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FACTORIZED U(n) SYMMETRIC S-MATRICES IN TWO DIMENSIONS

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S-matrices describing the scattering of solitons belonging to the fundamental representation of U(n) are classified

A variety of two-dimensional field theories have the property of being completely integrable at the classical level and describe interactions of solitons. In the cases investigated so far, the most studied example being the sine-Gordon theory (alias the Thirring model), the infinite number of conservation laws characterizing the soliton-like behaviour have been shown to survive quantization [1]. These soliton quantum field theories are then of special interest not only because of certain structural analogies to gauge theories in four dimensions, but because they are probably soluble by an S-matrix bootstrap method. They would provide the first examples of exact Wightman functions in theories with non-trivial (albeit simple) S-matrices and, hence, serve as good testing grounds for the validity of certain approximation schemes (e.g. semi-classical) apart from yielding further insight into the general problem of off-shell behaviour.

Knowledge of the exact S-matrix is the first step of the bootstrap program. The most important features of a general soliton S-matrix are exact elastic unitarity and factorization [2]. It is indeed remarkable that imposition of these properties on the scattering of a particle and its antiparticle fixes uniquely the minimal S-matrix with non-trivial backward scattering [3], where by the minimal S-matrix is meant the one with the minimum number of singularities and zeros on the physical sheet. This minimum solution depends on a single parameter and the solution can be interpreted as the exact S-matrix of the Thirring model. This situation is reminiscent of the

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fact that the requirement of the existence of an infinite number of conservation laws on a Lagrangian field theory of a single scalar field with non-derivative coupling fixes the sine-Gordon theory uniquely.

Another class of classical field theories which seem to possess [4] an analogous structure is that of the non-linear O(n) chiral models and the (O($2n$) symmetric) Gross-Neveu model. With a program of the type outlined above in mind Zamolodchikov and Zamolodchikov [5] have recently studied the imposition of elastic unitarity and factorization on the scattering of massive particles belonging to a vector multiplet of O(n) For $n = 2$ (O(2) $\simeq$ U(1)) the Thirring model solution with a single parameter is recovered For $n \geqslant 3$ the authors found a class of minimum solutions with non-trivial backward scattering with no free parameter but analytic in n and, furthermore, presented various arguments in the expansion in $1/n$ to support the connection of the minimal solution with the S-matrix of the non-linear chiral models and a particular non-minimal solution with the S-matrix of the Gross-Neveu models

The success of these attempts encourages the search for new models by this simple method. In this paper we investigate the factorized S-matrices describing scattering of a U(n) multiplet of particles belonging to the fundamental representation with their antiparticles There are a variety of interesting classes of solutions which are summarized in table 1 A common feature is that the minimum S-matrices are all free from parameters As expected there is also a subclass of solutions with the higher symmetry O($2n$) corresponding to those of Zamolodchikov Many of the other solutions also have a higher symmetry. For example the non-trivial solution with vanishing backward scattering for the case $n = 2$ actually possesses an SO(4) symmetry The problem remains, however, that we have not yet succeeded in proposing associated field theories to the new classes. U(n) Thirring models other than the Gross-Neveu model provide possible candidates. The existence of the solutions does at least suggest that there are many models of interest to be discovered and that a more complete classification of solutions corresponding to various groups and their representations would be highly desirable

We consider the general $2 \to 2$ scattering of a particle P_α belonging to the fundamental representation of U(n) with its antiparticle A_α. The S-matrix element is given by

$$^{\mathrm{out}}\langle P_\beta(\tilde p_1)\,A_\delta(\tilde p_2)|P_\alpha(p_1)\,A_\gamma(p_2)\rangle^{\mathrm{in}} \tag{1}$$

$$= {}_{\alpha\gamma}F_{\beta\delta}(\theta)\,\delta(\tilde p^1_1 - p^1_1)\,\delta(\tilde p^1_2 - p^1_2) - {}_{\alpha\gamma}B_{\delta\beta}(\theta)\,\delta(\tilde p^1_1 - p^1_2)\,\delta(\tilde p^1_2 - \tilde p^1_1),$$

with the forward and backward amplitudes each given by two invariant amplitudes

$${}_{\alpha\gamma}F_{\beta\delta}(\theta) = t_1(\theta)\,\delta_{\alpha\beta}\delta_{\gamma\delta} + t_2(\theta)\,\delta_{\alpha\gamma}\delta_{\beta\delta}\,,$$

$${}_{\alpha\gamma}B_{\delta\beta}(\theta) = r_1(\theta)\,\delta_{\alpha\beta}\delta_{\gamma\delta} + r_2(\theta)\,\delta_{\alpha\gamma}\delta_{\beta\delta}\,, \tag{2}$$

