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THE FORM FACTOR PROGRAM: A REVIEW AND NEW RESULTS, THE NESTED $SU(N)$ OFF-SHELL BETHE ANSATZ AND THE $1/N$ EXPANSION

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The purpose of the "bootstrap program" for integrable quantum field theories in 1+1 dimensions is to construct a model explicitly in terms of its Wightman functions. We illustrate this program here mainly in terms of the $SU(N)$ Gross–Neveu model. We construct the nested off-shell Bethe ansatz for an $SU(N)$ factoring S-matrix and consider the problem of how to sum over intermediate states in the short-distance limit of the two-point Wightman function for the sinh-Gordon model.

Keywords: integrable quantum field theory, form factor

1. Introduction

The bootstrap program to formulate particle physics in terms of the scattering data, i.e., in terms of the S-matrix, goes back to Heisenberg [1] and Chew [2]. This approach works very well for integrable quantum field theories in 1+1 dimensions [3]–[8]. The program does not start with any classical Lagrangian. Instead, it classifies integrable quantum field theory models and in addition provides their explicit exact solutions in terms of all Wightman functions. We achieve contact with the classical models only at the end, when we compare our exact results with Feynman graph (or other) expansions that are usually based on Lagrangians.

One of us (M. K.) and others formulated the on-shell program in [4], i.e., the exact determination of the scattering matrix using the Yang–Baxter equations. The concept of generalized form factors was introduced in [7], where consistency equations were formulated that are expected to be satisfied by these quantities. Thereafter, this approach was developed further and studied in the context of several explicit models in [9]. Here, we apply the form factor program to an $SU(N)$-invariant S-matrix (see [10], [11]). We must apply the nested "off-shell"[1] Bethe ansatz to obtain the vector part of the form factors. This gives the missing link in Smirnov's [9] discussion of $SU(N)$ form factors, where the vectors were given by an "indirect definition" characterized by necessary properties but not provided explicitly. We compare the $1/N$ expansion for the chiral $SU(N)$ Gross–Neveu model [12] with our exact results for the form factors. This model is interesting in particular because the particles are anyons and the $1/N$ expansion is problematic [13]–[15] and moreover exhibits the asymptotic freedom. The Wightman functions are finally obtained by integrating and summing over intermediate states. The explicit evaluation of all these integrals and sums remains an open challenge for almost all models except the Ising model.

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[1] "Off-shell" in the context of the Bethe ansatz means that the spectral parameters in the algebraic Bethe ansatz state are not fixed by the Bethe ansatz equations in order to obtain an eigenstate of a Hamiltonian, but they are integrated over.

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2. The bootstrap program

The bootstrap program for integrable quantum field theories in 1+1 dimensions provides the solution of a model in terms of all its Wightman functions. The result is obtained in three steps:

1. The S-matrix is calculated using general properties such as unitarity and crossing, the Yang–Baxter equations (which are a consequence of integrability), and the additional assumption of "maximum analyticity." This means that the two-particle S-matrix is an analytic function in the physical plane (of the Mandelstam variable $(p_1+p_2)^2$) and has only poles of physical origin there. The only dependence on the model is the assumption that there is a particle spectrum with an underlining symmetry. All S-matrices with the given properties are classified.

2. Generalized form factors, which are matrix elements of the local operators

$$^{\mathrm{out}}\big\langle \theta'_m, \dots, \theta'_1 \big| \mathcal{O}(x) |\theta_1, \dots, \theta_n\big\rangle^{\mathrm{in}},$$

   are calculated using the S-matrix. More precisely, equations listed in *a*–*e* in Sec. 3 are solved.

3. The Wightman functions are obtained by inserting a complete set of intermediate states. In particular, the two-point function for a Hermitian operator $\mathcal{O}(x)$ is given by

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{4\pi} \cdots \big|\langle 0|\mathcal{O}(0)|\theta_1, \dots, \theta_n\big\rangle^{\mathrm{in}}\big|^2 \exp\Big(-ix \sum p_i\Big).$$

   Up to now, a direct proof that these sums converge exists only for the scaling Ising model [8], [16]–[19].

It was recently shown in [20] that models with factoring S-matrices exist in the framework of algebraic quantum field theory.

