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EXACT FORM FACTORS IN (1 + 1)-DIMENSIONAL FIELD THEORETIC MODELS WITH SOLITON BEHAVIOUR

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A system of equations is derived which must be satisfied by multiparticle matrix elements of any local operator in field theories with soliton behaviour. Form factors of various operators of interest are calculated exactly by means of the known exact S-matrices in the sine-Gordon, massive Thirring, non-linear σ-, and Gross-Neveu models. The finite sine-Gordon wave function renormalization constant is determined exactly.

1. Introduction

Two-dimensional field theories have for some time played the role of testing grounds for general hypotheses and approximation schemes as well as acting as catalysts for new ideas and sources of inspiration. Their recent investigation has mainly involved studies of non(standard)-perturbation techniques, for example semiclassical methods [1], $1/N$ expansions [2] and topological structures [3]. The various ideas have motivated studies of a wide class of models and many new results have been established. The wonderful duality between the quantum sine-Gordon (SG) and the massive Thirring (MT) models, described by the Lagrangians

$$\mathcal{L}^{\mathrm{SG}} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{\alpha}{\beta^2}(\cos\beta\phi - 1)\,, \tag{1.1}$$

$$\mathcal{L}^{\mathrm{MT}} = \bar{\psi}(i\partial\!\!\!/ - m)\,\psi - \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)^2\,, \tag{1.2}$$

respectively, first convincingly demonstrated by Coleman [4], has been one of the highlights of these studies. Among other results was the unanticipated fact that many of the models possess infinitely many conservation laws [5a] * also at the quantum level.

The first such conservation laws were discovered in the SG and MT models [6] and recently they have also been established [7] in the O(N) non-linear sigma (NLS)

* For a review see ref. [5b].

model, the Gross-Neveu (GN) model, described by the Lagrangians [8]

$$\mathcal{L}^{\mathrm{NLS}} = \tfrac{1}{2}\sum_{i=1}^{N} (\partial_\mu n_i)^2 \,, \qquad \text{with} \qquad g\sum_{i=1}^{N} n_i^2 = 1 \,, \tag{1.3}$$

$$\mathcal{L}^{\mathrm{GN}} = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha i\partial\!\!\!/ \psi_\alpha + \tfrac{1}{2}g^2\Big(\sum_{\alpha=1}^{N} \bar{\psi}_\alpha\psi_\alpha\Big)^2 \,, \tag{1.4}$$

respectively, and supersymmetric generalizations thereof. These conservation laws have drastic effects for on-shell matrix elements and express the fact that the particles behave as solitons: that is in any scattering process the set of momenta is conserved [9] and secondly the S-matrix factorizes [10]. There are only special minimal S-matrices consistent with these properties. They depend on the symmetry of the model. For U(1), O(N) ($N > 2$) and U(N) ($N > 1$) there exist a one-parametric set [11], a uniquely determined [12], and a finite set [13] of minimal S-matrices, respec tively. By minimal is meant that S-matrix having the minimal number of poles and zeros in the physical sheet. Once the spectrum of the model is ascertained from other studies, then the exact S-matrix can be postulated, making minimality assumptions on zeros and redundant poles (analogous to properties of a general class of non-relativistic S-matrices [14]). This has been done for the massive Thirring model [11] and recently for the NLS and GN models by Zamolodchikov and Zamolodchikov [12], and various perturbative checks have so far always been found in agreement [15].

There is the hope that such soliton quantum field theories are in fact soluble. One possible method to go off-shell is first to obtain all multiparticle form factors of various operators of interest and then ultimately sum over intermediate states to obtain the full Green functions. Despite the fact that the on-shell behaviour of such models is very simple, the off-shell behaviour is comparatively rich and complicated. The method of deriving the form factors involves solving generalized Watson's theorems [16], subject to proper analytic properties and incorporating the exact S-matrices. This is discussed in sect. 2. Again minimality assumptions are made and the proposed exact form factors are subjected to various perturbative tests. The full problem when many external particles and many degrees of freedom are involved has not yet been completely solved. In this paper we just present various examples which illustrate a variety of interesting behaviours.

