Problem Set 3
Quantum Field Theory and Many Body Physics (SoSe2018)

Due: Friday, May 11, 2018 before the beginning of the class

In this problem set, we first consider the quantum mechanics of the harmonic chain using a traditional operator approach based on canonical quantization. Then, we discuss another (classical) field theory which you already know, namely classical electrodynamics, and discuss its Lagrangian and Hamiltonian formulations. There are some subtleties that result from the gauge freedom of the theory.

Problem 1: Canonical quantization of the harmonic chain (5+5+5+10 points)

In this problem, we want to consider the quantum mechanics of the harmonic chain, following the general prescription of canonical quantization. We start with the Hamiltonian

$$H = \int \mathrm{d}r \left\{ \frac{1}{2\rho}\pi^2(r) + \frac{1}{2}\rho c^2 (\partial_x \phi(r))^2 \right\} \tag{1}$$

of the harmonic chain. We know that the fields $\phi(x)$ and $\pi(x)$ are canonically conjugate so that they obey the fundamental Poisson brackets

$$\{\phi(x), \phi(x')\} = \{\pi(x), \pi(x')\} = 0 \quad ; \quad \{\pi(x), \phi(x')\} = \delta(x - x'). \tag{2}$$

Canonical quantization tells us to interpret the classical variables as quantum operators, with their commutators being equal to $-i\hbar$ times the corresponding Poisson bracket, i.e.,

$$[\hat{\phi}(x), \hat{\phi}(x')] = [\hat{\pi}(x), \hat{\pi}(x')] = 0 \quad ; \quad [\hat{\pi}(x), \hat{\phi}(x')] = \frac{\hbar}{i}\delta(x - x'). \tag{3}$$

Here, we denote the quantum field operators with a hat in order to distinguish them from the coresponding classical fields. We will now determine the eigenstates and the spectrum of the resulting quantum theory.

In conventional quantum mechanics with canonical operators $\hat{x}$ and $\hat{p}$, we would now find a representation of the canonical commutation relations, say in position representation,

$$\langle x|\hat{x}|x'\rangle = x\delta(x - x') \quad ; \quad \langle x|\hat{p}|x'\rangle = \frac{\hbar}{i}\delta(x - x'), \tag{4}$$

which turns the eigenvalue problem for the Hamiltonian into a differential equation, i.e., the Schrodinger equation. This approach would lead to a functional differential equation for our field theory which is not very useful. Instead, it is much more useful to use an approach which follows the ideas of Dirac's solution of the quantum harmonic oscillator by raising and lowering operators.

(a) Because of translational invariance, it is useful to Fourier transform the fields $\hat{\phi}(x)$ and $\hat{\pi}(x)$,

$$\hat{\phi}(x) = \frac{1}{L}\sum_q \hat{\phi}_q e^{iqx} \quad ; \quad \hat{\pi}(x) = \frac{1}{L}\sum_q \hat{\pi}_q e^{iqx} \tag{5}$$

with the inverse transformation

$$\hat{\phi}_q = \int \mathrm{d}x\, \hat{\phi}(x) e^{-iqx} \quad ; \quad \hat{\pi}_q = \int \mathrm{d}x\, \hat{\pi}(x) e^{-iqx}. \tag{6}$$

Now, establish that the Fourier transforms $\hat{\phi}_q$ and $\hat{\pi}_q$ satisfy the commutation relations

$$[\hat{\pi}_q, \hat{\phi}_{q'}] = -i\hbar L \delta_{q,-q'} \quad ; \quad [\hat{\phi}_q, \hat{\phi}_{q'}] = [\hat{\pi}_q, \hat{\pi}_{q'}] = 0 \tag{7}$$

and express the Hamiltonian in terms of these Fourier components $\hat{\phi}_q$ and $\hat{\pi}_q$,

$$H = \frac{1}{L}\sum_q \left[\frac{1}{2\rho}\hat{\pi}_q\hat{\pi}_{-q} + \frac{1}{2}\rho c^2 q^2 \hat{\phi}_q\hat{\phi}_{-q}\right]. \tag{8}$$

It is important to note that unlike the fields $\hat{\phi}(x)$ and $\hat{\pi}(x)$ as a function of position, the Fourier transforms are no longer hermitian operators. Instead, we have $(\hat{\pi}_q)^\dagger = \hat{\pi}_{-q}$ and $(\hat{\phi}_q)^\dagger = \hat{\phi}_{-q}$.

