An exact relativistic S-matrix in 1+1 dimensions: The on-shell solution of the massive Thirring model and the quantum Sine-Gordon equation*

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Abstract:

In the classical massive Thirring model and for the Sine-Gordon equation there exists an infinite sequence of conservation laws which imply for scattering processes: i) no particle production, ii) only momentum exchange, and iii) factorization of the N particle S-matrix into two-particle S-matrices. These conservation laws are claimed to survive quantization. The properties i), ii), and iii) together with unitarity, crossing symmetry, and T-invariance determine uniquely the S-matrix and the bound state spectrum which agrees with that one calculated in WKB approximation. Hence, in 1+1 dimensions all relativistic models describing a particle-antiparticle pair are equivalent if the above mentioned conservation laws hold true. The exact complete S-matrix is given explicitly.

Contents

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1 Introduction

1.1 The results [1,2]

We shall calculate an S-matrix element like

$$^{out}\langle\, f(p'),\ldots,\bar f(q'),\ldots,b_i(k'),\ldots \mid f(p),\ldots,\bar f(q),\ldots,b_j(k),\ldots\rangle^{in}$$

for the scattering of arbitrary numbers of particles (fermions) f, antiparticles (antifermions) $\bar f$, and different kinds of bound states $b_1,\ldots,b_n$, where b_i can either be considered as a bound state of $(f\bar f)$ or $(b_j b_{i-j})$ or $(b_j b_k b_{i-j-k})$ etc.

We shall see that the S-matrix in 1+1 dimensions is uniquely determined by some general properties which are valid, e.g., in the case of the massive Thirring model and the Sine-Gordon equation.

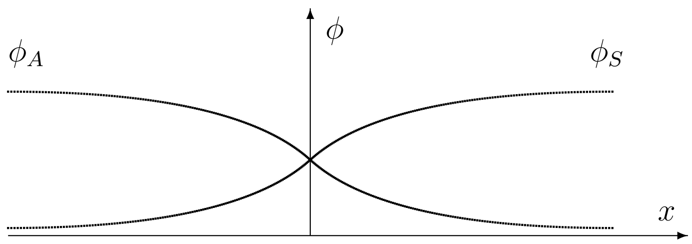
1.2 The Sine-Gordon equation

Let us consider the relativistic wave equation in 1+1 dimensions:

$$\Box\,\phi + \frac{\alpha}{\beta}\sin\beta\phi = 0 \tag{1}$$

where $\phi(x,t)$ is a classical field, $\sqrt{\alpha}$ a mass parameter, and β a coupling constant. This equation is completely soluble by means of the inverse scattering method [3]. There is a static solution called "soliton"

$$\phi_S(x) = \frac{4}{\beta}\,\arctan e^{\sqrt{\alpha}x}$$



with energy or mass $m = 8\sqrt{\alpha}/\beta^2$ and the antisoliton $\phi_A(x) = \phi_S(-x)$. Another set of solutions, oscillating in time, are the "breathers" which may be interpreted as soliton-antisoliton bound states. They have a continuous (center of mass) energy spectrum with $0 < E < 2m$.

Now we come to the most important property of the Sine-Gordon equation: there exists an infinite sequence of local conservation laws [4]

$$\partial_\mu J^\mu_n(x) = 0 \qquad n = 1,3,5,\ldots$$

The corresponding charges of asymptotic states $|\, p_1, \ldots, p_N \,\rangle^{in}_{out}$ are

$$\sum_{i=1}^{N} \left(p_{i+}\right)^n$$

where $p_\pm = p_0 \pm p_1$. (Similar currents exist for p_-.) The conservation of these charges implies for scattering

i) no particle production;

ii) only momentum exchange, i.e., the sets of incoming and outgoing momenta are equal;

iii) factorization of an N-particle S-matrix into two-particle ones:

$$S^{(N)}(p_1, \ldots, p_N) = \prod_{i<j} S^{(2)}(p_i, p_j)\,.$$

(Since the factors on the right hand side do not commute in general the ordering has to be specified, see below.)

