SUPPLEMENTAL MATERIAL

Robust spectral π pairing in the random-field Floquet quantum Ising model

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I. FLOQUET QUANTUM ISING MODEL IN THE ABSENCE OF RANDOM FIELDS

We review the mapping of the one-dimensional Floquet quantum Ising model to a Floquet Kitaev chain in the absence of disorder. The Floquet operator of the quantum Ising model [Eq. (1) of the main text] obeys spin-flip symmetry (so that eigenstates can be classified into even and odd with respect to the spin-flip operator $P = \prod_{j=1}^{N} X_j$). For even N, it obeys a charge-conjugation symmetry (operator $C = i^{\frac{N}{2}}(\prod_{j=1}^{\frac{N}{2}} Y_{2j-1}Z_{2j})\mathcal{K}$ involving complex conjugation $\mathcal{K}$, implying that eigenvalues come in complex-conjugate pairs). Moreover, the isospectral symmetrized Floquet operator $U^s_{F,0} = U_{g/2}U_J U_{g/2}$ obeys a time-reversal symmetry (operator $\mathcal{K}$, implying that one can choose a real eigenbasis).

The Jordan-Wigner transformation

$$\sigma_j^- = e^{i\pi\sum_{l<j} c_l^\dagger c_l} c_j, \qquad X_j = 1 - 2c_j^\dagger c_j, \tag{S1}$$

with $\sigma_j^\pm = \frac{1}{2}(Z_j \pm iY_j)$ maps the spin operators in Eq. (1) to fermions c_j. The Floquet operator $U_{F,0}$ maps to the Floquet Kitaev chain

$$U_{F,0} = e^{i\frac{\pi g}{2}\sum\limits_{j=1}^{N}(1-2c_j^\dagger c_j)} e^{i\frac{\pi J}{2}\sum\limits_{j=1}^{N-1}(c_{j+1}+c_{j+1}^\dagger)(c_j - c_j^\dagger)}. \tag{S2}$$

In the fermionic formulation, the spin-flip symmetry translates to conservation of fermion parity [operator $P = \prod_{j=1}^{N}(1 - 2c_j^\dagger c_j)$].

To work out the time evolution $c_j(t+1) = U^\dagger_{F,0} c_j(t) U_{F,0}$ of the fermion operators, one writes the fermion operators $c_j = \frac{1}{2}(a_{2j-1} + ia_{2j})$ in terms of Majorana operators a_j, so that

$$U_{F,0} = \prod_{j=1}^{N} e^{\frac{\pi g}{2}a_{2j-1}a_{2j}} \prod_{j=1}^{N-1} e^{\frac{\pi J}{2}a_{2j}a_{2j+1}} = \prod_{j=1}^{N}\left(\cos\frac{\pi g}{2} + a_{2j-1}a_{2j}\sin\frac{\pi g}{2}\right)\prod_{j=1}^{N-1}\left(\cos\frac{\pi J}{2} + a_{2j}a_{2j+1}\sin\frac{\pi J}{2}\right). \tag{S3}$$

We can find single-particle Bogoliubov operators satisfying

$$U^\dagger_{F,0}\gamma_\alpha U_{F,0} = e^{-i\epsilon_\alpha}\gamma_\alpha \tag{S4}$$

(see main text), where the operators γ_α are linear combinations of the c_j and $c_j^\dagger$ and the ϵ_α define the single-particle eigenphases. For periodic boundary conditions, we pass to the momentum representation $c_k = \frac{1}{\sqrt{N}}\sum_j e^{ikj}c_j$. Introducing the two-component operator $\phi_k = [c_k, c_{-k}^\dagger]^T$, the time evolution takes the form

$$\phi_k(t+1) = U_{\mathrm{BdG}}\phi_k(t), \tag{S5}$$

with the Bogoliubov-de Gennes Floquet operator

$$U_{\mathrm{BdG}}(k) = \begin{pmatrix} e^{-i\pi g}\big(\cos(\pi J) + i\sin(\pi J)\cos k\big) & e^{-i\pi g}\sin(\pi J)\sin k \\ -e^{i\pi g}\sin(\pi J)\sin k & e^{i\pi g}\big(\cos(\pi J) - i\sin(\pi J)\cos k\big)\end{pmatrix}. \tag{S6}$$

Diagonalizing $U_{\mathrm{BdG}} = D^\dagger\Lambda D$ with a diagonal matrix Λ, we have $\phi'_k(t+1) = \Lambda\phi'_k(t)$ with $\phi'_k = D\phi_k$. We can thus identify the entries of $\phi'_k = [\gamma_k, \gamma_{-k}^\dagger]^T$ with the Bogoliubov operators. An explicit calculation gives the particle-hole symmetric single-particle spectrum

$$\cos\epsilon_k = \cos(\pi J)\cos(\pi g) + \sin(\pi J)\sin(\pi g)\cos k, \tag{S7}$$

with the eigenphases ϵ_k defined modulo 2π.