Integrability. Integrability in (quantum) field theories means that there exist infinitely many local conservation laws

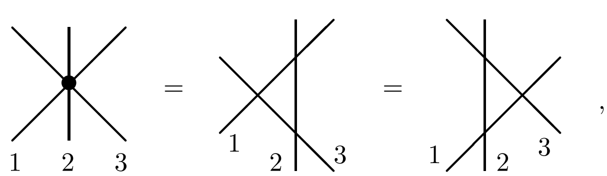
$$\partial_\mu J_L^\mu(t,x) = 0, \quad L = \pm 1, \pm 3, \dots .$$

A consequence of such conservation laws in 1+1 dimensions is that there is no particle production and the n-particle S-matrix is a product of two-particle S-matrices:

$$S^{(n)}(p_1, \dots, p_n) = \prod_{i<j} S_{ij}(p_i, p_j).$$

If backward scattering occurs, then the two-particle S-matrices do not commute, and their order must be specified. In particular, there are two possibilities for the three-particle S-matrix,

$$S^{(3)} = S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12},$$



which yield the Yang–Baxter equation.

The *two-particle S-matrix* has the form $S_{\alpha\beta}^{\beta'\alpha'}(\theta_{12})$, where α, β, etc., denote the type of the particles and the rapidity difference $\theta_{12} = \theta_1 - \theta_2$ is defined by $p_i = m_i(\cosh\theta_i, \sinh\theta_i)$. We also use the short notation $S_{12}(\theta_{12})$. It satisfies the unitarity and crossing relations

$$S_{21}(\theta_{21})S_{12}(\theta_{12}) = 1,$$

$$S_{12}(\theta_1 - \theta_2) = \mathbf{C}^{2\bar{2}} S_{\bar{2}1}(\theta_2 + i\pi - \theta_1)\mathbf{C}_{\bar{2}2} = \mathbf{C}^{1\bar{1}} S_{2\bar{1}}\big(\theta_2 - (\theta_1 - i\pi)\big)\mathbf{C}^{\bar{1}1},$$

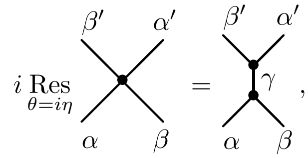
where $\mathbf{C}^{1\bar{1}}$ and $\mathbf{C}_{1\bar{1}}$ are charge conjugation matrices.

Bound states. Let γ be a bound state of particles α and β with the mass

$$m_\gamma = \sqrt{m_\alpha^2 + m_\beta^2 + 2m_\alpha m_\beta \cos\eta}, \quad 0 < \eta < \pi.$$

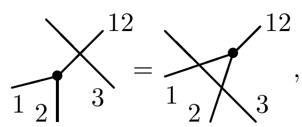
Then the two-particle S-matrix has a pole such that

$$i \operatorname*{Res}_{\theta=i\eta} S_{\alpha\beta}^{\beta'\alpha'}(\theta) = \Gamma_\gamma^{\beta'\alpha'} \Gamma_{\alpha\beta}^\gamma, \tag{1}$$



where η is called the *fusion angle* and $\Gamma_{\alpha\beta}^\gamma$ is the "bound-state intertwiner" [21], [22]. The *bound-state S-matrix*, which is the scattering matrix of a bound state (12) with a particle 3, is obtained by the *bootstrap equation* [21]

$$S_{(12)3}(\theta_{(12)3})\Gamma_{12}^{(12)} = \Gamma_{12}^{(12)} S_{13}(\theta_{13}) S_{23}(\theta_{23}),$$



where we use the usual short notation for matrices acting in the spaces corresponding to the particles 1, 2, 3, and (12).

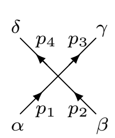
An example of an integrable model in 1+1 dimensions is the $SU(N)$ Gross–Neveu model [12] described by the Lagrangian

$$\mathcal{L} = \bar{\psi} i\gamma\partial\psi + \frac{g^2}{2}\big((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2\big),$$

where the Fermi fields form an $SU(N)$ multiplet. Other integrable quantum field theories are the sine-Gordon, the Toda, the scaling $Z(N)$ Ising, the nonlinear σ, the $O(N)$ Gross–Neveu models, etc.