In sect. 3 we derive form factors for the SG solitons. The electromagnetic form factor agrees with semiclassical approximations and MT model perturbation theory. The form factor is asymptotically dynamical-power behaved. Moreover, we use the form factor to obtain the exact wave function renormalization constant of the sine-Gordon field. As mentioned above, the form factors involving more than two solitons in the external states require solution of a matrix problem. The three-soliton form factors should prove prototypes for the general n-particle problem.

In sect. 4 we consider form factors of the sine-Gordon (elementary) breathers.

As an example we propose the form factor of the operator: ϕ^2: (x) between two breathers and perform checks in perturbation theory. This form factor tends to a constant asymptotically. We also consider the three-breather form factor of the fundamental SG field; we check that it yields the correct S-matrix in the appropriate on-shell limit and perform additional perturbative tests. Finally, we use this exact form factor to obtain transition form factors from higher breathers.

Our final examples in sect. 5 involve form factors in the GN and NLS models. Firstly we consider the two-particle matrix elements of the O(N) currents and perform tests in the $1/N$ expansion. In both models the form factors fall as (N-dependent) powers of logarithms, characteristic of asymptotically free theories. Finally we consider the form factor of the mass operator $\bar{\psi}\psi(x)$ in the GN model.

2. Generalized Watson's theorem [16]

In this section we derive a set of equations for matrix elements of local operators which follow from CPT invariance, crossing symmetry, unitarity and factorization [10] of the S-matrix. The latter special property is fulfilled in field theoretic models with an infinite set of conservation laws like the sine-Gordon (SG), the massive Thirring (MT), the Gross-Neveu (GN) and the nonlinear σ-model (NLS). Factorization means that for a scattering process the sets of incoming and outgoing particle momenta are equal:

$$\{p_1, ..., p_n\}^{\text{in}} = \{p'_1, ..., p'_{n'}\}^{\text{out}} , \tag{2.1}$$

and the n-particle S-matrix is a product of two-particle ones

$$S^{(n)}(p_1, ..., p_n) = \prod_{i<j} S^{(2)}(p_i, p_j) . \tag{2.2}$$

The matrix $S^{(n)}$ is defined by

$$S|\alpha_1(p_1), ...\rangle^{\text{in}} = |\alpha'_1(p_1), ...\rangle^{\text{in}}{}_{\alpha'_1 ...}\, S^{(n)}_{\alpha_1 ...}(p_1, ...) . \tag{2.3}$$

For non-vanishing backward scattering the factors on the r.h.s. of eq. (2.2) will not commute and the ordering has to be specified [17]. For simplicity we consider a theory with N different kinds of self-conjugate bosons α_k, $k = 1, ..., N$ and a hermitian local scalar operator $\mathcal{O}(x) = \mathcal{O}^\dagger(x)$. The generalization for charged particles, fermions and more complicated operators is straightforward.

We discuss an arbitrary matrix element like

$$^{\text{out}}\langle\alpha_1(p_1), ..., \alpha_m(p_m)|\mathcal{O}(x)|\alpha_{m+1}(p_{m+1}), ..., \alpha_n(p_n)\rangle^{\text{in}} , \tag{2.4}$$

which we call a "generalized form factor". The absence of particle production implies that these matrix elements have simple analytic behaviour in terms of rapidity difference variables defined by

$$p_i p_j + i\epsilon = m_i m_j \operatorname{ch} \theta_{ij} , \tag{2.5}$$

with $\theta_{ij} = |\theta_i - \theta_j|$ and $p_k = m_k(\mathrm{ch}\theta_k, \mathrm{sh}\,\theta_k)$. We define a vector function $F_\alpha(\theta)$, where $\alpha = (\alpha_1, ..., \alpha_n)$, $\theta = (\theta_{ij}; 1 \leqslant i < j \leqslant n)$ (there are, of course, only $n-1$ independent variables θ_{ij}) by

$$\langle 0|\mathcal{O}(x)|\alpha_1(p_1), ..., \alpha_n(p_n)\rangle^{\mathrm{in}} = \mathrm{e}^{-ix(p_1+...+p_n)}\, F_\alpha(\theta)\,. \tag{2.6}$$

