A Higher Conserved Current in the Quantized Massive Thirring Model (*).

B. BERG, M. KAROWSKI and H. J. THUN

Institut für Theoretische Physik der Freien Universität Berlin - Berlin, Germany

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Summary. — A higher conserved current in the quantized massive Thirring model is explicitly constructed.

In the classical limit of the massive Thirring model (*i.e.* in the tree approximation) the fields are elements of a Grassmann algebra. Recently, higher conserved currents (corresponding to higher powers of particle momenta) were found for this classical model ([1,2]).

With the field equation

$$(i\partial\!\!\!/ - m)\psi = g\gamma_\mu\psi\,\bar{\psi}\gamma^\mu\psi\,,$$

which reads, in terms of light-cone variables $x^\pm = (1/\sqrt{2})(x^0 \pm x^1)$,

$$\left\{\begin{aligned} i\sqrt{2}\,\partial_-\psi_1 &= m\psi_2 + 2g\,\psi_1\psi_2^+\psi_2\,,\\ i\sqrt{2}\,\partial_+\psi_2 &= m\psi_1 + 2g\,\psi_2\psi_1^+\psi_1\,,\end{aligned}\right. \tag{1}$$

an infinite set $j^\mu = \sum\limits_0^\infty \varepsilon^n j_n^\mu$ of conserved currents, due to KULISH and NISSIMOW ([2]), is given by

$$\partial_\mu j^\mu = \partial_-(i\psi_1^+\chi + \text{h.c.}) + \partial_+(\varepsilon m\psi_2^+\chi + \text{h.c.}) = 0\,, \tag{2}$$

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.

([1]) B. BERG, M. KAROWSKI and H. J. THUN: *Phys. Lett.*, **64** B, 286 (1976); R. FLUME, P. K. MITTER and N. PAPANICOLAOU: *Phys. Lett.*, **64** B, 289 (1976).

([2]) P. KULISH and E. NISSIMOW: preprint (June 1976) (submitted to *JETP Lett.*).

where $\chi = \sum_0^\infty \varepsilon^n x_n$ is defined by the equation

$$\varepsilon\sqrt{2}\partial_+\chi - \chi + 2i\varepsilon g\chi\psi_1^+\psi_1 + 4i\varepsilon g\psi_1\chi^+\chi + \psi_1 = 0 \,. \tag{3}$$

By means of the field equation (1) one obtains

$$\sqrt{2}\partial_-\chi + m^2\varepsilon\chi + 2ig\chi\psi_2^+\psi_2 + 4\varepsilon gm\psi_2\chi^+\chi + im\psi_2 = 0 \,. \tag{4}$$

The conservation (2) follows from (1), (3) and (4).

For the massive Thirring model in the BPHZ renormalization scheme [3] the effective Lagrangian is

$$\mathscr{L} = (1+b)\frac{i}{2}\bar\psi\partial\!\!\!/\psi - (m-a)\bar\psi\psi - \tfrac{1}{2}(g-c)\bar\psi\gamma_\mu\psi\bar\psi\gamma^\mu\psi \,,$$

where a, b, c are finite renormalization constants which are determined by the usual normalization conditions. The quantum equation of motion inside of a normal product reads [4]

$$\begin{aligned}\langle TN_{d+\frac{3}{2}}[Qi\partial\!\!\!/\psi](x)X\rangle &= \langle TN_{d+\frac{3}{2}}[\{Q\}(m-a)\psi](x)X\rangle + \\ &+ \langle TN_{d+\frac{3}{2}}[Q(-bi\partial\!\!\!/\psi + (g-c)\gamma_\mu\psi\bar\psi\gamma^\mu\psi)](x)X\rangle + i\left\langle TN_d[Q](x)\frac{\delta}{\delta\bar\psi(x)}X\right\rangle ,\end{aligned} \tag{5}$$

where Q is an arbitrary operator of dimension d and X a product of basic fields $\bar\psi$, ψ. The first term on the r.h.s. of eq. (5) is an anisotropic normal product [4], *i.e.* subgraphs which do not contain the line ψ are subtracted minimally and those which contain the line ψ are subtracted according to $d+\frac{3}{2}$. In general, anisotropies may destroy the conservation of a classical current. The equivalence [5] to the sine-Gordon theory and known results [6] for the sine-Gordon theory, however, suggest the existence of conserved currents in the quantized Thirring model. We treat an explicit example by the method explained in the following.