Table 1
Table of minimal solutions

Classes	Parameter	$t_1(\theta)$	$u_1(\theta)$	$r_1(\theta)$	$u_2(\theta)$	$r_2(\theta)$	$t_2(\theta)$
		$\uparrow \begin{smallmatrix}\bar{j} & & i\\ & \times & \\ i & & \bar{j}\end{smallmatrix}$	$\begin{smallmatrix}j & & i\\ & \times & \\ i & & j\end{smallmatrix}$	$\begin{smallmatrix}i & & \bar{j}\\ & \times & \\ i & & \bar{j}\end{smallmatrix}$	$\begin{smallmatrix}i & & j\\ & \times & \\ i & & j\end{smallmatrix}$	$\begin{smallmatrix}j & & \bar{j}\\ & \times & \\ i & & \bar{i}\end{smallmatrix}$	$\begin{smallmatrix}\bar{j} & & j\\ & \times & \\ i & & \bar{i}\end{smallmatrix}$
I		1	1	0	0	0	0
II	$\lambda = \frac{2}{n}$	$f(\theta,\lambda)$	$t_1(i\pi-\theta)$	0	$-\lambda \frac{i\pi}{\theta} u_1(\theta)$	0	$-\lambda \frac{i\pi}{i\pi-\theta} t_1(\theta)$
III	$\lambda = \frac{1}{n-1}$	$f(\theta,\lambda) f(i\pi-\theta,\lambda)$	$t_1(\theta)$	$-\lambda \frac{i\pi}{\theta} t_1(\theta)$	$r_1(\theta)$	$-\lambda \frac{i\pi}{i\pi-\theta} t_1(\theta)$	$r_2(\theta)$
IV	$\lambda = \frac{1}{n+1}$	$f(\theta,\lambda) f(i\pi-\theta,\lambda)\, i \operatorname{th} \frac{1}{2}(\theta+\frac{1}{2} i\pi)$	$-t_1(\theta)$	$\lambda \frac{i\pi}{\theta} t_1(\theta)$	$r_1(\theta)$	$-\lambda \frac{i\pi}{i\pi-\theta} t_1(\theta)$	$r_2(\theta)$
V	$\operatorname{ch} \pi\mu = n$	0	0	$\prod_{k=-\infty}^{\infty} \frac{f(\theta, k/2\mu i)}{f(\theta, k/2\mu i+\frac{1}{2})}$	$r_1(\theta)$	$\frac{\sin \mu(i\pi-\theta)}{\sin \mu\theta} r_1(\theta)$	$r_2(\theta)$
VI	$e^{\pi\mu} = n$	0	0	$\prod_{-\infty}^{\infty} \frac{f(\theta, k/2\mu i)}{f(\theta, k/2\mu i+\frac{1}{2})}$	$e^{i\mu\theta} r_1(\theta)$	$\frac{\sin \mu(i\pi-\theta)}{\sin \mu\theta} r_1(\theta)$	$e^{i\mu(i\pi-\theta)} r_2(\theta)$

and where θ is the rapidity variable given by $p_1 p_2/M^2 = \cosh\theta$. The particle-particle S-matrix element is given by

$$
{}^{\text{out}}\langle P_\beta(\tilde{p}_1)\, P_\delta(\tilde{p}_2)|P_\alpha(p_1)\, P_\gamma(p_2)\rangle^{\text{in}}
$$
$$
= {}_{\alpha\gamma}S_{\beta\delta}(\theta)\delta(\tilde{p}_1^1 - p_1^1)\,\delta(\tilde{p}_2^1 - p_2^1) - \delta(\tilde{p}_1^1 - p_2^1)\,\delta(\tilde{p}_2^1 - p_1^1)_{\alpha\gamma}\,S_{\delta\beta}(\theta)\,, \quad (3)
$$

with

$$
{}_{\alpha\gamma}S_{\beta\delta}(\theta) = u_1(\theta)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + u_2(\theta)\,\delta_{\alpha\delta}\,\delta_{\gamma\beta}\,.
$$