CPT invariance implies (since $p_ip_j - i\epsilon$ corresponds to $-\theta_{ij}^* = -\theta_{ij}$)

$$\langle 0|\mathcal{O}(0)|\alpha_1(p_1), ..., \alpha_n(p_n)\rangle^{\mathrm{out}} = F_\alpha(-\theta)\,, \tag{2.7}$$

and crossing implies (since $-p_ip_j - i\epsilon$ corresponds to $\hat{\theta}_{ij} = i\pi - \theta_{ij}$)

$$^{\mathrm{out}}\langle\alpha_1(p_1), ..., \alpha_m)|\mathcal{O}(0)|\, ..., \alpha_n(p_n)\rangle^{\mathrm{in}} = F_\alpha(\theta_{ij}, \hat{\theta}_{rs}, \theta_{kl})\,, \tag{2.8}$$

where $1 \leqslant i < j \leqslant m$, $1 \leqslant r \leqslant m < s \leqslant n$, and $m < k < l \leqslant n$.

Let us first derive Watson's equations for the case $n = 2$. Using unitarity we have

$$F_{\alpha_1\alpha_2}(\theta_{12}) = \langle 0|\mathcal{O}(0)|\alpha_1(p_1), \alpha_2(p_2)\rangle^{\mathrm{in}}$$

$$= \sum_{n'} \langle 0|\mathcal{O}(0)|n'\rangle^{\mathrm{out}}\ {}^{\mathrm{out}}\langle n'|\alpha_1(p_1), \alpha_2(p_2)\rangle^{\mathrm{in}}\,. \tag{2.9}$$

With *CPT* invariance, eq. (2.7), factorization, eq. (2.1), and eq. (2.3) we then obtain

$$F_{\alpha_1\alpha_2}(\theta_{12}) = F_{\alpha_1'\alpha_2'}(-\theta_{12})\, {}_{\alpha_1'\alpha_2'}S^{(2)}_{\alpha_1\alpha_2}(\theta_{12}) \tag{2.10}$$

and

$$F_{\alpha_1\alpha_2}(\hat{\theta}_{12}) = F_{\alpha_1\alpha_2}(-\hat{\theta}_{12}^*) \tag{2.11}$$

or

$$F_{\alpha_1\alpha_2}(i\pi - \theta_{12}) = F_{\alpha_1\alpha_2}(i\pi + \theta_{12})\,.$$

The last equation (2.11) is obtained by starting with $\langle\alpha_1(p_1)|\mathcal{O}(0)|\alpha_2(p_2)\rangle$ using the crossing relation, eq. (2.8), and the fact that the one-particle S-matrix is trivial.

For arbitrary n the generalized Watson theorem reads for $1 \leqslant m \leqslant n$

$$F_\alpha(\theta_{ij}, \hat{\theta}_{rs}, \theta_{kl})$$
$$= {}_{\alpha_1...\alpha_m}S^{(m)}_{\alpha_1'...\alpha_m'}(\theta_{ij})\, F_{\alpha'}(-\theta_{ij}, -\hat{\theta}^*_{rs}, -\theta_{kl})_{\alpha_{m+1}'...\alpha_n'}S^{(n-m)}_{\alpha_{m+1}...\alpha_n}(\theta_{kl})\,, \tag{2.12}$$

where the conventions are the same as in eq. (2.8).

We do not yet know methods to find solutions of the set of matrix equations (2.12) for the form factor functions $F_\alpha(\theta)$ in the most general case. In this paper we discuss two simple cases: $n = 2, N =$ arbitrary and $n = 3, N = 1$. For $n = 2$ and $N =$ arbitrary the matrix equations (2.12) can be simplified by diagonalizing the two-particle S-matrix. This results in a set of decoupled equations with the S-matrix eigen-

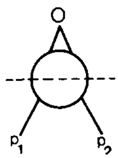

Fig. 1. General two-particle form-factor diagram.

values $S(\theta)$

$$\begin{aligned} F(\theta) &= F(-\theta)\, S(\theta)\,, \\ F(i\pi - \theta) &= F(i\pi + \theta)\,. \end{aligned} \tag{2.13}$$