(b) Now introduce the operators

$$\hat{a}_q = \sqrt{\frac{\rho c|q|}{2L\hbar}}\left(\hat{\phi}_q + \frac{i}{\rho c|q|}\hat{\pi}_q\right) \quad \text{with} \quad \hat{a}_q^\dagger = \sqrt{\frac{\rho c|q|}{2L\hbar}}\left(\hat{\phi}_{-q} - \frac{i}{\rho c|q|}\hat{\pi}_{-q}\right). \tag{9}$$

To understand the idea behind this definition, you may want to go back and remind yourself of the definition of raising and lowering operators for the quantum harmonic oscillator.

Invert this and express the original fields ϕ_q and π_q in terms of these operators. You should find

$$\hat{\phi}_q = \sqrt{\frac{L\hbar}{2\rho c|q|}}(\hat{a}_q + \hat{a}_{-q}^\dagger) \quad ; \quad \hat{\pi}_q = \sqrt{\frac{L\hbar\rho c|q|}{2}}(\hat{a}_q - \hat{a}_{-q}^\dagger). \tag{10}$$

Then, compute the commutation relations of the new operators,

$$[\hat{a}_q, \hat{a}_{q'}] = [\hat{a}_q^\dagger, \hat{a}_{q'}^\dagger] = 0 \quad ; \quad [\hat{a}_q, \hat{a}_{q'}^\dagger] = \delta_{qq'}. \tag{11}$$

and express the Hamiltonian in terms of these operators,

$$H = \sum_q \hbar c|q|\left(\hat{a}_q^\dagger\hat{a}_q + \frac{1}{2}\right). \tag{12}$$

(c) In (b), you showed that the Hamiltonian reduces to a set of independent harmonic oscillator Hamiltonians, one for each q. Explain how to derive the spectrum and the eigenstates using the creation and annihilation operators $a_q^\dagger$ and a_q.

You should find a particle-like spectrum with linear dispersion relation. These particles are called phonons. Each mode can be occupied by arbitrarily many particles and the energy of many-particle states is just the sum of the energies of the individual particles, i.e., phonons in the harmonic chain are non-interacting bosons. The continuum limit of the harmonic chain is the simplest example of a free bosonic quantum field theory.

(d) Now use these results to find the result

$$\left\langle \hat{\phi}_q\hat{\phi}_{-q}\right\rangle = \frac{\hbar L}{2\rho c\,|q|}\coth\left(\frac{\beta\hbar c\,|q|}{2}\right). \tag{13}$$

Remember that the quantum expectation value of an operator $\hat{O}$ at finite temperature T is given by

$$\langle\hat{O}\rangle = \frac{\mathrm{tr}\rho\hat{O}}{\mathrm{tr}\hat{\rho}}, \tag{14}$$

where the density operator is $\rho = \exp(-\beta H)$ and $\beta = 1/k_B T$. Check that this reduces to the result for the classical harmonic chain when $T \gg \hbar c|q|$ (classical limit). Then give the result in the quantum limit $T \ll \hbar c\,|q|$ and discuss the sum over q to compute $\langle[\phi(x) - \phi(x')]^2\rangle$. How does the stability of a crystal against quantum fluctuations depend on dimensionality?

