha_i} \\ -\mathrm{i}e^{-\mathrm{i}\alpha_i} & 0 \end{pmatrix} \tag{15}$$

where α_i is the phase of the order parameter of a superconductor coupled to the vertex. This scattering matrix can be obtained from the Bogoliubov-de-Gennes equation in one dimension where $\Delta(x) = e^{\mathrm{i}\alpha_i}|\Delta_0|\theta(x)$ in the limit $\mu \ll \Delta_0 \ll E$. The vertex scattering matrices obey time reversal invariance if either $\alpha_i = 0$ or $\alpha_i = \pi$. We assume that the central vertex preserves the particle type and that scattering between all bonds has the same amplitude (this avoids the localization effects found in usual star graphs with Neumann boundary conditions at the central vertex [16])– this is achieved by choosing [17]

$$S_{C,kl} = \frac{1}{\sqrt{N}}\, e^{2\pi\mathrm{i}\frac{kl}{N}} \tag{16}$$

for the electron-electron scattering. The electron-hole symmetry then demands that one chooses $S^*_{C,kl}$ for hole-hole scattering at the central vertex. This choice also obeys time reversal symmetry $S_C = S_C^T$. Andreev star graphs of this type will generally belong to the symmetry class C – only when *all* peripheral vertex scattering matrices obey time reversal symmetry they belong to class CI. Accordingly, we build ensembles corresponding to the symmetry classes C (uncorrelated Andreev phases α_i with uniform distributions in the interval $[0, 2\pi)$) and CI (uncorrelated Andreev phases taking values $\alpha_i = 0$ or $\alpha_i = \pi$ with equal probability). Numerically computed ensemble averages are in excellent agreement with random matrix results from C-GE and CI-GE as shown in figure 2.

The quantization condition for the Andreev star graph has the form

$$\det\left(\mathcal{S}(k) - \ \right) = 0 \tag{17}$$

where k is the quantized wave number and $\mathcal{S}(k)$ is the unitary $N \times N$ matrix

$$\mathcal{S}(k) = S_C \mathcal{L} D_- \mathcal{L} S_C^* \mathcal{L} D_+ \mathcal{L}. \tag{18}$$

Here S_C (S_C^*) is the central scattering matrix for an electron (hole) given by equation (16). The matrix $\mathcal{L} = e^{\mathrm{i}kL}$ contains the phases accumulated when the quasiparticle propagates along the bonds (k is the wavenumber measured from the Fermi wavenumber). Finally, $D_\pm = -\mathrm{i}\,\mathrm{diag}(e^{\mp \mathrm{i}\alpha_i})$ contains the *Andreev* phases accumulated at the vertices. Since

$$\mathcal{S}(k) = e^{\mathrm{i}4kL}\mathcal{S}_0 \tag{19}$$

where

$$\mathcal{S}_0 = S_C D_- S_C^* D_+ \tag{20}$$

does not depend on k, the k-spectrum is periodic with period $\frac{\pi}{2L}$. As we are interested in the spectrum near $k = 0$ on the scale of the mean level spacing $\Delta k = \frac{\pi}{2NL}$ this periodicity has no effect on our results for $N \ll 1$.

Following previous work on quantum graphs [15], we write the density of states in k-space as

$$\rho(k) = \rho_{\mathrm{Weyl}} + \delta\rho(k) \tag{21}$$

with

$$\rho_{\mathrm{Weyl}} = \frac{2NL}{\pi} \tag{22}$$

and obtain the exact trace formula

$$\begin{aligned} \delta\rho(k) &= \mathrm{Im}\,\frac{d}{dk}\sum_{m=1}^{\infty}\frac{1}{m\pi}\mathrm{tr}\,\mathcal{S}(k)^m \\ &= \frac{1}{\pi}\mathrm{Re}\sum_p t_p A_p e^{\mathrm{i}S_p + \mathrm{i}\chi} \end{aligned} \tag{23a}$$

as a semiclassical sum over periodic orbits p of the graph. Here, periodic orbits are defined as a sequence $i_1, i_2, \ldots, i_l$ of peripheral vertices, with cyclic permutations identified. Since the particle type changes at the peripheral vertices, the sequences must have even length $l = 2m$. The primitive traversal 'time' of a periodic orbit is $t_p = 4mL/r$ (where r is the repetition number). Note that the bond length L is a dimensionless quantity. We will refer to it nevertheless as time. The stability amplitude is $A_p = 1/N^m$ and the action is $S_p = 4mkL + \sum_{j=1}^{2m}(-1)^{j+1}2\pi i_j i_{j+1}/N$. The accumulated Andreev phase is $\chi = -m\pi - \sum_{j=1}^{2m}(-1)^{j+1}\alpha_{i_j}$. The trace formula (23a) depends trivially on k. With $k = \frac{\pi}{2NL}\kappa$ we measure the wave number in units of the mean level spacing and have

$$\delta\rho(\kappa) = \frac{2}{N}\sum_{m=1}^{\infty}\cos\left(2\pi\kappa\frac{m}{N}\right)\mathrm{tr}\,\mathcal{S}_0^m \tag{23b}$$