Anisotropic normal products can be evaluated in terms of ordinary ones by the Zimmermann identity

$$N_{d+\frac{3}{2}}[\{Q\}\psi] = N_{d+\frac{3}{2}}[Q\psi] + \sum_i z_i N_{d+\frac{3}{2}}[Q_i\psi] \,, \tag{6}$$

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(3) W. ZIMMERMANN: *Ann. of Phys.*, **77**, 536 (1973).

(4) M. GOMES and J. H. LOWENSTEIN: *Phys. Rev. D*, **7**, 550 (1973).

(5) S. COLEMAN: *Phys. Rev. D*, **11**, 2088 (1975). A nonperturbative proof has been given by B. SCHROER and T. TRUONG: FUB-preprint, HEP 6 (1976); R. SEILER and D. UHLENBROCK: *Les méthodes mathématiques de la théorie quantique des champs*. No. 248, p. 363, FUB/HEP 8/76, to appear in *Ann. of Phys.*

(6) P. KULISH and E. NISSIMOV: to appear in *Teor. Math. Phyz.* (1976); E. NISSIMOV: to appear in *Bulg. Journ. Phys.* (1976); R. FLUME: *Phys. Lett.*, **62** B, 93 (1976), and Erratum, *Phys. Lett.*, B, to be published.

where the operators Q_i of dimension $d+1$ and the coefficients are those which appear in the Zimmermann identity

$$N_d[Q] = N_{d+1}[Q] + \sum_i z_i N_{d+1}[Q_i] .$$

After application of the field equation (5) to the r.h.s. of (6), one obtains a system of equations for the anisotropies. For the current j_3 we show explicitly that all anisotropies can be expressed as divergences and contact terms.

For simplicity, we rewrite the field equation (6) as

$$\text{(7a)} \qquad \langle TN_{d+\frac{3}{2}}[\{Q\}F](x)X\rangle = i\left\langle N_d[Q](x)\frac{\delta}{\delta\bar{\psi}(x)}X\right\rangle ,$$

where

$$\text{(7b)} \qquad \begin{cases} F_1 = i\partial_+\psi_2 - \psi_1 - \psi_2\psi_1^+\psi_1 , \\ F_2 = i\partial_-\psi_1 - \psi_2 - \psi_1\psi_2^+\psi_2 \end{cases}$$

are obtained by the substitution

$$\partial \to \frac{m-a}{\sqrt{2}(1+b)}\partial , \quad \psi \to \sqrt{\frac{m-a}{2(g-c)}}\psi \quad \text{and} \quad F \to \frac{(m-a)^{\frac{3}{2}}}{(2(g-c))^{\frac{1}{2}}}F .$$

To construct the current j_3, we consider the operator

$$\text{(8)} \qquad D = D_0 + \xi_1 D_1 + \xi_2 D_2 + \xi_3 D_3 ,$$

where

$$\text{(9)} \qquad \begin{cases} D_0 = \partial_- N_4[-i\psi_1^+\partial_+^3\psi_1] + \text{h.c.} , \\ D_1 = \partial_- N_4[-\partial_+\psi_1^+\psi_1^+\psi_1\partial_+\psi_1] + \text{h.c.} , \\ D_2 = \partial_+ N_3[-\psi_2^+\partial_+^2\psi_1] + \text{h.c.} , \\ D_3 = \partial_+ N_3[i\psi_2^+\psi_1^+\psi_1\partial_+\psi_1] + \text{h.c.} , \end{cases}$$

and ξ_1, ξ_2 and ξ_3 are real constants which finally will be chosen such that one obtains a conserved current.