Crossing requires

$$
{}_{\alpha\gamma}S_{\beta\delta}(i\pi - \theta) = {}_{\alpha\delta}F_{\beta\gamma}(\theta)\,,
$$
$$
{}_{\alpha\gamma}B_{\delta\beta}(i\pi - \theta) = {}_{\alpha\delta}B_{\gamma\beta}(\theta)\,, \quad (4)
$$

which in terms of the amplitudes reads

$$
u_1(\theta) = t_1(i\pi - \theta)\,, \qquad u_2(\theta) = t_2(i\pi - \theta)\,, \qquad r_1(\theta) = r_2(i\pi - \theta)\,, \quad (5)
$$

As explained by Zamolodchikov it is convenient to associate asymptotic states with products of symbols $P_\alpha(\theta_\alpha)$ for particles (similarly $A_\alpha(\theta_\alpha)$ for antiparticles) corresponding to the αth member of the multiplet with rapidity θ_α. In (out) states are identified with products in which all symbols are arranged in order of decreasing (increasing) θ. The product relations

$$
P_\alpha(\theta_1)\, P_\beta(\theta_2) = {}_{\alpha\beta}S_{\gamma\delta}(\theta_1 - \theta_2)\, P_\delta(\theta_2)\, P_\gamma(\theta_1)\,,
$$
$$
P_\alpha(\theta_1) A_\beta(\theta_2) = {}_{\alpha\beta}F_{\gamma\delta}(\theta_1 - \theta_2) A_\delta(\theta_2)\, P_\gamma(\theta_1)
$$
$$
+ {}_{\alpha\beta}B_{\gamma\delta}(\theta_1 - \theta_2)\, P_\delta(\theta_2) A_\gamma(\theta_1)\,, \quad (6)
$$

incorporate the relations between in and out states Unitarity can then be expressed algebraically as the consistency of the product, yielding

$$
{}_{\alpha\beta}S_{\gamma\delta}(\theta)\ {}_{\delta\gamma}S_{\epsilon\eta}(-\theta) = \delta_{\alpha\eta}\,\delta_{\beta\epsilon}\,,
$$
$$
{}_{\alpha\beta}F_{\gamma\delta}(\theta)\ {}_{\delta\gamma}F_{\epsilon\eta}(-\theta) + {}_{\alpha\beta}B_{\gamma\delta}(\theta)\ {}_{\delta\gamma}B_{\epsilon\eta}(-\theta) = \delta_{\alpha\eta}\,\delta_{\beta\epsilon}\,,
$$
$$
{}_{\alpha\beta}F_{\gamma\delta}(\theta)\ {}_{\delta\gamma}B_{\epsilon\eta}(-\theta) + {}_{\alpha\beta}B_{\gamma\delta}(\theta)\ {}_{\delta\gamma}F_{\epsilon\eta}(-\theta) = 0\,, \quad (7)
$$

which in terms of invariant s-channel amplitudes reads

$$M_{a\pm}(\theta)\, M_{a\pm}(-\theta) = 1 \;, \qquad a = 1,2,3 \;, \tag{8}$$

with

$$M_{1\pm} = u_1 \pm u_2 \;, \qquad M_{2\pm} = t_1 \pm r_1 \;, \qquad M_{3\pm} = t_1 \pm r_1 + n(t_2 \pm r_2) \;, \tag{9}$$

The requirement of factorization of the S-matrix [3] in the formulation [5] now becomes equivalent to the requirement of associativity of the product which results in various equations. A detailed study shows the independent equations are given by

$$\begin{aligned} & S \cdot F \cdot F = S * F * F \;, \\ & S \cdot F \cdot B = B * B * F + F * S * B \;, \\ & F \cdot S \cdot F + B \cdot B \quad B = F * S * F + B * B * B \;, \end{aligned} \tag{10}$$

where

$$A \cdot B \cdot C \qquad \text{means} \qquad {}_{\alpha\beta}A_{\delta\epsilon}(\theta_{12})\; {}_{\delta\gamma}B_{\kappa\lambda}(\theta_{13})\; {}_{\epsilon\lambda}C_{\mu\nu}(\theta_{23}) \;,$$

$$A * B * C \qquad \text{means} \qquad {}_{\delta\epsilon}A_{\kappa\mu}(\theta_{12})\; {}_{\alpha\lambda}B_{\delta\nu}(\theta_{13})\; {}_{\beta\delta}C_{\epsilon\lambda}(\theta_{23}) \,.$$