The $SU(N)$ S-matrix. All solutions of an $U(N)$-invariant S-matrix satisfying unitarity, crossing, and the Yang–Baxter equation were obtained in [23]. Following [24], [25], we adopt the standpoint that in the $SU(N)$ Gross–Neveu model, the antiparticles are bound states of $N-1$ particles. This implies that there is no particle–antiparticle backward scattering and that the $SU(N)$ S-matrix should be given by solution II in [23]. The scattering of the fundamental particles that form a multiplet corresponding to the vector representation of $SU(N)$ is (see [14], [15], [26] and for $N = 2$ [27])

$$S_{\alpha\beta}^{\delta\gamma}(\theta) =$$



$$= \delta_{\alpha\gamma}\delta_{\beta\delta} b(\theta) + \delta_{\alpha\delta}\delta_{\beta\gamma} c(\theta), \tag{2}$$

where because of the Yang–Baxter equation, the relation $c(\theta) = -2\pi i b(\theta)/(N\theta)$ holds and the highest-weight amplitude is given as

$$a(\theta) = b(\theta) + c(\theta) = -\frac{\Gamma(1-\theta/(2\pi i))\Gamma(1-1/N+\theta/(2\pi i))}{\Gamma(1+\theta/(2\pi i))\Gamma(1-1/N-\theta/(2\pi i))}. \tag{3}$$

There is a bound-state pole at $\theta = i\eta = 2\pi i/N$ in the antisymmetric tensor sector, which agrees with the results in [24] that the bound state of $N-1$ particles is to be identified with the antiparticle. Similarly to the scaling $Z(N)$ Ising and $A(N-1)$ Toda models [28], [29], this is to be used to construct form factors in the $SU(N)$ model [10].

3. Form factors

For a local operator $\mathcal{O}(x)$, the generalized form factors [7] are defined as

$$F^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1,\dots,\theta_n) = \langle 0|\mathcal{O}(0)|p_1,\dots,p_n\rangle^{\text{in}}_{\alpha_1\dots\alpha_n} \tag{4}$$

for $\theta_1 > \dots > \theta_n$. For other orders of the rapidities, they are defined by analytic continuation. The index α_i denotes the type of the particle with momentum p_i. We also use the short notation $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ or $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$.

For the $SU(N)$ Gross–Neveu model, α denotes the types of particles belonging to all fundamental representations of $SU(N)$ with dimensions $\binom{N}{r}$, $r = 1, 2, \dots, N-1$. In most of the formulas, we restrict $\alpha = 1, 2, \dots, N$ to the vector representation multiplet. As with the S-matrix, maximum analyticity for generalized form factors means that they are meromorphic and all poles in the physical strips $0 \le \operatorname{Im}\theta_{ij} \le \pi$ have a physical interpretation. Together with the usual LSZ assumptions [30] of local quantum field theory, the following form factor equations can be derived:

a. Watson's equations describe the symmetry property under the simultaneous permutation of both the variables θ_i, θ_j and the spaces $i, j = i+1$:

$$F^{\mathcal{O}}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = F^{\mathcal{O}}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)S_{ij}(\theta_{ij})$$

for all possible arrangements of the θ.

b. The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces,

$$\begin{aligned}&{}^{\text{out},\bar{1}}\langle p_1|\mathcal{O}(0)|p_2,\dots,p_n\rangle^{\text{in,conn}}_{2\dots n} = \\ &\qquad = \mathbf{C}^{\bar{1}1}\sigma^{\mathcal{O}}_1 F^{\mathcal{O}}_{1\dots n}(\theta_1+i\pi,\theta_2,\dots,\theta_n) = F^{\mathcal{O}}_{2\dots n1}(\theta_2,\dots,\theta_n,\theta_1-i\pi)\mathbf{C}^{1\bar{1}},\end{aligned}$$

where $\sigma^{\mathcal{O}}_\alpha$ takes the statistics of the particle α with respect to $\mathcal{O}$ into account (e.g., $\sigma^{\mathcal{O}}_\alpha = -1$ if α and $\mathcal{O}$ are both fermionic; these numbers can be more general for anyonic or order and disorder fields; see [29]).