Making use of the field equation (7) we obtain

$$\text{(10)} \qquad \langle TDX\rangle = \langle T(-2B_1 + 4\xi_1 B_2 + 2\xi_1 B_3 + \xi_2 B_1' + \xi_3 B_2' + (\xi_3 - \xi_1)B_3' - \\ - 2O_{11} + 2\xi_1 A_9 - 4\xi_1 A_{10} - \xi_3 A' + D_4 - D_5 + D_6 + 2\xi_1 D_7 + \xi_2 D_{17})X\rangle + \\ + \text{contact terms} .$$

The operators B_i are given by

$$(11)\qquad \begin{cases} B_1 = N_5[-\psi_2^+\partial_+^3\psi_1] + \text{h.c.}\,, \\ B_2 = N_5[i\psi_2^+\partial_+\psi_1^+\psi_1\partial_+\psi_1] + \text{h.c.}\,, \\ B_3 = N_5[i\psi_2^+\psi_1^+\psi_1\partial_+^2\psi_1] + \text{h.c.}\,, \end{cases}$$

and the operators B'_i are obtained from B_i by replacing the normal product N_5 by N_4. The operators A', A_i, O'_i, O_i, D_i are listed in the appendix. There we also prove by means of Zimmermann identities (6) and further application of the field equation (7) that the operators A', $A_i\,O'_i$ and O_i can be expressed as linear combinations of the operator B_i and the total derivatives D_i.

Therefore, (apart from contact terms) we obtain

$$(12)\qquad \langle T\partial_\mu j_3^\mu(x)X\rangle = \langle T(D - \sum d_i D_i)X\rangle = \sum b_i \langle TB_iX\rangle\,.$$

The coefficients b_i, d_i are rational functions of Zimmermann coefficients. The requirement

$$(13)\qquad b_i = 0 \qquad \text{for } i = 1, 2, 3$$

is a system of linear equations for the coefficients ξ_1, ξ_2 and ξ_3. In the tree approximation, where no anisotropies arise, the solution ξ_i^{tr} of the above system of equations (13) leads to a current which is (up to total divergencies) identical with the classical current j_3 as given by eq. (2).

The quantized solution differs from ξ^{tr} by terms of higher order in the coupling constant and in $\hbar$.

For another recent discussion of this conservation law see (7).

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APPENDIX

In this appendix we prove that all operators occurring in eq. (10) can be expressed in terms of the three operators B_i and of the total derivatives D_i. Let us first give the definition of the operators A', A_i, O'_i, O_i, D_i. The operator A' is a normal product of degree 4:

$$(\text{A.1})\qquad A' = N_4[\partial_+\psi_1^+\psi_1^+\psi_1\delta_1] + \text{h.c.}$$

(7) R. FLUME and S. MEYER: CERN preprint TH 2243.

The set $\{A_i\}$ consists of 10 operators which are normal products of degree 5:

$$\text{(A.2)}\qquad \begin{cases} A_1 = N_5[i\partial_+^2\psi_2^+\psi_1^+\psi_1\delta_1] + \text{h.c.}\,, \\ A_2 = N_5[i\partial_+\psi_2^+\partial_+\psi_1^+\psi_1\delta_1] + \text{h.c.}\,, \\ A_3 = N_5[i\partial_+\psi_2^+\psi_1^+\partial_+\psi_1\delta_1] + \text{h.c.}\,, \\ A_4 = N_5[i\psi_2^+\partial_+^2\psi_1^+\psi_1\delta_1] + \text{h.c.}\,, \\ A_5 = N_5[i\psi_2^+\partial_+\psi_1^+\partial_+\psi_1\delta_1] + \text{h.c.}\,, \\ A_6 = N_5[i\psi_2^+\psi_1^+\partial_+^2\psi_1\delta_1] + \text{h.c.}\,, \\ A_7 = N_5[i\partial_+^2\psi_1^+\psi_1^+\psi_2\delta_1] + \text{h.c.}\,, \\ A_8 = N_5[i\partial_+\psi_1^+\psi_1\partial_+\psi_2\delta_1] + \text{h.c.}\,, \\ A_9 = N_5[i\partial_+^2\psi_1^+\psi_1^+\psi_1\delta_2] + \text{h.c.}\,, \\ A_{10} = N_5[i\partial_+\psi_1^+\psi_1^+\partial_+\psi_1\delta_2] + \text{h.c.} \end{cases}$$