Bulk gap closings signal phase transitions and occur at $\epsilon = 0$ or $\epsilon = \pi$. Due to the invariance of $U_{F,0}$ under $g \to g+2$ and $J \to J+2$ as well as $g \to -g$ and $J \to -J$, we can restrict attention to $0 \le g, J \le 1$. For these parameters, the spectral gap $\Delta_0 = \pi(g - J)$ at zero energy is topological for $g < J$. Likewise, the spectral gap $\Delta_\pi = \pi(g + J - 1)$ at π is topological for $1 - g > J$. This gives the phase diagram in Fig. 1(b) of the main text.

In the fermion model with open boundary conditions, bulk gap closings indicate transitions between phases with and without localized Majorana modes at the ends. Majorana zero modes (MZMs) commute with the Floquet drive, while Majorana π modes (MPMs) anticommute

$$U_{F,0}^\dagger \gamma_0 U_{F,0} = \gamma_0, \qquad U_{F,0}^\dagger \gamma_\pi U_{F,0} = -\gamma_\pi. \tag{S8}$$

The Majorana operators are odd under fermion parity P. Using the transfer-matrix technique, one can construct explicit Majorana operators for semi-infinite chains [11]. In particular, one finds the localization lengths

$$\xi_{0,\pi} = -\frac{1}{\ln \lambda_{0,\pi}}, \quad \lambda_0 = \frac{\tan\frac{\pi g}{2}}{\tan\frac{\pi J}{2}}, \quad \lambda_\pi = \frac{\cot\frac{\pi g}{2}}{\tan\frac{\pi J}{2}} \tag{S9}$$

of the MZM (ξ_0) and MPM (ξ_π) modes. In a finite chain, Majorana hybridization leads to a splitting away from zero or π, which is exponentially small in the chain length, $\delta_{0,\pi} \propto e^{-N/\xi_{0,\pi}}$. One notices that the correlation lengths ξ_0 and ξ_π map onto each other under $g \leftrightarrow 1 - g$, explaining the symmetry of the Majorana splittings in Fig. 1(c) of the main text. We note that the Majorana splittings are smaller than the average many-body level spacing $2\pi/2^N$ provided that $\xi_{0,\pi} < 1/\ln 2$. For $J = 0.5$, this is true provided that $g > 0.71$.

II. RANDOM TRANSVERSE FIELDS

We find that the lognormal splitting distributions for a random transverse field as shown in Fig. 2(a),(b) are very well fit by

$$\overline{\ln\frac{\delta_{0,\pi}}{\Delta}} = -\frac{N}{\xi_{0,\pi}}, \qquad \mathrm{var}\ln\frac{\delta_{0,\pi}}{\Delta} = \frac{N}{\ell}. \tag{S10}$$

Here, $\xi_{0,\pi}$ is the Majorana localization length in Eq. (S9) and the mean free path ℓ can be accurately fit by

$$\ell = \frac{3}{\pi^2(dg)^2}\sin^2(\pi g) \tag{S11}$$

across both phases. (Here, dg denotes the width of the distribution of the random transverse field, see main text.) These results are closely analogous to results for the corresponding Hamiltonian problem [26].