An interesting feature is that $n = 2$ does not turn out to be a special case as it was in the O(n) analysis The resulting functional equations for the invariant amplitudes are themselves not very illuminating. Suffice it to say, however, that they are sufficiently restrictive to permit a complete classification of the solutions, since the equations either cause certain amplitudes to vanish or lead to simple (often linear) functional equations involving ratios (of amplitudes) in which these amplitudes appear as denominators

There exists a wide variety of solutions, six distinct classes in all, which are presented in table 1. In each class the minimal solutions contain only n as a parameter. The classes are characterized by having additional symmetry properties, typically expressed as the vanishing of amplitudes in various channels. The Zamolodchikov solutions correspond to class III To see this we consider the O($2n$) S-matrix of Zamolodchikov describing scattering of a (self-conjugate) vector multiplet (V_i).

$${}_{ik}\mathcal{S}_{jl}(\theta) = \sigma_1(\theta)\, \delta_{ik}\delta_{jl} + \sigma_2(\theta)\, \delta_{ij}\delta_{kl} + \sigma_3(\theta)\, \delta_{il}\delta_{jk} \;. \tag{11}$$

We identify

$$P_\alpha = \sqrt{\tfrac{1}{2}}(V_{2\alpha-1} + i\,V_{2\alpha}) \;, \qquad \alpha = 1 \,..\, n \;,$$

and thus obtain the conditions for O(2n) symmetry.

$$u_1 = t_1 = \sigma_2 \,, \qquad u_2 = r_1 = \sigma_3 \tag{12}$$

Note that for the case $n = 4$ one can reduce the symmetry under consideration from O(4) to SO(4) by adding a further term to ${}_{ik}\mathcal{S}_{jl}(\theta)$ of the form $\epsilon_{ijkl}\sigma_4(\theta)$. Re-performing the analysis of Zamolodchikov for this case turns out to be rather lengthy and we just state the result. Minimum solutions for $\sigma_4 \neq 0$ (note that crossing requires $\sigma_4(\theta) = -\sigma_4(i\pi - \theta)$) are given by

$$\sigma_4(\theta) = \pm\sigma_3(\theta) \,, \qquad \sigma_2(\theta) = \sigma_3(\theta)(1 - 2\theta/i\pi) \,, \tag{13}$$

with

$$\sigma_2(\theta) = R(\theta)\,R(i\pi - \theta) \,, \qquad R(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{3}{4} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(1 - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{4} - \frac{i\theta}{2\pi}\right)} \,. \tag{14}$$

The solution with $\sigma_3 = \sigma_4$ can be identified with the U(n) solution II in the case $n = 2$, since with the extra amplitude σ_4 the condition for SO(4) symmetry becomes

$$u_1 = \sigma_2 - \sigma_4 \,, \qquad u_2 = \sigma_3 + \sigma_4 \,, \qquad r_1 = \sigma_3 - \sigma_4 \tag{15}$$

All classes of U(n) symmetric S-matrices (*cf.* table 1) can be expressed in terms of a function $f(\theta, \lambda)$ meromorphic in θ for Re $\lambda \geqslant 0$ which is uniquely defined by the following conditions

$$f(\theta)\,f(-\theta) = 1 \,, \qquad f(i\pi - \theta)\,f(i\pi + \theta) = \frac{\theta}{\theta + \pi^2\lambda^2} \,, \tag{16}$$

and $f(0) = 1, f(\theta) \neq 0$ and ∞ for $0 \leqslant \mathrm{Im}\,\theta < \pi$, $|\ln f(\theta)| \lesssim e^{\theta}$ for Re $\theta \to \pm\infty$

The auxiliary function $f(\theta, \lambda)$ is given explicitly by

$$f(i\pi\varphi, \lambda) = \prod_{l=0}^{\infty} \frac{(2l + 1 - \varphi)(2l + 1 + \lambda + \varphi)}{(2l + 1 + \varphi)(2l + 1 + \lambda - \varphi)}$$

$$= \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\varphi)\,\Gamma(\frac{1}{2} + \frac{1}{2}\lambda - \frac{1}{2}\varphi)}{\Gamma(\frac{1}{2} - \frac{1}{2}\varphi)\,\Gamma(\frac{1}{2} + \frac{1}{2}\lambda + \frac{1}{2}\varphi)} \,. \tag{17}$$