Theorem. If $F(\theta)$ is meromorphic in the physical strip $0 \leqslant \operatorname{Im} \theta \leqslant \pi$ with possible poles (or zeros) only on the imaginary axis and $F(\theta) = O(\exp \exp|\theta|)$ for $|\operatorname{Re} \theta| \to \infty$, the solutions of eqs. (2.13) are uniquely determined by the poles at $\theta = i\alpha_k$ (and zeros) up to a normalization constant. They can be written as

$$F(\theta) = K(\theta)\, F^{\min}(\theta)\,, \tag{2.14}$$

where the minimal solution of eqs. (2.13) $F^{\min}$ (with $F^{\min}(i\pi) = 1$) has no poles (and zeros) in the physical strip and $K(\theta)$ is a solution of

$$K(\theta) = K(-\theta) = K(2\pi i + \theta)\,. \tag{2.15}$$

Remarks. (a) Poles of $F(\theta)$ will be determined by one-particle states in the channel corresponding to the S-matrix eigenvalue denoted by the dashed line in fig. 1.

(b) There are necessarily zeros at threshold $\theta = 0$ if $S(0) = -1$.

For all examples we discuss in sects. 3, 4 and 5 we make the minimality assumption that there are no zeros away from threshold in the physical strip. We give the proof of the theorem for the case that there are only poles present: generalizations are simple.

Proof. Let $i\alpha_1, \ldots, i\alpha_L$ be the locations of poles of $F(\theta)$ in the physical strip and

$$K(\theta) = \text{const} \prod_{k=1}^{L} \left[\operatorname{sh} \tfrac{1}{2}(\theta - i\alpha_k)\, \operatorname{sh} \tfrac{1}{2}(\theta + i\alpha_k)\right]^{-1}\,, \tag{2.16}$$

which is a solution of eqs. (2.15), then $F^{\min}(\theta) = F(\theta)/K(\theta)$ is analytic in $0 \leqslant \operatorname{Im} \theta \leqslant \pi$ and satisfies eqs. (2.13). Cauchy's theorem implies if C is a contour enclosing the strip $0 \leqslant \operatorname{Im} \theta \leqslant 2\pi$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\ln F^{\min}(\theta)\right] &= \frac{1}{8\pi i} \int_{C} \frac{\mathrm{d}z}{\operatorname{sh}^2 \frac{1}{2}(z - \theta)} \ln F^{\min}(z) \\ &= \frac{1}{8\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\operatorname{sh}^2 \frac{1}{2}(z - \theta)} \ln \frac{F^{\min}(z)}{F^{\min}(z + 2\pi i)} \end{aligned}$$

$$= \frac{1}{8\pi i} \int_{-\infty}^{\infty} \frac{dz}{\mathrm{sh}^2 \frac{1}{2}(z-\theta)} \ln S(z) . \tag{2.17}$$

This means we can calculate the (normalized) minimal solution $F^{\min}(\theta)$ uniquely from the S-matrix (eigenvalue) $S(\theta)$. In practice one can use a simpler formula. If the S-matrix is given by an integral representation

$$S(\theta) = \exp \int_0^{\infty} dx\, f(x)\, \mathrm{sh}\, x \frac{\theta}{i\pi} , \tag{2.18}$$

then

$$F^{\min}(\theta) = \exp \int_0^{\infty} dx\, f(x) \frac{\sin^2 [(i\pi - \theta)/2\pi]}{\mathrm{sh}\, x} . \tag{2.19}$$

If the operator $\mathcal{O}(x)$ is connected with a "charge" such that $|\alpha(p)\rangle$ is an eigenstate, the normalization constant can be determined from

$$\langle \alpha(p)|\mathcal{O}(x)|\alpha(p)\rangle = F(i\pi) . \tag{2.20}$$

In the following sections we apply this theorem to calculate the SG soliton (i.e. the MTM fermion), GN and NLS-model form factors.

The other simple case for Watson's equations (2.12) is where n is arbitrary but $N = 1$ (i.e. there is only one kind of particle). Then the two-particle S-matrix is only a number and the factors in eq. (2.2) commute. The solutions of the generalized Watson equations are

$$F(\theta_{12}, ...) = K(\theta_{12}, ...) \prod_{i<j} F^{\min}(\theta_{ij}) , \tag{2.21}$$

where K satisfies eqs. (2.12) with $S = 1$, and $F^{\min}$ is given by eq. (2.17) or (2.19). The poles of K will again be determined by one-particle states in all subchannels $(\alpha_k, ..., \alpha_l) \subset (\alpha_1, ..., \alpha_n)$, *cf.* fig. 2.