The usual quantization of a field theory – by defining Green's functions in renormalized perturbation theory by means of the Gell-Mann-Low expansion – is not very useful for the Sine-Gordon equation if one wants to describe solitons. Since ϕ_S is very far away from the vacuum $\phi = 0$, which would be the starting point for ordinary perturbation theory. The quantization of classical solutions can be attacked by semiclassical methods [5]. Dashen, Hasslacher, and Neveu obtained in WKB approximation the bound state spectrum which is now discrete, of course,

$$m_k = 2m \sin \frac{k\pi}{2\lambda} \qquad k = 1, 2, \ldots < \lambda$$

where

$$\lambda = \frac{8\pi}{\beta^2} - 1$$

and

$$m = \frac{8\sqrt{\alpha}}{\beta^2} \left(1 - \frac{\beta^2}{8\pi}\right)$$

is the soliton mass. This spectrum was claimed to be exact.

Coleman [6] gave arguments based on perturbation theory that the quantized Sine-Gordon theory must be equivalent to another field theory in 1+1 dimensions, the massive Thirring model. Other approaches to this problem were presented in [7].

1.3 The Thirring model

The (massless) Thirring model [8] describes a self-interacting Dirac field in 1+1 dimension:

$$\mathcal{L}_0 = \bar{\psi} i\gamma\partial\psi - \frac{1}{2} g j_\mu j^\mu$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$ and g is a coupling constant. The operator solution of this model is known since a long while and the Wightman functions

$$\langle\, \psi(x_1)\ldots\bar{\psi}(x_N)\,\rangle$$

can be calculated [8]. This field theory, however, is rather simple. Because of infra-red effects there are no one-particle poles in momentum space and there is no scattering in this model. The massive Thirring model defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - m\bar{\psi}\psi$$

is much more complicated. Coleman's equivalence now says: identify the two fundamental Thirring particles f, $\bar{f}$ with the soliton, antisoliton of the Sine-Gordon equation and relate the coupling constants of both models by

$$1 + \frac{2g}{\pi} = \frac{8\pi}{\beta^2} - 1\,.$$

Our approach is, in some sense, an alternative proof of this equivalence and also a proof of the exactness of the WKB Sine-Gordon spectrum.

2 The massive Thirring model

Our treatment of the massive Thirring model starts with the observation that for the classical case there exists again an infinite sequence of local conservation laws [9]:

$$\partial_\mu J_n^\mu(x) = 0 \quad n = 1, 3, 5, \ldots.$$

The fields ψ, $\bar{\psi}$ are anticommuting objects in the classical model; we consider it as the tree approximation of the corresponding quantized model. The consequences of these conservation laws are the same as for the Sine-Gordon equations mentioned above. This can be seen as follows:

$$F := (i\gamma\partial - m)\psi - g\, j_\mu\gamma^\mu\psi = 0\,.$$

In general, we have

$$\partial_\mu J_n^\mu = \bar{\triangle}_n F + h.c.$$

where $\bar{\triangle}_n$ is a local operator which may be expressed by powers of derivatives of the fields. The integrated Ward identity with $X = \psi(x_1)\dots\psi(x_N)$ in tree approximation reads

$$
\begin{aligned}
0 &= \textstyle\int d^2x \langle T \partial_\mu J_n^\mu(x)\, X\,\rangle_{tr} \\
&= \textstyle\int d^2x \langle T \left(\bar{\triangle}_n F(x) + h.c.\right) X\,\rangle_{tr} \\
&= \int d^2x \langle T \left(\bar{\triangle}_n i \frac{\delta}{\delta\bar{\psi}(x)} + h.c.\right) X\,\rangle_{tr} \\
&= \langle\, T \triangle_n(x_1)\dots\bar{\psi}(x_N)\,\rangle_{tr} + \dots + \langle\, T\, \psi(x_1)\dots\bar{\triangle}_n(x_N)\,\rangle_{tr}\,.
\end{aligned}
$$