III. STROBOSCOPIC FLOQUET PERTURBATION THEORY

We derive the perturbative expressions given in Eq. (3) in the main text. Guided by Hamiltonian perturbation theory, we expand eigenvalues and eigenstates of Floquet operators

$$e^{-i\lambda V} U_0 |n\rangle = e^{-iE_n} |n\rangle \tag{S12}$$

in powers of the perturbation V as counted by powers of λ. Inserting the expansions

$$E_n = E_{n,0} + \lambda E_{n,1} + \lambda^2 E_{n,2} + \dots, \qquad |n\rangle = |n_0\rangle + \lambda|n_1\rangle + \lambda^2|n_2\rangle + \dots \tag{S13}$$

in Eq. (S12), we find to quadratic order

$$\begin{aligned}&\Big(1 - i\lambda V - \frac{1}{2}\lambda^2 V^2 + \dots\Big) U_0 \big(|n_0\rangle + \lambda|n_1\rangle + \lambda^2|n_2\rangle \dots\big)\\ &\quad = e^{-iE_{n,0}}\Big(1 - i\lambda E_{n,1} - \lambda^2\big(iE_{n,2} + \frac{1}{2}E_{n,1}^2\big) + \dots\Big)\big(|n_0\rangle + \lambda|n_1\rangle + \lambda^2|n_2\rangle \dots\big).\end{aligned} \tag{S14}$$

We compare terms on both sides order by order in λ. At zeroth order, we recover

$$U_0 |n_0\rangle = e^{-iE_{n,0}} |n_0\rangle . \tag{S15}$$

At first order, we obtain (exploiting the orthogonality $\langle n_0|n_1\rangle = 0$) the first order shift

$$E_{n,1} = \langle n_0|V|n_0\rangle \tag{S16}$$

as well as the first-order correction of the eigenstate,

$$|n_1\rangle = ie^{-iE_{n,0}} \sum_{m\neq n} \frac{\langle m_0|V|n_0\rangle}{e^{-iE_{n,0}} - e^{-iE_{m,0}}} |m_0\rangle . \tag{S17}$$

The second-order correction to the eigenphases follows from

$$U_0 |n_2\rangle - iVU_0 |n_1\rangle - \frac{1}{2}V^2 U_0 |n_0\rangle = e^{-iE_{n,0}} \left\{ |n_2\rangle - iE_{n,1} |n_1\rangle - \left(iE_{n,2} + \frac{E_{n,1}^2}{2} \right) |n_0\rangle \right\} . \tag{S18}$$

Projecting this expression on $\langle n_0|$ and using $\langle n_0|n_2\rangle = 0$ as well as the lower-order results, this yields

$$E_{n,2} = \sum_{m\neq n} \frac{|\langle n_0|V|m_0\rangle|^2}{2\tan\frac{E_{n,0}-E_{m,0}}{2}} . \tag{S19}$$

This reduces to the standard expressions of Hamiltonian perturbation theory for a pair of close levels. However, it differs drastically from Hamiltonian perturbation theory when the difference between the unperturbed eigenphases is close to π, where the eigenphase denominator diverges.

IV. SPLITTINGS OF PAIRED MANY-BODY STATES

Here, we provide more details on Eqs. (4) and (5), which apply stroboscopic Floquet perturbation theory to the splittings of paired many-body eigenstates with $U_0 = U_g U_J$ and $e^{iV} = U_h$. In the absence of a random longitudinal field, the paired states differ in their occupations of the fermion mode constructed from the Majorana (zero or π) modes. Thus, the (even and odd) states differ in the corresponding Majorana parity,

$$(-i\gamma_L\gamma_R) |n^{\mathrm{e}}\rangle = |n^{\mathrm{e}}\rangle , \quad (-i\gamma_L\gamma_R) |n^{\mathrm{o}}\rangle = -|n^{\mathrm{o}}\rangle , \tag{S20}$$

with $\gamma_{L/R}$ denoting the Majorana operators at the left and right ends. Paired states convert into each other by application of the Majorana operators, e.g., $\gamma_L |n^{\mathrm{e}}\rangle = |n^{\mathrm{o}}\rangle$ and have identical occupations of all non-Majorana modes. The energies E_n^{e} and E_n^{o} of the paired states differ by

$$E_n^{\mathrm{o}} = E_n^{\mathrm{e}} - \delta_0 \quad \text{or} \quad E_n^{\mathrm{o}} = E_n^{\mathrm{e}} + \pi - \delta_\pi , \tag{S21}$$

in the case of MZM or MPM phases, respectively. The splittings $\delta_{0,\pi}$ are exponentially small in the length of the chain, $\delta_{0,\pi} \sim e^{-N/\xi_{0,\pi}}$ and identical for all pairs.

The random longitudinal field

$$V = \sum_j h_j Z_j , \tag{S22}$$

is odd under total fermion parity. It thus couples unperturbed states, which have different total parity P, but may have identical or different occupations of the Majorana mode.