Then, the form factor becomes

$$\begin{aligned} K(\tau) &= 2\int_{-\infty}^{\infty} d\kappa\, e^{-\mathrm{i}2\pi\kappa\tau}\langle\delta\rho(\kappa)\rangle \\ &= \frac{1}{N}\sum_{m=1}^{\infty} K_m\,\delta(\tau - \frac{m}{N}). \end{aligned} \tag{24a}$$

Here, $\langle\cdot\rangle$ denotes the average over Andreev phases. The coefficients can be written as a sum over periodic orbits p_m with $2m$ Andreev reflections,

$$K_m = 2\left\langle\mathrm{tr}\,\mathcal{S}_0^m\right\rangle = 2\sum_{p_m}\frac{m}{r}\left\langle A_p e^{\mathrm{i}S_p(k=0)+\mathrm{i}\chi}\right\rangle. \tag{24b}$$

K_m can be viewed as a form factor in discrete time m/N. A smooth function can be obtained by averaging over a small time interval $1 \ll \Delta\tau \ll \frac{1}{N}$ – this gives

$$\overline{K(\tau)} = 2\sum_{m:\tau<m/N<\tau+\Delta\tau}\frac{K_m}{\Delta\tau N}. \tag{24c}$$

IV. UNIVERSAL FORM FACTOR FOR ANDREEV GRAPHS

We now turn to compute the generalized form factor (24a). Let us first focus on the Andreev graphs with broken time reversal (symmetry class C). The average over Andreev phases reduces the number of contributing periodic orbits. Only those periodic orbits survive the average that visit each peripheral vertex an even number of times – half as incoming electron and half as incoming hole. In the spirit of Berry's diagonal approximation we now look for the orbits that contribute coherently i.e the orbits whose total phase due to the scattering matrix of the central vertex vanishes. As the phase factors due to scattering between bonds i and j for electrons and for holes are complex conjugates of one another, this requires that the periodic orbits contain equal numbers of scatterings from i to j as electron and hole. This leads to the orbits sketched in figure 4: An *odd* number of peripheral vertices are visited twice, once as an

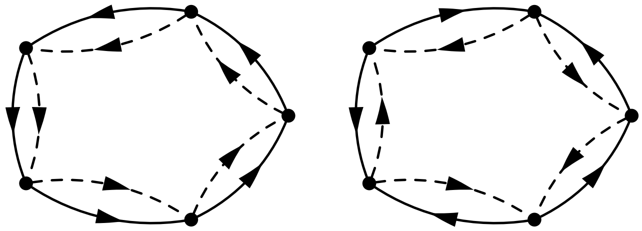

FIG. 4: Periodic orbits contributing in the self-dual approximation (at $m = 5$). The vertices in the diagram correspond to peripheral vertices of the original star graph, full and dashed lines represent electron and hole propagation. In the universality class C-GE, only the left diagram contributes. In class CI-GE, the right diagram gives m additional contributions as the turning point can be any of the m vertices.

electron and once as a hole. First, the peripheral vertices are visited once, alternating between electrons and holes, and subsequently the vertices are visited again in the same order but with the roles of electrons and holes interchanged. Such orbits are also singled out by invariance under electron-hole conjugation and will be called self-dual periodic orbits for this reason. The self-dual approximation to the form factor takes only these orbits into account. For a self dual orbit contributing to K_m we have $A_p = 1/N^m$, $S_p = 2\pi\kappa\frac{m}{N}$, and $\chi = m\pi$ (since m is odd $e^{i\chi} = -1$). The number of such orbits of length $2m$ is N^m/m, where the denominator m reflects the identification of cyclic permutations of peripheral vertices. With these ingredients, we find the short-time result

$$K_{m,\text{s.d.}} = -1 + (-1)^m \quad \Rightarrow \quad \overline{K}_{\text{s.d.}}(\tau) = -1, \qquad (25)$$

where $\overline{K}(t)$ is the time averaged form factor. This reproduces the random matrix result from C-GE.