c. There are poles determined by one-particle states in each subchannel given by a subset of particles of the state in (4). In particular, the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n) = 2i\mathbf{C}_{12}F^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n)(\mathbf{1} - \sigma^{\mathcal{O}}_2 S_{2n}\cdots S_{23}).$$

d. If there are also bound states in the model, then the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. For instance, if particles 1 and 2 form a bound state (12), then there is a pole at $\theta_{12} = i\eta$, $0 < \eta < \pi$, such that

$$\operatorname*{Res}_{\theta_{12}=\eta} F^{\mathcal{O}}_{12\ldots n}(\theta_1, \theta_2, \ldots, \theta_n) = F^{\mathcal{O}}_{(12)\ldots n}(\theta_{(12)}, \ldots, \theta_n)\sqrt{2}\,\Gamma^{(12)}_{12},$$

where the bound state intertwiner $\Gamma^{(12)}_{12}$ is defined by (1) (see [21], [22]).

e. Because we work with relativistic quantum field theories, we naturally also have

$$F^{\mathcal{O}}_{1\ldots n}(\theta_1 + \mu, \ldots, \theta_n + \mu) = e^{s\mu} F^{\mathcal{O}}_{1\ldots n}(\theta_1, \ldots, \theta_n)$$

if the local operator transforms under Lorentz transformations as $F^{\mathcal{O}} \to e^{s\mu} F^{\mathcal{O}}$, where s is the "spin" of $\mathcal{O}$.

These equations were proposed in [9] as generalizations of the equations derived in the original articles [7], [8], [31]. They were proved [32] using the LSZ assumptions and maximal analyticity.

We now provide a constructive and systematic way to solve the equations in *a*–*e* for the covector-valued function $F^{\mathcal{O}}_{1\ldots n}$ if the scattering matrix is given.

3.1. Two-particle form factors. For two-particle form factors, the form factor equations in *a* and *b* are

$$\begin{aligned} F(\theta) &= F(-\theta) S(\theta), \\ F(i\pi - \theta) &= F(i\pi + \theta) \end{aligned} \tag{5}$$

for all eigenvalues of the two-particle S-matrix. In general theories, Watson's equations [33] hold only below the particle production thresholds. But in integrable theories, there is no particle production, and they therefore hold for all complex values of θ. It was shown [7] that these equations together with maximal analyticity have a unique solution.

As an example, we write the (highest-weight) $SU(N)$ form factor function [10]

$$F(\theta) = c \exp\left[\int_0^\infty \frac{dt}{t \sinh^2 t} e^{t/N} \sinh t\left(1 - \frac{1}{N}\right)\left(1 - \cosh t\left(1 - \frac{\theta}{i\pi}\right)\right)\right],$$

which is the minimal solution of (5) with $S(\theta) = a(\theta)$ as given by (3).

3.2. The general form factor formula. As usual (see [7]), we separate the minimal part and write the form factor for n particles as

$$F^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1, \ldots, \theta_n) = K^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij}). \tag{6}$$

Using the *off-shell Bethe ansatz* for the (covector-valued) K-function

$$K^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta}, \underline{z}) p^{\mathcal{O}}(\underline{\theta}, \underline{z}) \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta}, \underline{z}), \tag{7}$$

we transform the complicated form factor equations in *a*–*e* into simple ones for the p-functions, which are scalar and other simple functions of $e^{\pm z_i}$. The off-shell Bethe ansatz state $\Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta}, \underline{z})$ is obtained as a product of S-matrix elements; the integration contour $\mathcal{C}_{\underline{\theta}}$ depends on the model (see below for the $SU(N)$ model). The scalar functions

$$h(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n} \prod_{j=1}^{m} \phi(\theta_i - z_j) \prod_{1\le i<j\le m} \tau(z_i - z_j), \qquad \tau(z) = \frac{1}{\phi(z)\phi(-z)} \tag{8}$$

depend only on $S(\theta)$ (see (10) below), i.e., on the S-matrix, whereas the p-function $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ depends on the operator.