In eqs. (A.1) and (A.2) the symbol δ_k is defined by the relation

$$\text{(A.3)}\qquad N_d[Q\delta_k] = N_d[\{Q\}\psi_k] - N_d[Q\psi_k]\,,$$

i.e. A' and A_i are differences of anisotropic operators and of the corresponding isotropic ones. They vanish in the classical limit. The set $\{O'_i\}$ contains 4 isotropic operators of degree 4:

$$\text{(A.4)}\qquad \begin{cases} O'_1 = N_4[i\psi_2^+\partial_+\psi_1^+\partial_+\psi_1\psi_1] + \text{h.c.}\,, \\ O'_2 = N_4[i\partial_+^2\psi_1^+\psi_1^+\psi_2\psi_1] + \text{h.c.}\,, \\ O'_3 = N_4[i\partial_+\psi_1^+\psi_1^+\partial_+\psi_2\psi_1] + \text{h.c.}\,, \\ O'_4 = N_4[-\partial_+^3\psi_2^+\psi_1] + \text{h.c.}\,, \end{cases}$$

whereas the set $\{O_i\}$ consists of 21 isotropic operators of degree 5:

$$\text{(A.5)}\qquad \begin{cases} O_1 = N_5[\partial_-\partial_+\psi_1^+\psi_1^+\partial_+\psi_1\psi_1] + \text{h.c.}\,, \\ O_2 = N_5[\partial_-\psi_1^+\partial_+\psi_1^+\partial_+\psi_1\psi_1] + \text{h.c.}\,, \\ O_3 = N_5[\partial_-\psi_1^+\psi_1^+\partial_+^2\psi_1\psi_1] + \text{h.c.}\,, \\ O_4 = N_5[\partial_+^2\psi_2^+\psi_2^+\partial_+\psi_1\psi_1] + \text{h.c.}\,, \\ O_5 = N_5[\partial_+\psi_2^+\psi_2^+\partial_+^2\psi_1\psi_1] + \text{h.c.}\,, \\ O_6 = N_5[\partial_+^3\psi_2^+\psi_1^+\psi_2\psi_1] + \text{h.c.}\,, \\ O_7 = N_5[\partial_+^2\psi_2^+\partial_+\psi_1^+\psi_2\psi_1] + \text{h.c.}\,, \\ O_8 = N_5[\partial_+^2\psi_2^+\psi_1^+\partial_+\psi_2\psi_1] + \text{h.c.}\,, \\ O_9 = N_5[\partial_+\psi_2^+\partial_+^2\psi_1^+\psi_2\psi_1] + \text{h.c.}\,, \\ O_{10} = N_5[\partial_+\psi_2^+\partial_+\psi_1^+\partial_+\psi_2\psi_1] + \text{h.c.}\,, \end{cases}$$