For the ensemble of Andreev graphs of symmetry class CI, the average over Andreev phases (over signs) is survived by periodic orbits with an even number of visits to each vertex. This condition is weaker than the condition we had before in class C. Thus there are more periodic orbits contributing to the form factor. The self-dual orbits are again defined to be the orbits that are invariant under the symmetries that define the symmetry class: electron-hole conjugation and time reversal. In the self-dual approximation, this leads to additional orbits (see figure 4). Self-dual periodic orbits either visit the same peripheral vertices in the same sequence twice with electrons and holes interchanged (these orbits are also self-dual in class C) or they visit the same peripheral vertices twice but in opposite sequence (these orbits are invariant under the combination of time reversal and electron-hole conjugation). As discussed above the first type of self-dual orbits only exists if the number m of visited peripheral vertices is odd and their contribution to the form factor $K_m$is $-1+(-1)^m$. The second kind of orbits exists for any number of visited peripheral vertices – however for any fixed choice of m peripheral vertices there are now m different periodic orbits (the turning point can be any of the m vertices). Since $e^{i\chi} = 1$ for even M and $e^{i\chi} = -1$ for odd m the contribution to K_m of these orbits is $(-1)^m 2m$. Altogether one has

$$K_{m,\text{s.d.}} = -1+(-1)^m(2m+1) \quad \Rightarrow \quad \overline{K}_{\text{sd}}(\tau) = -1, \qquad (26)$$

again his is in accordance with the random matrix result from CI-GE for short times.

The results can be extended to other new symmetry classes. We also note that our results remain valid for a rather large class of central scattering matrices S_C. Finally, by going beyond the diagonal approximation, it is possible to extract the orbits contributing to the form factor to linear order in τ (weak localization corrections). These extensions will be discussed elsewhere [18].

V. CONCLUSIONS

We considered the universal spectral statistics for ergodic Andreev graphs belonging to the new symmetry classes, in the semiclassical approximation. While it was known that semiclassics has problems in some types of Andreev systems [10, 11], we showed here for quantum graphs that the universal spectral statistics is correctly reproduced by a semiclassical analysis of the appropriate generalized form factor. The universal spectral statistics is predicted by the random-matrix ensembles C-GE and CI-GE that describe systems of the symmetry classes C and CI in the ergodic limit. One important ingredient is the classically chaotic dynamics of quantum graphs. Our results clarify under which conditions to expect spectral statistics described by the Gaussian random-matrix ensembles C-GE and CI-GE.

We thank R. Klesse, S. Nonnenmacher, H. Schanz, U. Smilansky, and especially P. Brouwer ans M. Zirnbauer for helpful discussions. This work was supported in part by SFB 290.

[1] E.P. Wigner, Ann. Math. **67**, 325 (1958).
[2] F.J. Dyson, J. Math. Phys. **3**, 140, 1199 (1962).
[3] For a review, see T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[4] M.R. Zirnbauer, J. Math. Phys. **37**, 4989 (1996).
[5] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993); J.J.M. Verbaarschot, *ibid* **72**, 2531 (1994).
[6] A. Altland, B.D. Simons, and M.R. Zirnbauer, Phys. Rep. **359**, 283 (2002).
[7] A. Altland and M.R. Zirnbauer, Phys. Rev. B **55**, 1142

(1997).
[8] S. Gnutzmann, B. Seif, F. von Oppen and M.R. Zirnbauer, cond-mat/0207388 (2002).
[9] M.V. Berry, Proc. Roy. Soc. London Ser. A **400**, 229 (1985).
[10] J. Melsen, P. Brouwer, K. Frahm, and C. Beenakker, Europhys. Lett. **35**, 7 (1996).
[11] D. Taras-Semchuk and A. Altland, Phys. Rev. B **64**, 014512 (2001).
[12] W. Ihra, M. Leadbeater, J.L. Vega, and K. Richter, Eur. Phys. J. B **21**, 425 (2001).
[13] M.C. Gutzwiller, J. Math. Phys. **12**, 343 (1971)
[14] J.H. Hannay and A.M. Ozorio de Almeida, J. Phys. A **17**, 3429 (1984).
[15] T. Kottos and U. Smilansky, Phys. Rev. Lett **79**, 4794 (1997); Ann. Phys. **274**, 76 (1999).
[16] G. Berkolaiko and J.P. Keating, J. Phys. A **32**, 7827 (1999).
[17] G. Tanner, J. Phys. A **34**, 8485 (2001).
[18] S. Gnutzmann, B. Seif, F. von Oppen and M.R. Zirnbauer, unpublished.