In sect. 4 we consider the case $n = 3, N = 1$ to calculate the matrix element of the sine-Gordon field with three elementary bosons:

$$\langle 0|\phi(x)|b_1(p_1)\, b_1(p_2)\, b_1(p_3)\rangle^{\text{in}} .$$

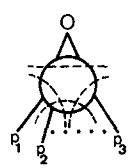

Fig. 2. General n-particle form-factor diagram.

3. Sine-Gordon soliton form factor and wave function renormalization constant

The sine-Gordon (SG) soliton form factor (i.e. the form factor of the fermion f in the massive Thirring model (MT)) has been calculated previously [18]. The diagonalized two-particle MT model S-matrix is [19]

$$S^{\mathrm{MT}}(\theta,\lambda)=\begin{bmatrix} S_{\mathrm{ff}} & & & 0 \\ & S_{\mathrm{f\bar f}}^{(+)} & & \\ & & S_{\mathrm{f\bar f}}^{(-)} & \\ 0 & & & S_{\mathrm{\bar f\bar f}} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\dfrac{\mathrm{sh}\frac{1}{2}\lambda(\theta+i\pi)}{\mathrm{sh}\frac{1}{2}\lambda(\theta-i\pi)} & 0 & 0 \\ 0 & 0 & -\dfrac{\mathrm{ch}\frac{1}{2}\lambda(\theta+i\pi)}{\mathrm{ch}\frac{1}{2}\lambda(\theta-i\pi)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times S_{\mathrm{ff}}(\theta,\lambda)\,, \tag{3.1}$$

where

$$S_{\mathrm{ff}}(\theta,\lambda)=\exp\int_0^\infty \frac{\mathrm{d}x}{\mathrm{d}}\,\frac{\mathrm{sh}\,\frac{1}{2}x(1-1/\lambda)}{\mathrm{sh}(x/2\lambda)\,\mathrm{ch}\,\frac{1}{2}x}\,\mathrm{sh}\,x\,\frac{\theta}{i\pi}\,. \tag{3.2}$$

The parameter λ is related to the MT and SG coupling constants g and β, respectively, by

$$\lambda=1+\frac{2g}{\pi}=\frac{8\pi}{\beta^2}-1\,. \tag{3.3}$$

The fermion-antifermion S-matrix eigenvalues $S_{\mathrm{f\bar f}}^{(\pm)}$ correspond to positive (negative) C-parity. Watson's equations read

$$F^{\mathrm{MT}}(\theta,\lambda)=F^{\mathrm{MT}}(-\theta,\lambda)\,S^{\mathrm{MT}}(\theta,\lambda)\,,\qquad F^{\mathrm{MT}}(i\pi-\theta,\lambda)=F^{\mathrm{MT}}(i\pi+\theta,\lambda)\,. \tag{3.4}$$

For $g<0$ (i.e. $\lambda<1$) $F^{\mathrm{MT}}(\theta,\lambda)$ has no poles in the physical strip, since in the repulsive region $g<0$ there are no $\mathrm{f\bar f}$ bound states (we assume absence of redundant poles). Hence, for $\lambda<1, F^{\mathrm{MT}}(\theta,\lambda)=F^{\mathrm{MT,min}}(\theta,\lambda)$ and the solution of eqs. (3.4) is given by

$$\begin{aligned} F^{\mathrm{MT}}(\theta,\lambda)&=(F_{\mathrm{ff}}^{\mathrm{MT}},F_{+}^{\mathrm{MT}},F_{-}^{\mathrm{MT}},F_{\mathrm{\bar f\bar f}}^{\mathrm{MT}}) \\ &=\left(1\,,\ \frac{\mathrm{sh}\,\frac{1}{2}(i\pi-\theta)}{\mathrm{sh}\,\frac{1}{2}\lambda(i\pi-\theta)}\,,\ \frac{\mathrm{ch}\,\frac{1}{2}(i\pi-\theta)}{\mathrm{ch}\,\frac{1}{2}\lambda(i\pi-\theta)}\,,\ 1\right)F_{\mathrm{ff}}^{\mathrm{MT}}(\theta,\lambda)\,, \end{aligned} \tag{3.5}$$