A. Splittings of MZM modes

In discussing the perturbed splittings of MZM modes, we assume that the field is much smaller than the many-body level spacing. Then we can restrict attention to a pair of partner states. The coupling between the states is dominated by the effects of the fields h_1 and h_N acting on the spins at the ends of the chain. While the boundary spins Z_1 and Z_N are polarized by the random longitudinal field, the interior spins remain unpolarized due to the existence of mobile domain walls in generic eigenstates. Technically, this suppression arises from the string operators in Eq. (S1). Thus, we have

$$v = \langle n^{\mathrm{e}}|V|n^{\mathrm{o}}\rangle \approx \frac{\pi h_1}{2}\langle n^{\mathrm{e}}|Z_1|n^{\mathrm{o}}\rangle + \frac{\pi h_N}{2}\langle n^{\mathrm{e}}|Z_N|n^{\mathrm{o}}\rangle = \frac{\pi(h_1+h_N)\psi_M}{2} \tag{S23}$$

for the matrix elements entering the effective 2×2 Hamiltonian. Here, we used that $Z_1 = \psi_M\gamma_L + \dots$ and $Z_N = i\psi_M\gamma_R P + \dots$, where ψ_M is the Majorana wavefunction at the boundary sites, P denotes the fermion parity operator, and the ellipses stand for above-gap excitations. For uniformly distributed $h_1, h_N \in [-dh, dh]$, the matrix element v has a triangular distribution

$$p(v) = \frac{1}{v_0}\left(1 - \frac{|v|}{v_0}\right)\theta\left(v_0 - |v|\right). \tag{S24}$$

Here, $v_0 = 2\psi_M(\pi dh/2)$ is the maximal shift caused by the two boundary fields and $\theta(x)$ is the Heaviside function. The effective 2×2 Hamiltonian becomes

$$H_n^{\mathrm{eo}} = \frac{\delta_0}{2}\Big(|n^{\mathrm{e}}\rangle\langle n^{\mathrm{e}}| - |n^{\mathrm{o}}\rangle\langle n^{\mathrm{o}}|\Big) + v\Big(|n^{\mathrm{e}}\rangle\langle n^{\mathrm{o}}| + |n^{\mathrm{o}}\rangle\langle n^{\mathrm{e}}|\Big), \tag{S25}$$

where $\delta_0 > 0$ is the bare splitting. Note that the Hamiltonian takes the same form for all pairs n, so that the splitting remains uniform across the many-body spectrum (to leading order) and varies only between disorder realizations. The eigenenergies $E_\pm = \pm\frac{1}{2}\sqrt{\delta_0^2 + 4v^2}$, yield the perturbed splittings

$$\delta_0' = E_+ - E_- = \sqrt{\delta_0^2 + 4v^2}, \tag{S26}$$

which are larger than the bare splittings. Using $p(v)$, we arrive at

$$p\left(\delta_0'\right) = \frac{|\delta_0'|}{2v_0}\left(\frac{1}{\sqrt{\delta_0'^2 - \delta_0^2}} - \frac{1}{2v_0}\right)\theta\left(|\delta_0'| - \delta_0\right)\theta\left(\sqrt{\delta_0^2 + 4v_0^2} - |\delta_0'|\right). \tag{S27}$$

At $\delta_0' = \delta_0$ we find a square-root singularity, which – unlike the bulk of the distribution – is insensitive to the specific choice of distribution of the random fields.

B. Splittings of MPM modes

In the MPM phase, the Majorana modes do not induce degeneracies in the many-body spectrum of $U_{F,0}$. Hence, the random longitudinal field affects the spectrum only in second-order perturbation theory. We write

$$E_n^{\mathrm{o}} = E_n^{\mathrm{e}} + \pi - \delta_n. \tag{S28}$$

The random field shifts the splitting δ_n away from the bare splitting δ_π by an amount $\Delta\delta_n$,