Problem 2: Maxwell action (5+5+5+5+5+5+5 points)

Hamilton's principle is not restricted to mechanical systems, but can be extended to field theories. The field theory can be defined by a Lagrangian (or, equivalently, the corresponding action), which, through Hamilton's principle, leads to the correct equations of motion of the field when varying the field configurations.

In this problem, we discuss the Lagrange formalism for the electromagnetic field whose equations of motion are Maxwell's equations (setting $c = 1$),

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \quad ; \quad \nabla \cdot \mathbf{B} = 0 \quad ; \quad \nabla \cdot \mathbf{E} = 4\pi\rho\,, \quad ; \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j}\,. \tag{15}$$

The first two equations are referred to as homogeneous Maxwell equations.

(a) The homogeneous Maxwell equations are solved by the introducing the scalar and vector potentials, $\phi(x,t)$ and $\mathbf{A}(x,t)$. Write the solution of Eq. (15) explicitly in terms of ϕ, $\mathbf{A}$.

(b) While the six components of the $\mathbf{E}$ and $\mathbf{B}$ fields are not independent and thus not suitable as generalized coordinates, the potentials are. For this reason, write the Lagrangian of the magnetic field

$$L = \int \mathrm{d}\mathbf{r}\mathcal{L} = \int \mathrm{d}\mathbf{r} \left[\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\phi + \mathbf{j}\mathbf{A} \right] \tag{16}$$

in terms of ϕ and $\mathbf{A}$, using the expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and $\mathbf{A}$. ($\mathcal{L}$ is referred to as Lagrangian density.) Then, use Hamilton's principle, applied to the action $S = \int dt\, L$, to derive the inhomogeneous Maxwell equations, i.e., Eq. (15).

(c) Show that the Lagrangian for the electromagnetic field can equivalently be written as

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \tag{17}$$

in terms of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the four-potential $A^\mu = (\phi, \mathbf{A})$, and the four-current $j^\mu = (\rho, \mathbf{j})$.

This expression for the Lagrangian can be deduced (up to the numerical coefficients) by demanding that (i) the action is a Lorentz scalar, (ii) the free-field term is quadratic in the velocity, i.e., in $\partial^\beta A^\alpha$, and (iii) the coupling to the source j^μ is linear in agreement with the Lagrangian of a particle in the electromagnetic field.

(d) Now consider the Lagrangian of a charged particle in an electromagnetic field, described by a scalar potential $\phi(\mathbf{r},t)$ and a vector potential $\mathbf{A}(\mathbf{r},t)$,

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\phi(\mathbf{r},t) + e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r},t) \tag{18}$$

Show that the corresponding Euler-Lagrange equation takes the form

$$m\ddot{\mathbf{r}} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \tag{19}$$

in terms of the electric field $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

(e) Use Hamilton's principle to show that the Euler-Lagrange equations (say, for a single particle) remain invariant when adding a total time derivative to the Lagrangian,

$$L \to L + \frac{\mathrm{d}}{\mathrm{d}t}\Lambda(\mathbf{r},t). \tag{20}$$

(f) Show directly from the Lagrangian that the result formulated in (b) implies that the equations of motion are invariant under the gauge transformation

$$\phi(\mathbf{r},t) \to \phi(\mathbf{r},t) - \frac{\partial \Lambda(\mathbf{r},t)}{\partial t} \quad ; \quad \mathbf{A}(\mathbf{r},t) \to \mathbf{A}(\mathbf{r},t) + \nabla\Lambda(\mathbf{r},t) \tag{21}$$

of the potentials. In addition check explicitly that this leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged.

(g) Derive the corresponding Hamiltonian $H(\mathbf{p},\mathbf{r})$. Is the canonical momentum $\mathbf{p}$ gauge invariant? How about the kinematic momentum $\boldsymbol{\pi} = m\dot{\mathbf{r}}$?