$$
\text{(A.5)}\quad \begin{cases}
O_{11} = N_5[\psi_1^+ \partial_+^3 \psi_1^+ \psi_2 \psi_1] & + \text{h.c.}\,, \\
O_{12} = N_5[\psi_2^+ \partial_+^2 \psi_1^+ \partial_+ \psi_2 \psi_1] & + \text{h.c.}\,, \\
O_{13} = N_5[\psi_2^+ \partial_+ \psi_1^+ \partial_+^2 \psi_2 \psi_1] & + \text{h.c.}\,, \\
O_{14} = N_5[\partial_+ \psi_2^+ \partial_+ \psi_1^+ \partial_+ \psi_1 \psi_2] & + \text{h.c.}\,, \\
O_{15} = N_5[\psi_2^+ \partial_+^2 \psi_1^+ \partial_+ \psi_1 \psi_2] & + \text{h.c.}\,, \\
O_{16} = N_5[i\partial_- \partial_+^3 \psi_1^+ \psi_1] & + \text{h.c.}\,, \\
O_{17} = N_5[i\partial_+^4 \psi_2^+ \psi_2] & + \text{h.c.}\,, \\
O_{18} = N_5[i\partial_- \partial_+^2 \psi_1^+ \partial_+ \psi_1] & + \text{h.c.}\,, \\
O_{19} = N_5[i\partial_- \partial_+ \psi_1^+ \partial_+^2 \psi_1] & + \text{h.c.}\,, \\
O_{20} = N_5[i\partial_- \psi_1^+ \partial_+^3 \psi_1] & + \text{h.c.}\,, \\
O_{21} = N_5[i\partial_+^3 \psi_2^+ \partial_+ \psi_2] & + \text{h.c.}
\end{cases}
$$

Finally there is a set $\{D_i\}$ of 20 total derivatives:

$$
\text{(A.6)}\quad \begin{cases}
D_0 = \partial_- N_4[i\partial_+^3 \psi_1^+ \psi_1] & + \text{h.c.}\,, \\
D_1 = \partial_- N_4[-\partial_+ \psi_1^+ \psi_1^+ \psi_1 \partial_+ \psi] & + \text{h.c.}\,, \\
D_2 = \partial_+ N_3[-\psi_2^+ \partial_+^2 \psi_1] & + \text{h.c.}\,, \\
D_3 = \partial_+ N_3[i\psi_2^+ \psi_1^+ \psi_1 \partial_+ \psi_1] & + \text{h.c.}\,, \\
D_4 = \partial_+ N_4[-i\psi_1^+ \partial_+^2 \partial_- \psi_1] & + \text{h.c.}\,, \\
D_5 = \partial_+ N_4[-i\partial_+ \psi_1^+ \partial_+ \partial_- \psi_1] & + \text{h.c.}\,, \\
D_6 = \partial_+ N_4[-i\partial_+^2 \psi_1^+ \partial_+ \partial_- \psi_1] & + \text{h.c.}\,, \\
D_7 = \partial_+ N_4[-\partial_- \psi_1^+ \psi_1^+ \psi_1 \partial_+ \psi_1] & + \text{h.c.}\,, \\
D_8 = \partial_+ N_4[-i\partial_+^3 \psi_2^+ \psi_2] & + \text{h.c.}\,, \\
D_9 = \partial_+ N_4[-i\partial_+^2 \psi_2^+ \partial_+ \psi_2] & + \text{h.c.}\,, \\
D_{10} = \partial_+ N_4[-\psi_2^+ \partial_+ \psi_1^+ \partial_+ \psi_1 \psi_2]\,, & \\
D_{11} = \partial_+ N_4[-\psi_2^+ \partial_+^2 \psi_1^+ \psi_1 \psi_2] & + \text{h.c.}\,, \\
D_{12} = \partial_+ N_4[i\psi_2^+ \psi_1^+ \partial_+ \psi_1 \delta_1] & + \text{h.c.}\,, \\
D_{13} = \partial_+ N_4[i\psi_2^+ \partial_+ \psi_1^+ \psi_1 \delta_1] & + \text{h.c.}\,, \\
D_{14} = \partial_+ N_4[i\partial_+(\psi_2^+ \psi_1^+ \psi_1)\delta_1] & + \text{h.c.}\,, \\
D_{15} = \partial_+ N_4[i\partial_+(\psi_2^+ \psi_1^+ \psi_1 \delta_1)] & + \text{h.c.}\,, \\
D_{16} = \partial_+ N_4[i\partial_+ \psi_1^+ \psi_1^+ \psi_2 \delta_1] & + \text{h.c.}\,, \\
D_{17} = \partial_+ N_3[i\partial_+ \psi_1^+ \psi_1] & + \text{h.c.}\,, \\
D_{18} = \partial_+ N_3[-\partial_+^2 \psi_2^+ \psi_1] & + \text{h.c.}\,, \\
D_{19} = \partial_+ N_3[-\partial_+ \psi_2^+ \partial_+ \psi_1] & + \text{h.c.}
\end{cases}
$$