In momentum space on the mass shell after amputation only the parts in $\triangle_n$ contribute which are linear in the fields. These vanish for even n and are given for odd n by

$$\triangle_n = i\partial_+^n\psi + \dots$$

where $\partial_+ = \partial_0 + \partial_1$. Applying the LSZ reduction technique we obtain

$$0 = {}^{out}\langle\, p_1', \dots, p_M')\,|\, p_1, \dots, p_{N-M}\,\rangle^{in} \left(\sum_{i=1}^{M}\left(p_{i+}'\right)^n - \sum_{i=1}^{N-M}\left(p_{i+}\right)^n\right)\,.$$

This means, an the S-matrix element vanishes unless $\sum_{i=1}^{M}(p_{i+})^n,\ n = 1, 3, 5, \dots$ is conserved. It follows that the set of incoming and outgoing momenta are equal:

$$\{p_1', \dots, p_M'\} = \{p_1, \dots, p_{N-M}\}\,.$$

Hence we have the properties:

i) no particle production

ii) only momentum exchange

iii) factorization of S.

One believes [10] that i) and ii) imply iii).

The question is now whether the conservation laws survive quantization. The apparent occurrence of anomalies may be seen as follows: In the BPHZ scheme the integrated Ward identity reads [11]

$$
\begin{aligned}
0 &= \textstyle\int d^2x \langle\, T\, N_d\,[\,\partial_\mu J_n^\mu\,]\,(x)\, X\,\rangle \\
&= \textstyle\int d^2x \langle\, T\left(N_d\left[\,\bar{\triangle}_n F\,\right](x) + h.c.\right) X\,\rangle \\
&= \int d^2x \langle\, T \left(N_{d-3/2}\left[\,\bar{\triangle}_n\,\right](x)\, i\frac{\delta}{\delta\bar{\psi}(x)} + h.c.\right) X\,\rangle \\
&\quad - \textstyle\int d^2x \langle\, T\left(N_d\left[\,\left(\bar{\triangle}_n - \{\bar{\triangle}_n\}\right) m_0\psi\,\right](x) + h.c.\right) X\,\rangle\,.
\end{aligned}
$$

The dimension $d = n+2$ of the operator $\partial_\mu J_n^\mu$ determines the subtraction degree of the normal product $N_d\left[\partial_\mu J_n^\mu\right](x)$. The unrenormalized mass m_0 differs from m by a counter term which is finite in the BPHZ scheme and is determined by normalization conditions. The reason for the extra term on the right-hand side is that $m_0\psi$ in F has lower dimension. Hence, in the quantum equation of motion inside of a normal product, $N_d\left[\bar{\triangle}_n m_0\psi\right](x)$ has to be replaced by the anisotropic normal product $N_d\left[\{\bar{\triangle}_n\} m_0\psi\right](x)$. This means that subgraphs which contain the line ψ are subtracted according to d and those which do not contain the line ψ are subtracted minimally $(d-1)$. Therefore, the extra term will, in general, be different from zero if $\triangle_n$ is not linear in the fields which is true for $n \geq 3$. One expects [12] that the currents J_n^μ can be redefined such that these anomalies cancel. This was explicitly shown for $n = 3$ [13]. We believe that the conservation laws and the properties i), ii), and iii) hold true in the quantized massive Thirring model.

3 The S-matrix

Our main result can be expressed as follows:

Theorem: *If a relativistic model in 1+1 dimensions with a particle-antiparticle pair $f, \bar{f}$ fulfills the assumptions:*

1. *i) no particle production*

   ii) only momentum exchange

   iii) factorization of the S-matrix;

2. *unitarity, crossing and T-invariance;*

3. *non-vanishing backward particle-antiparticle scattering (and some more technical properties concerning analytic and asymptotic behaviour which will show up in the proof),*

then the S-matrix is uniquely determined. I depends on two parameters: the mass m and a "coupling" constant λ.