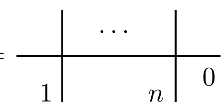
The $SU(N)$ form factors. The form factors for n fundamental particles (in the vector representation of $SU(N)$) are given by (6)–(8), where the *nested Bethe ansatz* is needed. This means that Ψ has the form

$$\Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z}) = L_{\beta_1\ldots\beta_m}(\underline{z})\Phi^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z}), \tag{9}$$

where the indices α_i take the values $\alpha_i = 1,2,\ldots,N$ and the summations range $\beta_i = 2,3,\ldots,N$. The off-shell Bethe ansatz state $\Phi^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z})$ is obtained using the technique of the algebraic Bethe ansatz as follows.

We consider a state with n particles and define the monodromy matrix

$$T_{1\ldots n,0}(\underline{\theta},\theta_0) = S_{10}(\theta_{10})\cdots S_{n0}(\theta_{n0}) =$$



as a matrix acting in the tensor product of the *quantum space*, a space of n particles (with respect to their quantum numbers) $V^{1\ldots n} = V^1\otimes\cdots\otimes V^n$ and the auxiliary space V^0. All vector spaces V^i are isomorphic to a space V whose basis vectors label all kinds of particles. We here take $V \cong \mathbb{C}^N$ as the space of the vector representation of $SU(N)$. The Yang–Baxter algebra relation for the S-matrix yields

$$T_{1\ldots n,a}(\underline{\theta},\theta_a)T_{1\ldots n,b}(\underline{\theta},\theta_b)S_{ab}(\theta_a-\theta_b) = S_{ab}(\theta_a-\theta_b)T_{1\ldots n,b}(\underline{\theta},\theta_b)T_{1\ldots n,a}(\underline{\theta},\theta_a),$$

which in turn implies the basic algebraic properties of the submatrices A, B, C, and D with respect to the auxiliary space defined by

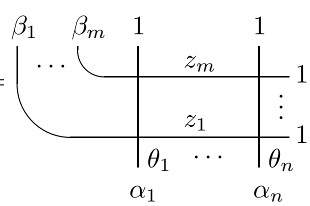
$$T_{1\ldots n,0}(\underline{\theta},z) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{\theta},z) & B_{1\ldots n,\beta}(\underline{\theta},z) \\ C^{\beta}_{1\ldots n}(\underline{\theta},z) & D^{\beta'}_{1\ldots n,\beta}(\underline{\theta},z)\end{pmatrix}, \quad \beta,\beta' = 2,3,\ldots,N.$$

The basic Bethe ansatz covectors Φ of (9) are obtained by applying the operators C to a pseudovacuum state

$$\Phi^{\underline{\beta}}_{1\ldots n}(\underline{\theta},\underline{z}) = \Omega_{1\ldots n}C^{\beta_m}_{1\ldots n}(\underline{\theta},z_m)\cdots C^{\beta_1}_{1\ldots n}(\underline{\theta},z_1).$$

This can be depicted as

$$\Phi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z}) =$$



$$\text{for } \beta_i = 2,3,\ldots,N, \quad \alpha_i = 1,2,\ldots,N. \tag{10}$$

This means that $\Phi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z})$ is a product of S-matrix elements as given by (10), with S-matrix (2) at each crossing point of lines and where the sum over all indices of internal lines is to be taken. The pseudovacuum is the highest-weight covector (with the weight $w = (n,0,\ldots,0)$)

$$\Omega_{1\ldots n} = e(1)\otimes\cdots\otimes e(1),$$

where the unit vector $e(\alpha)$, $\alpha = 1,2,\ldots,N$, corresponds to the particle of type α in the vector representation of $SU(N)$. The technique of the nested Bethe ansatz means that for the covector-valued function $L_{\beta_1\ldots\beta_m}(\underline{z})$

given by (9), the second-level Bethe ansatz is to be constructed. This ansatz has the same form as (7) except that the range of the indices is reduced by one. Iterating this nesting procedure finally yields a scalar function. The integration contour $\mathcal{C}_{\underline{\theta}}$ depends on the $\underline{\theta}$ (see [29], [10]).