After having defined so many operators, now we come to the main purpose of this appendix, *i.e.* to prove that all these operators can be expressed linearly in terms of B_i and D_i. Let us first consider A', which is a difference of an anisotropic operator and of the corresponding isotropically oversubtracted one. A' is related to the operators O'_i by a Zimmermann identity, cf. eq. (6):

$$A' = \sum_{i=1}^{4} z'_i O'_i \,. \tag{A.7}$$

The operators O'_i in turn can be expressed in terms of B'_i and total derivatives D'_i:

$$\left\{\begin{aligned} O'_1 &= -B'_2 \,, \\ O'_2 &= B'_3 \,, \\ O'_3 &= -B'_2 - B'_3 - D_3 \,, \\ O'_4 &= -B'_1 + D_{18} - D_{19} + D_2 \,. \end{aligned}\right. \tag{A.8}$$

Next we again apply Zimmermann identities which express the operators B'_i and A_i in terms of isotropic normal products of degree 5:

$$B'_i = B_i + \sum_{j=1}^{21} z'_{ij} O_j \,, \tag{A.9}$$

$$A_i = \sum_{j=1}^{21} z_{ij} O_i \,. \tag{A.10}$$

By applying the field equation (7*a*) the O_i give rise to anisotropic A_i's, isotropic B_i's and total derivatives D_i:

$$\left\{\begin{aligned} O_1 &= A_{10} - B_2 + D_1 \,, \\ O_2 &= -A_{10} - B_2 \,, \\ O_3 &= -A_9 - B_3 \,, \\ O_4 &= -A_7 - A_8 + B_2 - D_{16} \,, \\ O_5 &= -A_7 + B_3 \,, \\ O_6 &= -A_1 - 2A_2 - 2A_3 - A_4 - 2A_5 - A_6 + B_3 - 2D_{14} - D_{15} \,, \\ O_7 &= -A_3 - A_5 - A_6 - D_{12} \,, \\ O_8 &= -A_1 \,, \\ O_9 &= -A_6 - B_3 \,, \\ O_{10} &= -A_2 \,, \\ O_{11} &= A_4 - A_5 + A_6 - B_2 + B_3 + D_{11} - D_{10} \,, \\ O_{12} &= -A_4 \,, \\ O_{13} &= -A_2 - A_4 - A_5 + B_2 - D_{13} \,, \end{aligned}\right. \tag{A.11}$$

$$(\text{A.11})\quad \begin{cases} O_{14} = \quad A_5 + B_2\,, \\ O_{15} = -A_5 - B_2 - D_{10}\,, \\ O_{16} = \quad D_0 + D_4 - D_5 + D_6\,, \\ O_{17} = -D_8 + D_9\,, \\ O_{18} = -2D_4 + D_5 - D_6 - D_0\,, \\ O_{19} = -2D_5 + 2D_4 + D_6 + D_0\,, \\ O_{20} = \quad 2D_0 + D_4 - D_5 + D_6\,, \\ O_{21} = -D_9\,. \end{cases}$$

From eqs. (A.10) and (A.11) we obtain an inhomogeneous linear system of equations for the A_i which can be solved for small coupling, as the determinant is $1 + O(g, \hbar)$, and from eqs. (A.11) and (A.9) we also get O_i and B_i' in terms of B_i and D_i.

● RIASSUNTO (*)

Si costruisce esplicitamente una corrente conservata più elevata nel modello massivo quantizzato di Thirring.

(*) *Traduzione a cura della Redazione.*

Сохраняющийся ток высшего порядка в квантованной массивной модели Тирринга.

Резюме (*). — В явном виде конструируется сохраняющийся ток высшего порядка в квантованной массивной модели Тирринга.

(*) *Переведено редакцией.*