Note that assumption 1. is a consequence of the conservation laws considered in the previous section.

<u>Proof:</u> An S-matrix obeying 1. can be formulated as follows [2]:

$$
\begin{aligned}
&{}^{out}\langle\, \alpha'_1(p'_1), \ldots, \alpha'_N(p'_N) \,|\, \alpha_1(p_1), \ldots, \alpha_N(p_N)\,\rangle^{in} \\
&\qquad = \langle\, \alpha'_1(p'_1), \ldots, \alpha'_N(p'_N) \,|\, S^{(n)} \,|\, \alpha_1(p_1), \ldots, \alpha_N(p_N)\,\rangle
\end{aligned}
$$

where the α_i denote the different kinds of particles $f, \bar{f}, b_1, b_2, \ldots$. The N-particle S-matrix is a product of all two-particle ones in a special order which, for example for $p_1 > \ldots > p_N$ is given by:

$$S^{(N)}(p_1, \ldots, p_N) = \prod_{i=1}^{N-1} \left(\prod_{i<j\leq N} S^{(2)}(p_i, p_j) \right)$$

where the two-particle S-operator is defined by

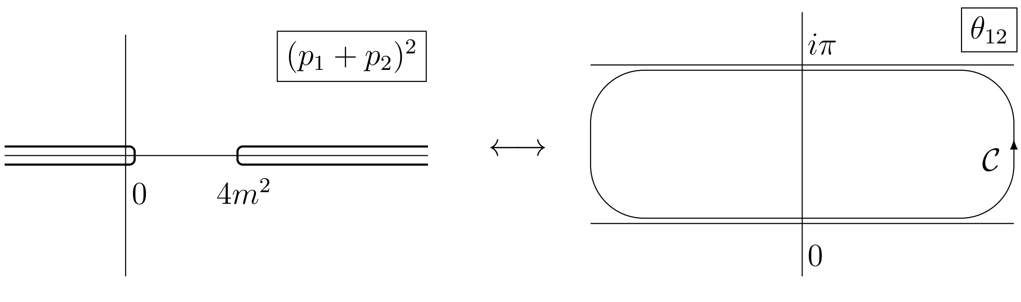
$$S^{(2)} \,|\, \ldots, \alpha_i(p_i), \ldots, \alpha_j(p_j) \ldots \rangle = t_{\alpha_i \alpha_j} \,|\, \ldots, \alpha_i(p_i), \ldots, \alpha_j(p_j) \ldots \rangle \\ + r_{\alpha_i \alpha_j} \,|\, \ldots, \alpha_j(p_i), \ldots, \alpha_i(p_j) \ldots \rangle \,.$$

The transmission and the reflection amplitudes are t and r, respectively. Because of ii), reflection only appears for different particles with equal mass, as f and $\bar{f}$. Let us consider first "repulsive" couplings ($g < 0,\ \beta^2 > 4\pi$) such that there are no bound states. The general case will be obtained by analytic continuation with respect to the coupling. Then $S^{(2)}$ contains four functions: $t_{ff}, t_{\bar{f}\bar{f}}, t_{f\bar{f}} := t$ and $r_{f\bar{f}} := r$.

Let us introduce the rapidity difference θ_{12} by

$$(p_1 + p_2)^2 = 2m^2 \left(1 + \cosh \theta_{12}\right).$$

The physical plane is mapped onto the strip $0 < \mathrm{Im}\, \theta < \pi$ and the physical region for $f\bar{f}$ scattering $(p_1 + p_2)^2 - i\epsilon > 4m^2$ correspond to $\theta_{12} > 0$.