Swieca's picture is that the bound state of $N-1$ fundamental particles is to be identified with the antiparticle and together with the form factor recurrence relations in c and d leads to the equation for the function $\phi(z)$ (see [29], [10])

$$\prod_{k=0}^{N-2}\phi(z+ki\eta)\prod_{k=0}^{N-1}F(z+ki\eta)=1,\qquad \eta=\frac{2\pi}{N},$$

with the solution

$$\phi(\theta)=\Gamma\left(\frac{\theta}{2\pi i}\right)\Gamma\left(1-\frac{1}{N}-\frac{\theta}{2\pi i}\right).$$

It was shown in [10] that the form factors given by (6) and (7) satisfy the form factor equations in a–e if some simple equations for the p-function are satisfied. We note that the form factors of this model were also calculated in [9], [34] using other techniques. In [9], the vectors were given by an indirect definition characterized by necessary properties but not provided explicitly. This missing link is given by the nested off-shell Bethe ansatz. Another procedure for constructing nested $SU(N)$ off-shell Bethe vectors was discussed in [35].

3.3. Examples. We present some simple examples illustrating our general results.

The energy–momentum tensor. For the local operator $\mathcal{O}(x)=T^{\rho\sigma}(x)$, where $\rho,\sigma=\pm$ denote the light-cone components, the p-function, as for the sine-Gordon model in [22], is

$$p^{T^{\rho\sigma}}(\underline{\theta},\underline{z})=\sum_{i=1}^{n}e^{\rho\theta_i}\sum_{i=1}^{m}e^{\sigma z_i}.$$

For the ($n{=}N$)-particle form factor, there are $n_l=N-l$ integrations at the lth level of the off-shell Bethe ansatz. We calculate the form factor of the particle α and the bound state $(\underline{\beta})=(\beta_1,\dots,\beta_{N-1})$ of $N-1$ particles. At each level, all integrations up to one can be performed iteratively using the bound-state relation in d. All the remaining integrations at the higher levels can then be done. The result for the form factor of the particle α and the bound state $(\underline{\beta})$ is given by

$$F^{T^{\rho\sigma}}_{\alpha(\underline{\beta})}(\theta_1,\theta_2)=K^{T^{\rho\sigma}}_{\alpha(\underline{\beta})}(\theta_1,\theta_2)G(\theta_{12}),\tag{11}$$

$$K^{T^{\rho\sigma}}_{\alpha(\underline{\beta})}(\theta_1,\theta_2)=N_2^{T^{\rho\sigma}}(e^{\rho\theta_1}+e^{\rho\theta_2})\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz}{R}\phi(\theta_1-z)e^{\sigma z}L(\theta_2-z)\epsilon_{\delta\underline{\gamma}}S^{\delta 1}_{\alpha\epsilon}(\theta_1-z)S^{\epsilon(\underline{\gamma})}_{(\underline{\beta})1}(\theta_2-z),$$

where the summation is over $\underline{\gamma}$ and $\delta>1$ and $G(\theta)$ is the minimal form factor function of two particles of the ranks $r=1$ and $r=N-1$. The functions $G(\theta)$ and $L(\theta)$ are given by

$$G(i\pi-\theta)F(\theta)\phi(\theta)=1,\qquad L(\theta)=\frac{\Gamma(1/2+\theta/(2\pi i))\Gamma(-1/2+1/N-\theta/(2\pi i))}{b(i\pi-\theta)}.$$

The remaining integral in (11) can be evaluated (similarly to [22]) with the result

$$\langle 0|T^{\rho\sigma}(0)|\theta_1,\theta_2\rangle^{\text{in}}_{\alpha(\underline{\beta})}=4M^2\epsilon_{\alpha\underline{\beta}}e^{(\rho+\sigma)(\theta_1+\theta_2+i\pi)/2}\,\frac{\sinh\bigl((\theta_{12}-i\pi)/2\bigr)}{\theta_{12}-i\pi}G(\theta_{12}).$$

Similarly to [22], we can prove the eigenvalue equation

$$\left(\int dx\,T^{\pm 0}(x)-\sum_{i=1}^{n}p_i^{\pm}\right)|\theta_1,\dots,\theta_n\rangle^{\text{in}}_{\underline{\alpha}}=0$$

for arbitrary states.