Problem 3: Hamiltonian formulation for the electromagnetic field (5+10+10 points)

Next we discuss the Hamiltonian formulation of electrodynamics, starting with the Langrangian

$$L = \int \mathrm{d}\mathbf{r}\mathcal{L} = \int \mathrm{d}\mathbf{r} \left[\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\phi + \mathbf{j}\mathbf{A}\right]. \tag{22}$$

This is a necessary step in the canonical quantization of electromagnetism. In this case, quantization is not really straight-forward because of the gauge freedom of the theory.

(a) Compute the canonically conjugate momentum fields to the potentials ϕ and $\mathbf{A}$. Show that $\pi_{\mathbf{A}} = -\mathbf{E}/4\pi$ is canonically conjugate to $\mathbf{A}$ and that the canonically conjugate π_ϕ of ϕ vanishes. Use your result to derive the Hamiltonian

$$H = \int \mathrm{d}\mathbf{r}\mathcal{H} = \int \mathrm{d}\mathbf{r} \left[\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \frac{1}{4\pi}\mathbf{E}\cdot\nabla\phi + \rho\phi - \mathbf{j}\mathbf{A}\right]. \tag{23}$$

Note that this is essentially written in terms of the fields and the momenta as it does not contain any of the velocities (time derivatives of the potentials) and $\pi_{\mathbf{A}}$ is just proportional to $\mathbf{E}$.

(b) First consider the scalar potential ϕ. As the canonically conjugate momentum vanishes, we can obtain the corresponding Hamilton equation by minimizing the time integral of the Hamiltonian over ϕ. (Remember that Hamilton's equations follow from Hamilton's principle written in phase space variables.) Show that this reproduces the Maxwell equation

$$\nabla\cdot\mathbf{E} = 4\pi\rho. \tag{24}$$

This is not a dynamical equation of the theory but rather a constraint that must be satisfied for all times.

(c) Next consider Hamilton's equations for $\mathbf{A}$ and $\pi_{\mathbf{A}}$ and relate your results to Maxwell's equations.

(d) Now let's choose the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$. By choosing this gauge, we effectively fix one component of $\mathbf{A}$ so that $\mathbf{A}$ has only two independent components. Writing $\mathbf{A}$ in terms of its longitudinal and its transverse components, $\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp$ (defined[1] by $\nabla\times\mathbf{A}_\parallel = 0$ and $\nabla\cdot\mathbf{A}_\perp = 0$), we see that the gauge choice sets the parallel component to zero while the transverse component remains finite.

Since $\mathbf{A}$ has only two independent components, so should the cononically conjugate momentum. Thus, we also decompose $\mathbf{E}$ into its longitudinal and transverse parts, $\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp$ so that $\mathbf{E}_\parallel = -\nabla\phi$ and $\mathbf{E}_\perp = -\partial\mathbf{A}/\partial t$. Thus, it is the transverse part of $\mathbf{E}$ which is canonically conjugate to $\mathbf{A}$.

To discuss the longitudinal part, show that with our gauge choice, the constraint becomes $-\nabla^2\phi = 4\pi\rho$ and can be solved as

$$\phi(\mathbf{r},t) = \int \mathrm{d}\mathbf{r}' \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \tag{25}$$

Now, collect the terms in the Hamiltonian which depend on $\mathbf{E}_\parallel$ and express them using the constraint. Show that this yields the Hamiltonian

$$H = \int \mathrm{d}\mathbf{r} \left[\frac{\mathbf{E}_\perp^2 + \mathbf{B}^2}{8\pi} - \mathbf{j}\mathbf{A}\right] + \frac{1}{2}\int \mathrm{d}\mathbf{r}\int \mathrm{d}\mathbf{r}' \frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{26}$$

This form of the Hamiltonian makes the Coulomb interaction explicit and can be used as a starting point for quantizing the electromagnetic field.

[1] A simple way to think about this is in Fourier space, where $\mathbf{A}_\parallel$ is parallel to and $\mathbf{A}_\perp$ perpendicular to $\mathbf{q}$.