$$\delta_n = \delta_\pi + \Delta\delta_n. \tag{S29}$$

In second-order perturbation theory, Eq. (S19), the shift becomes

$$\Delta\delta_n = \sum_m \left[\frac{|v_{nm}^{\mathrm{eo}}|^2}{2\tan\frac{E_n^{\mathrm{e}} - E_m^{\mathrm{o}}}{2}} - \frac{|v_{nm}^{\mathrm{oe}}|^2}{2\tan\frac{E_n^{\mathrm{o}} - E_m^{\mathrm{e}}}{2}}\right] + \sum_{m\neq n}\left[\frac{|v_{nm}^{\mathrm{ee}}|^2}{2\tan\frac{E_n^{\mathrm{e}} - E_m^{\mathrm{e}}}{2}} - \frac{|v_{nm}^{\mathrm{oo}}|^2}{2\tan\frac{E_n^{\mathrm{o}} - E_m^{\mathrm{o}}}{2}}\right] \tag{S30}$$

with matrix elements

$$v_{nm}^{ab} = \left\langle n^a\middle|V\middle|m^b\right\rangle, \quad a, b \in \{\mathrm{e}, \mathrm{o}\}. \tag{S31}$$

The first term contains processes which change the Majorana parity $-i\gamma_L\gamma_R$, in addition to the global fermion parity P. Consequently, the bulk parity defined as $Q = (-i\gamma_L\gamma_R)P$ remains invariant. The second term contains processes which leave the Majorana parity $-i\gamma_L\gamma_R$ unchanged, implying that Q changes.

In the MPM phase, the coupling within the pairs is negligible due to the divergence of the eigenphase denominator as the eigenphase difference approaches π. Thus, the effect of the perturbation is controlled by the coupling between different pairs. Since there are many such couplings of similar magnitude, it is plausible that their effect can be approximated in a self-consistent scheme. For this reason, we made the perturbative expression in Eq. (S30) self-consistent (in analogy with the self-consistent Born approximation) by retaining the exact eigenenergies $E_n^{e/o}$ in the denominators.

Using that the splittings δ_n are small, we expand the right-hand side of Eq. (S30) for $\Delta\delta_n$ to linear order in the δ_n. This yields

$$\delta_n - \delta_\pi = -\sum_m \Sigma_{nm}\delta_m + \Lambda_n, \tag{S32}$$

with

$$\Sigma_{nm} = \left(\sum_l \frac{|v_{nl}^{\rm eo}|^2}{4\cos^2 \frac{E_n^{\rm e} - E_l^{\rm e}}{2}} - \sum_{l\neq n} \frac{|v_{nm}^{\rm oo}|^2}{4\sin^2 \frac{E_n^{\rm e} - E_l^{\rm e}}{2}} \right) \delta_{nm} + \frac{|v_{nm}^{\rm eo}|^2}{4\cos^2 \frac{E_n^{\rm e} - E_m^{\rm e}}{2}} + \frac{|v_{nm}^{\rm oo}|^2(1-\delta_{nm})}{4\sin^2 \frac{E_n^{\rm e} - E_m^{\rm e}}{2}} \tag{S33}$$

and

$$\Lambda_n = \sum_{m\neq n} \frac{|v_{nm}^{\rm ee}|^2 - |v_{nm}^{\rm oo}|^2}{2\tan \frac{E_n^{\rm e} - E_m^{\rm e}}{2}}. \tag{S34}$$

We can then express the vector δ'_π of splittings δ_n in matrix notation as

$$\delta'_\pi = \frac{1}{1+\Sigma}(\delta_\pi + \Lambda) \tag{S35}$$

Here, δ_π should also be interpreted as a vector, with all entries equal to the bare splitting δ_π.

To derive these expressions, it is convenient to decompose the matrix elements into symmetric and antisymmetric matrix contributions,

$$\Delta\delta_n = \Delta\delta_n^{\rm eo/oe} + \Delta\delta_n^{\rm ee/oo} \tag{S36}$$

with

$$\begin{aligned} \Delta\delta_n^{\rm eo/oe} = \sum_m \frac{|v_{nm}^{\rm eo}|^2 + |v_{nm}^{\rm oe}|^2}{2} \left[\frac{1}{2\tan \frac{E_n^{\rm e} - E_m^{\rm o}}{2}} - \frac{1}{2\tan \frac{E_n^{\rm o} - E_m^{\rm e}}{2}} \right] \\ + \sum_m \frac{|v_{nm}^{\rm eo}|^2 - |v_{nm}^{\rm oe}|^2}{2} \left[\frac{1}{2\tan \frac{E_n^{\rm e} - E_m^{\rm o}}{2}} + \frac{1}{2\tan \frac{E_n^{\rm o} - E_m^{\rm e}}{2}} \right] \end{aligned} \tag{S37}$$