The fields $\psi_\alpha(x)$. Because the Bethe ansatz yields highest-weight states, we obtain the matrix elements of the spinor field $\psi(x) = \psi_1(x)$. The p-function for the local operator $\psi^{(\pm)}(x)$ is (also see [32])

$$p^{\psi^{(\pm)}}(\underline{\theta}, \underline{z}) = \exp\biggl(\pm\frac{1}{2}\biggl(\sum_{i=1}^{m} z_i - \biggl(1 - \frac{1}{N}\biggr)\sum_{i=1}^{n}\theta_i\biggr)\biggr).$$

For example, the one-particle form factor is

$$\langle 0|\psi^{(\pm)}(0)|\theta\rangle_\alpha = \delta_{\alpha 1} e^{\mp(1-1/N)\theta/2}.$$

The last two formulas are consistent with the proposal in [24], [15] that the statistics of the fundamental particles in the chiral $SU(N)$ Gross–Neveu model should be $\sigma = e^{2\pi i s}$, where $s = (1 - 1/N)/2$ is the spin. For the (n=N+1)-particle form factor, there are again n_l=N−l integrations at the lth level of the off-shell Bethe ansatz. Similarly to the above, we obtain the two-particle and one-bound-state form factor

$$\begin{aligned}
&F^{\psi^{(\pm)}}_{\alpha\beta(\underline{\gamma})}(\theta_1, \theta_2, \theta_3) = K^{\psi^{(\pm)}}_{\alpha\beta(\underline{\gamma})}(\theta_1, \theta_2, \theta_3) F(\theta_{12}) G(\theta_{13}) G(\theta_{23}), \qquad (12)\\
&K^{\psi^{(\pm)}}_{\alpha\beta(\underline{\gamma})} = N^\psi \exp\biggl(\mp\frac{1}{2}\biggl(1 - \frac{1}{N}\biggr)\sum\theta_i\biggr)\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz}{R}\phi(\theta_1 - z)\phi(\theta_2 - z)L(\theta_3 - z)e^{\pm z/2} \times\\
&\qquad \times \epsilon_{\delta\underline{\gamma}} S^{\delta 1}_{\alpha_1\epsilon}(\theta_1 - z) S^{\epsilon 1}_{\alpha_2\zeta}(\theta_2 - z) S^{\zeta(\underline{\gamma})}_{(\underline{\beta})1}(\theta_3 - z).
\end{aligned}$$

We were not able to evaluate this integral. In [36], we will discuss the $1/N$ expansion and the physical interpretation of the results for the chiral Gross–Neveu model.

3.4. The $1/N$ expansion: The $SU(N)$ Gross–Neveu model. The Lagrangian is [12]

$$\mathcal{L} = \bar\psi i\gamma\partial\psi + \frac{g^2}{2}\bigl((\bar\psi\psi)^2 - (\bar\psi\gamma^5\psi)^2\bigr) = \bar\psi(i\gamma\partial - \sigma - i\gamma^5\pi)\psi - \frac{1}{2g^2}(\sigma^2 + \pi^2).$$

There are infrared divergences due to the "would-be-Goldstone boson" π [13]–[15]. Using the approach in [15], we, for example, obtain (see [36] for more details)

$$\begin{aligned}
&{}^{\gamma}\langle\theta_3|i(i\gamma\partial - m)\psi^\delta(0)|\theta_1, \theta_2\rangle^{\text{in}}_{\alpha\beta} =\\
&\qquad = -\frac{i\pi}{N}2m\delta^\gamma_\alpha\delta^\delta_\beta\biggl(\frac{u(p_2)}{\cosh(\theta_{13}/2)} - \frac{\gamma^5 u(p_2)}{\sinh(\theta_{13}/2)}\biggr)\frac{\sinh\theta_{13}}{\theta_{13}}u(p_2) + \cdots + O\biggl(\frac{1}{N^2}\biggr),
\end{aligned}$$

where the dots denote a term where 1 and 2 is exchanged and disconnected terms. The same result follows from our exact form factor (12).