Unitarity $S^\dagger S = S(-\theta)\, S(\theta) = 1$ and crossing symmetry

$$t_{ff}(\theta) = t_{\bar{f}\bar{f}}(\theta) = t(i\pi - \theta)$$

$$r(\theta) = r(i\pi - \theta)$$

yield

$$r^2 = t^2(1 - 1/|t|^2)\,.$$

Hence, there remains only one unknown function $t(\theta)$ which we shall determine now [1]. If $\ln t(\theta)$ is analytic in the physical strip and does not grow terribly for $|\,\mathrm{Re}\ \theta\,| \to \infty$, Chauchy's formula yields

$$
\begin{aligned}
\ln t(\theta) &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{\sinh(z-\theta)}\,\ln t(z) \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dz}{\sinh(z-\theta)}\,\ln\Big(t(z)\,t(i\pi+z)\Big)
\end{aligned}
$$

$\mathcal{C}$ being the contour enclosing the physical strip $0 < \mathrm{Im}\;\theta < \pi$. The unitarity and crossing relations imply

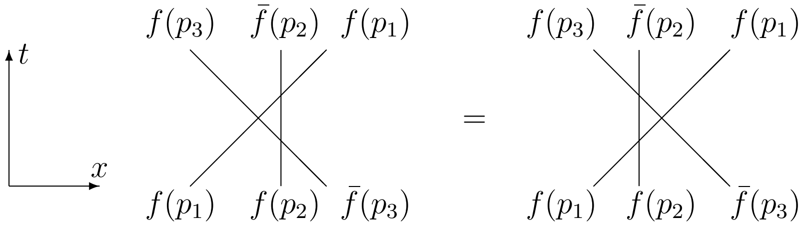
$$
t(z)\,t(i\pi+z) = \frac{h(z)}{h(i\pi-z)}
$$

where $h(z) = t(z)/r(z)$ which is well defined if $r \neq 0$. We shall now prove that $h(z)$ is determined by the factorization condition of the three-particle S-matrix and T-invariance: $S^{(3)}$ has to be symmetric

$$
S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}\,.
$$

This means, for example, that the amplitude for the transition

$$
f(p_1)\,f(p_2)\,\bar{f}(p_3) \;\rightarrow\; f(p_1)\,\bar{f}(p_2)\,f(p_3)
$$



has to be equal to that one of the reversed process.
Using $t(\theta) = r(\theta)\,h(\theta) = t_{ff}(\theta)\,h(\theta)/h(i\pi-\theta)$, we obtain the functional equation for $h(\theta)$

$$
h(\alpha+\beta) = h(i\pi+\alpha)\,h(\beta) + h(\alpha)\,h(i\pi-\beta)\,.
$$

The solutions of this equation are

$$
h(\theta) = \frac{\sinh\lambda\theta}{\sinh\lambda i\pi}
$$

where λ is a free parameter to be interpreted later. After some calculations we obtain:

$$
t(\theta) = \exp i\int_0^{\infty}\frac{dx}{x}\,\frac{\sinh\frac{x}{2}(1-\lambda)}{\sinh\frac{x}{2}\,\cosh\frac{x}{2}\lambda}\,\sin\frac{x\lambda}{\pi}(i\pi-\theta)
$$

or

$$
t(\theta) = F(\theta)/F(2\pi i-\theta)
$$

with

$$F(\theta) = \prod_{k=1}^{\infty} \prod_{l=0}^{\infty} \frac{\left(2l + \frac{k}{\lambda} + \frac{\theta}{i\pi}\right)\left(2l + \frac{k-1}{\lambda} + \frac{\theta}{i\pi}\right)}{\left(2l - 1 + \frac{k}{\lambda} + \frac{\theta}{i\pi}\right)\left(2l + 1 + \frac{k-1}{\lambda} + \frac{\theta}{i\pi}\right)} .$$

This amplitude was first proposed by Zamolodchikov [14] who used the extra assumptions (a) exactness of the WKB spectrum; (b) absence of resonances; and (c) for integer values of λ absence of reflection and validity of a formula for $t(\theta)$ due to Korepin and Faddeev [15]. In our treatment the properties (a) – (c) follow from the assumptions of the theorem.