and

$$\begin{aligned} \Delta\delta_n^{\rm ee/oo} = \sum_{m\neq n} \frac{|v_{nm}^{\rm ee}|^2 + |v_{nm}^{\rm oo}|^2}{2} \left[\frac{1}{2\tan \frac{E_n^{\rm e} - E_m^{\rm e}}{2}} - \frac{1}{2\tan \frac{E_n^{\rm o} - E_m^{\rm o}}{2}} \right] \\ + \sum_{m\neq n} \frac{|v_{nm}^{\rm ee}|^2 - |v_{nm}^{\rm oo}|^2}{2} \left[\frac{1}{2\tan \frac{E_n^{\rm e} - E_m^{\rm e}}{2}} + \frac{1}{2\tan \frac{E_n^{\rm o} - E_m^{\rm o}}{2}} \right]. \end{aligned} \tag{S38}$$

The eigenphase differences in the denominators can be written as

$$E_n^{\rm o} - E_m^{\rm o} = E_n^{\rm e} - E_m^{\rm e} - \delta_n + \delta_m, \qquad E_n^{\rm e} - E_m^{\rm o} = E_n^{\rm e} - E_m^{\rm e} - \pi + \delta_m. \tag{S39}$$

Using the expansions

$$\frac{1}{\tan\left(\frac{x-\pi+\delta_m}{2}\right)} - \frac{1}{\tan\left(\frac{x+\pi-\delta_n}{2}\right)} \simeq -\frac{(\delta_n+\delta_m)}{2\cos^2(\frac{x}{2})}, \tag{S40}$$

$$\frac{1}{\tan\left(\frac{x-\pi+\delta_m}{2}\right)} + \frac{1}{\tan\left(\frac{x+\pi-\delta_n}{2}\right)} \simeq \frac{2}{\tan\left(\frac{x-\pi}{2}\right)} - \frac{(\delta_n-\delta_m)}{2\cos^2(\frac{x}{2})} \tag{S41}$$

to linear order, one finds

$$\Delta\delta_n^{\mathrm{eo/oe}} = \sum_m \frac{|v_{nm}^{\mathrm{eo}}|^2 - |v_{nm}^{\mathrm{oe}}|^2}{2\tan\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}-\pi}{2}} - \sum_m \frac{(\delta_n+\delta_m)\left(|v_{nm}^{\mathrm{eo}}|^2+|v_{nm}^{\mathrm{oe}}|^2\right)}{8\cos^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} - \sum_m \frac{(\delta_n-\delta_m)\left(|v_{nm}^{\mathrm{eo}}|^2-|v_{nm}^{\mathrm{oe}}|^2\right)}{8\cos^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} \tag{S42}$$

as well as

$$\Delta\delta_n^{\mathrm{ee/oo}} = \sum_{m\neq n} \frac{|v_{nm}^{\mathrm{ee}}|^2 - |v_{nm}^{\mathrm{oo}}|^2}{2\tan\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} - \sum_{m\neq n} \frac{(\delta_n-\delta_m)\left(|v_{nm}^{\mathrm{ee}}|^2+|v_{nm}^{\mathrm{oo}}|^2\right)}{8\sin^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} + \sum_{m\neq n} \frac{(\delta_n-\delta_m)\left(|v_{nm}^{\mathrm{ee}}|^2-|v_{nm}^{\mathrm{oo}}|^2\right)}{8\sin^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}}. \tag{S43}$$

Unlike in the MZM case, the splittings in the MPM phase vary across the many-body spectrum, so that terms involving $\delta_n - \delta_m$ do not vanish. Collecting terms and using $|v_{nm}^{\mathrm{eo}}| = |v_{nm}^{\mathrm{oe}}|$, we find

$$\delta_n - \delta_\pi = -\sum_m \frac{(\delta_n+\delta_m)|v_{nm}^{\mathrm{eo}}|^2}{4\cos^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} - \sum_{m\neq n} \frac{(\delta_n-\delta_m)|v_{nm}^{\mathrm{oo}}|^2}{4\sin^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} + \sum_{m\neq n} \frac{|v_{nm}^{\mathrm{ee}}|^2 - |v_{nm}^{\mathrm{oo}}|^2}{2\tan\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}} \tag{S44}$$

and thus Eqs. (S32), (S33) and (S34). Note that the first term on the right-hand side involves matrix elements between states of different Majorana parities, while the second and third terms involve matrix elements between states of equal Majorana parities.