4. Wightman functions

As the simplest case, we consider the two-point function of a local scalar operator $\mathcal{O}(x)$:

$$w(x) = \langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle.$$

Inserting a complete set of in-states, we can write

$$w(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\int d\theta_1\cdots\int d\theta_n \exp\Bigl(-ix\sum p_i\Bigr) g_n(\underline{\theta}). \qquad (13)$$

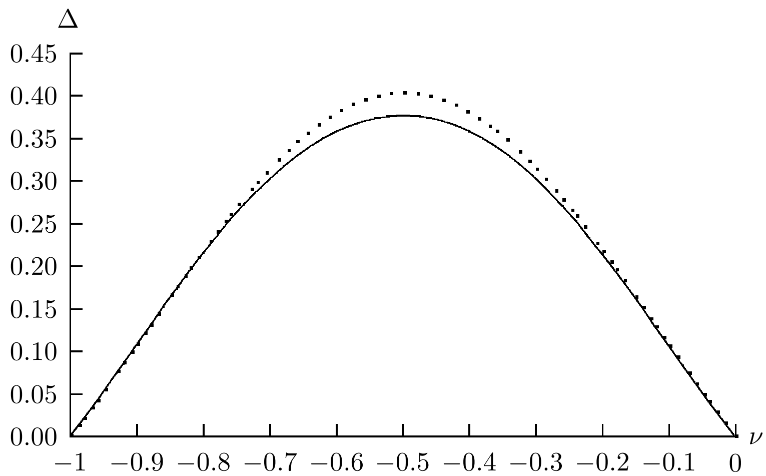


Fig. 1. Dimension of the exponential of the field for the sinh-Gordon model: one-particle (dotted line) and (1+2)-particle (solid line) intermediate state contributions.

We here introduce the functions

$$g_n(\underline{\theta}) = (4\pi)^{-n} F^{\mathcal{O}}(\theta_1, \ldots, \theta_n) F^{\mathcal{O}}(\theta_n + i\pi, \ldots, \theta_1 + i\pi),$$

where crossing is used. In particular, we consider exponentials of a scalar Bose field

$$\mathcal{O}(x) = {:}e^{i\gamma\varphi(x)}{:},$$

where $: \ldots :$ denotes normal ordering with respect to the physical vacuum, which amounts to $\langle 0|{:}e^{i\gamma\varphi(x)}{:}|0\rangle = 1$, and $g_0 = 1$ therefore holds.

The logarithm of the two-point function. For specific operators, for example, exponentials of Bose fields, it might be more convenient (see below) to consider sum (13) in a different form. For $g_0 = 1$, we can write (also see [37])

$$w(x) = \exp\biggl[\sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_n \, \exp\Bigl(-ix \sum p_i\Bigr) h_n(\underline{\theta})\biggr].$$

It is well known that the functions g_n and h_n are related by the cummulant formula

$$g_I = \sum_{I_1 \cup \cdots \cup I_k = I} h_{I_1} \cdots h_{I_k},$$

where we use the short notation $g_I = g_n(\theta_1, \ldots, \theta_n)$, $I = \{1, 2, \ldots, n\}$.

Because of the Lorentz invariance, it suffices to consider the value $x = (-i\tau, 0)$. Let $\mathcal{O}(x)$ be a scalar operator. Then the functions $h_n(\underline{\theta})$ depend only on the rapidity differences. We integrate once and obtain the relation for small τ [37], [38]. Therefore, the two-point Wightman function has a power-law behavior for short distances,

$$w(x) \approx C(m\tau)^{-4\Delta} \quad \text{as } \tau \to 0,$$

where the dimension is given by

$$\Delta = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots \int d\theta_{n-1} \, h_n(\theta_1, \ldots, \theta_{n-1}, 0)$$

if the integrals exist. This is true for the exponentials of Bose fields $\mathcal{O} = :e^{\gamma\varphi(x)}:$ because of the asymptotic behavior of h_n as $\operatorname{Re}\theta \to \infty$.

As an example, we consider the sinh-Gordon model. The dimension of the exponential of the field $\mathcal{O}(x) = :e^{\beta\varphi(x)}:$ can be calculated in the one- and (1+2)-particle intermediate state approximation

$$\Delta_{1+2} = \frac{1}{2}\left(h_1 + \frac{1}{2!}\int d\theta\, h_2(\theta, 0) + \dots\right).$$

The integral can be calculated exactly [38]. The result is shown in Fig. 1.

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