The poles of the transmission amplitude $t(\theta)$ in the physical strip

$$\theta_k = i\pi(1 - k/\lambda) \qquad k = 1, \ldots < \lambda$$

correspond to $(f\bar{f})$ bound states with masses

$$m_k = 2m \sin \frac{k\pi}{2\lambda} .$$

This spectrum coincides with the one calculated in WKB approximation for the Sine-Gordon equation [5], if we relate the free parameter λ to the coupling constants β and g by:

$$\lambda = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi} .$$

For $\lambda < 1$, where there are no bound states, the S-matrix is completely determined. For $\lambda > 1$ we have to calculate the scattering of bound states, i.e., the amplitudes $t_{b_k f}$ and $t_{b_k b_l}$ [2,16]. We consider the residues of $S^{(3)}(p_1, p_2, p_3)$ at $(p_1 + p_2)^2 = m_k^2$ and obtain after some calculations in agreement with previous results in the semiclassical limit [15]:

$$t_{b_k f}(\theta) = (-1)^k A_0 A_k \left(\prod_{j=1}^{k-1} A_j\right)^2 \quad \text{where} \quad A_j = \frac{\sin \frac{\pi}{2}\left(\frac{\theta}{i\pi} + \frac{k-2j}{2\lambda} + \frac{1}{2}\right)}{\sin \frac{\pi}{2}\left(\frac{\theta}{i\pi} - \frac{k-2j}{2\lambda} - \frac{1}{2}\right)}$$

correspondingly for $k \le l$:

$$t_{b_k b_l}(\theta) = B_0 B_k \left(\prod_{j=1}^{k-1} B_j\right)^2 \quad \text{where} \quad B_j = \frac{\tan \frac{\pi}{2}\left(\frac{\theta}{i\pi} + \frac{k+l-2j}{2\lambda}\right)}{\tan \frac{\pi}{2}\left(\frac{\theta}{i\pi} - \frac{k+l-2j}{2\lambda}\right)} .$$

So, finally, we can calculate the general S-matrix element written down in the introduction.

4 Off-shell quantities

Using Watson's theorem [17], one can calculate the soliton form factor [18]

$$\langle\, f(p_1) \,|\, j^\mu(0) \,|\, f(p_2) \,\rangle = \bar{u}(p_1)\gamma^\mu u(p_2)\, G(\theta_{12})$$

where

$$G(\theta_{12}) = \frac{\cosh\frac{\theta}{2}}{\cosh\frac{\theta}{2}\lambda} \exp - \int_0^\infty \frac{dx}{x} \frac{\sinh\frac{x}{2}(1-\lambda)}{\sinh\frac{x}{2}\,\cosh\frac{x}{2}\lambda\,\sinh x\lambda} \sin^2\frac{x\lambda}{2\pi}\theta\,.$$

In principle, matrix elements like

$$^{out}\langle\, f(p'), \dots, \bar{f}(q'), \dots, b_i(k'), \dots \,|\, \phi(x) \,|\, f(p), \dots, \bar{f}(q), \dots, b_j(k), \dots \rangle^{in}$$

can be calculated by means of a generalized Watson's theorem and methods similar to those used for the derivation of the S-matrix. This program, however, turns out to be rather complicated. The determination of Wightman functions like

$$\langle\, \phi(x)\, \phi(y) \,\rangle \quad \text{or} \quad \langle\, \psi(x_1) \dots \psi(x_N) \,\rangle$$

is still an open problem.

The latter must have the same small distance behaviour as those for the massless Thirring model [8] since the Callan-Symanzik function β in the massive Thirring model vanishes identically [19].

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