C. Implications

We find that terms involving matrix elements between states of equal Majorana parities can be neglected for $N < N^*(g)$. In this regime, the eigenphase differences in the denominators of the corresponding terms in Eqs. (S33) and (S34) remain large compared to the many-body level spacing. In fact, coupled states must have different bulk parities. For g close to unity, the eigenphase regions supporting states with different bulk parities do not overlap, so that the denominators remain large. This is a consequence of the small bandwidth $\propto (1-g)$ of the singe-particle excitations about the phase $\pm\pi/2$, see Fig. 1(a) of the main text. For zero single-particle bandwidth, the eigenphases of states with different bulk parities differ by an odd multiple of $\pi/2$. A finite single-particle bandwidth changes the many-body eigenphases by an amount of order $\propto N^{1/2}(1-g)$ (originating from summing over N terms with random signs). As long as this change remains small compared to unity, there are no small denominators in the expression for Λ_n. Thus, we conclude that $N^* \propto 1/(1-g)^2$.

When $N < N^*$, Eq. (S44) simplifies to

$$\delta_n - \delta_\pi = -\sum_m \frac{(\delta_n+\delta_m)|v_{nm}^{\mathrm{eo}}|^2}{4\cos^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}}, \tag{S45}$$

which corresponds to Eq. (5) of the main text. In the perturbative limit (bimodal regime), the δ_n remain close to δ_π and we find that the random longitudinal field reduces the splittings δ_n below δ_π. More generally, we can rewrite Eq. (S45) as

$$\sum_m \sigma_{nm}\delta_n = \delta_\pi \tag{S46}$$

with

$$\sigma_{nm} = \left(1 + \sum_l \frac{|v_{nm}^{\mathrm{eo}}|^2}{4\cos^2\frac{E_n^{\mathrm{e}}-E_l^{\mathrm{e}}}{2}}\right)\delta_{nm} + \frac{|v_{nm}^{\mathrm{eo}}|^2}{4\cos^2\frac{E_n^{\mathrm{e}}-E_m^{\mathrm{e}}}{2}}. \tag{S47}$$

We then find

$$\delta_n = \sum_m (\sigma^{-1})_{nm} \delta_\pi. \tag{S48}$$

When the perturbation becomes sufficiently large (Gaussian regime), σ is a "random" matrix far from the unit matrix with exclusively nonnegative entries. Then, one expects the matrix elements $(\sigma^{-1})_{nm}$ of the inverse matrix to have "random" signs. The approximately Gaussian distribution which we find numerically [see, e.g., the curve for $dh = 0.1$ in Fig. 2(d)], can then be roughly interpreted as a consequence of the central limit theorem. We note that the matrix elements of σ have a rather broad distribution as a consequence of near degeneracies of the eigenphase denominators. At the same time, the distribution of the matrix elements of σ^{-1} do not have long tails. However, the matrix elements of σ^{-1} are still rather structured. As a result, the central-limit argument is less accurate for a particular disorder realization, but applies with reasonable accuracy after averaging over disorder configurations. The transition between the bimodal and Gaussian regimes occurs when the σ_{nm} become of order unity. We find numerically that σ_{nm} is of order $dh^2 \exp(N/\zeta)$ with $\zeta \approx 1.65$, which depends only weakly on g. Thus, the transition occurs at $N^{**} \sim \ln\bigl(1/dh^2\bigr)$.

Conversely, for $N > N^*$, all terms in Eq. (S44) have to be retained when computing the splitting δ_n. In this regime, terms involving matrix elements between states of equal Majorana parities can be viewed as a sum over many terms of the form $1/x$ (with x representing the eigenphase denominators), where x has a distribution that remains nonzero for $x = 0$. Assuming that the terms are statistically independent, one then obtains a Lorentzian distribution for Λ_n. This follows since the distribution of $1/x$ has a long tail, with the Lorentzian being the relevant Levy stable distribution [27]. While we observe deviations from Lorentzian behavior for Λ_n, we find that the distribution of δ_n can be well fit by a Lorentzian. Possibly, the distribution of δ_n is less influenced by the lognormal distribution of matrix elements of the random field, as the matrix elements appear both in the numerator (via Λ_n) and the denominator (via Σ_{nm}).