For $-1 < \mathrm{Re}\,\varphi < 1$ we have the integral representation

$$f(i\pi\varphi, \lambda) = \exp \int_0^{\infty} \frac{dx}{x} \frac{e^{-\lambda x} - 1}{\mathrm{sh}\,x} \mathrm{sh}\,x\varphi \,, \tag{18}$$

and f has the asymptotic properties

$$f(i\pi\varphi, \lambda) = 1 + \tfrac{1}{2}i\pi\lambda \operatorname{th} \tfrac{1}{2}i\pi\varphi + O(\lambda^2) \qquad \text{for } \lambda \to 0\,,$$
$$= e^{i\pi\lambda/2 + O(1/\varphi)} \qquad \text{for } i\pi\varphi \to \infty\,. \tag{19}$$

For $t_1 \neq 0$ the unitarity, crossing and factorization relations lead to

$$t_1(i\pi - \theta)\, t_1(i\pi + \theta) = \frac{\theta^2}{\theta^2 + \pi^2\lambda}\,, \tag{20}$$

with $\lambda = 0, 2/n, 1/(n-1), 1/(n+1)$ for solution I, II, III, IV, respectively. For class I, II we have furthermore

$$t_1(\theta)\, t_1(-\theta) = 1\,, \tag{21}$$

and for III, IV

$$t_1(\theta) = t_1(i\pi - \theta)\,, \qquad t_1(\theta) = -t_1(i\pi - \theta)\,, \qquad \text{respectively.}$$

The minimal solutions of these equations which have no zeros and a minimal number of poles in the physical strip $0 < \operatorname{Im}\theta < \pi$ are given in table 1 The amplitudes of class I, II, III are meromorphic in θ with poles and zeros outside of the physical strip on the imaginary axis The amplitude t_1, odd under crossing, corresponding to IV has in addition a pole at $\theta = \frac{1}{2}i\pi$.

For $t_1 = 0$ (solution class V, VI) we obtain

$$r_1(i\pi - \theta)\; r_1(i\pi + \theta) = \frac{-\sin^2\mu\theta}{\sin\mu(i\pi - \theta)\sin\mu(i\pi + \theta)}\,, \tag{22}$$

and $r_1(\theta)r_1(-\theta) = 1$, with $\operatorname{ch}\pi\mu = n$, $e^{\pi\mu} = n$ for classes V, VI, respectively. The minimal solutions of the equations listed in table 1 are meromorphic with poles and zeros on the lines $\operatorname{Im}\theta/\pi = k = \text{integer} \neq 0$.

The general factorizing U(n) symmetric S-matrices are obtained from the minimal solutions by multiplication by a factor which contains CDD-like poles θ_i on the imaginary axis in the physical strip

$$\prod_{i=1}^{L} \frac{\operatorname{sh}\frac{1}{2}(\theta + \theta_i)}{\operatorname{sh}\frac{1}{2}(\theta - \theta_i)}$$

Note that, except for class I, II, the poles have to appear pairwise $\theta_i = i\pi - \theta_j$ because of crossing symmetry. The SO(4) symmetric S-matrix with $\sigma_3 = \sigma_4$ is obtained by solution II with an extra pole at $\frac{1}{2}i\pi$.

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References

[1] P.P Kulish and E.R. Nissimov, JETP Lett. 24 (1976) 247,
R. Flume, Phys. Lett. 62B (1976) 93,
B. Berg, M. Karowski and H.-J. Thun, Nuovo Cim 38A (1977) 11,
R. Flume and S. Meyer, Nuovo Cim. Lett. 18 (1977) 238,
M. Luscher, Nucl. Phys. B117 (1976) 475.
[2] R. Flume, V Glaser and D. Jagolnitzer, unpublished,
P P Kulish, Teor. Mat. Fiz 26 (1976) 198
[3] M Karowski, H.-J. Thun, T.T. Truong and P H Weisz, Phys. Lett 67B (1977) 321.
[4] M. Luscher and K Pohlmeyer, DESY preprint, in press.
[5] A.B. Zamolodchikov and A B Zamolodchikov, Nucl. Phys B133 (1978) 525, Preprint ITEP-112.