where

$$F_{\mathrm{ff}}^{\mathrm{MT}}(\theta,\lambda)=\exp\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\mathrm{sh}\,\frac{1}{2}x(1-1/\lambda)}{\mathrm{sh}(x/2\lambda)\,\mathrm{ch}\,\frac{1}{2}x}\,\frac{\sin^2(x\hat\theta/2\pi)}{\mathrm{sh}\,x}\,, \tag{3.6}$$

and for $\lambda > 1$ by means of analytic continuation.

The electromagnetic current form factor is given by

$$\langle f(p_1)|j^\mu(0)|f(p_2)\rangle = \bar{u}(p_1)\,\gamma^\mu u(p_2)\,F_-^{\mathrm{MT}}(i\pi - \theta_{12}, \lambda)\,, \tag{3.7}$$

since j^μ is odd under charge conjugation. This formula has been checked in perturbation theory around $\lambda = 1$ and in the classical limit $\lambda \to \infty$ [18]. As interesting properties we note:

(i) the asymptotic ($t \to -\infty$) behaviour of the form factor is

$$F_-^{\mathrm{MT}} \sim (-t)^{g/2\pi}\,;$$

(ii) the charge radius is given by

$$R^2 \equiv \frac{\mathrm{d}}{\mathrm{d}t} F_-^{\mathrm{MT}}|_{t=0} = \frac{1}{8m^2}\left(\lambda^2 - \frac{1}{\pi^2}\int_0^\infty \frac{\mathrm{d}x\, x\, \mathrm{sh}\,\frac{1}{2}x(1-1/\lambda)}{\mathrm{sh}\,x\;\mathrm{sh}(x/2\lambda)\;\mathrm{ch}\,\frac{1}{2}x} - 1\right), \tag{3.8}$$

with expected behaviour $R^2 \sim \lambda^2/8m^2$ in the limit $\lambda \to \infty$.

The lowest fermion-antifermion bound state b_1 with mass $m_1 = 2m\sin(\pi/2\lambda)$ (existing for $\lambda > 1$), which is the elementary boson corresponding to the sine-Gordon field $\phi(x)$, can be built up only by an $\mathrm{f}\bar{\mathrm{f}}$ pair [17]. Hence, the one-particle pole of the two-point function $\langle T\,\phi(x)\,\phi(y)\rangle$ in momentum space at $k^2 = m_1^2$ comes only from $\mathrm{f}\bar{\mathrm{f}}$ intermediate states. By means of eqs. (3.5), (3.6) and (3.7) we can calculate these contributions exactly, since Coleman's correspondence [4] relates ϕ and j_μ by

$$\epsilon^{\mu\nu}\partial_\nu\phi = -\frac{2\pi}{\beta}j^\mu\,. \tag{3.9}$$

The quantized sine-Gordon field theory is finite after "tadpole" subtractions by normal ordering (*cf.* appendix A). We determine exactly the SG wave-function renormalization constant, defined for $\lambda > 1$ by $\langle 0|\phi(0)|b_1\rangle = \sqrt{Z(\lambda)}$ or

$$\langle \phi(x)\,\phi(y)\rangle = Z(\lambda)\,\Delta_+(x-y, m_1^2) + \text{contributions from larger masses}\,, \tag{3.10}$$

or in momentum space

$$\langle \mathrm{T}\,\tilde{\phi}(k)\,\phi(0)\rangle \approx Z(\lambda)\frac{i}{k^2 - m_1^2} \quad \text{at } k^2 \approx m_1^2\,.$$

We consider first the case $\lambda < 1$ where the bound state b_1 does not exist and the SG-propagator has no pole but only a cut starting at $k^2 = 4m^2$. The contribution from fermion-antifermion intermediate states is given by

$$\langle \phi(x)\,\phi(y)\rangle^{\mathrm{f}\bar{\mathrm{f}}} = \int \frac{\mathrm{d}p_1\,\mathrm{d}p_2}{4\pi\omega_1 4\pi\omega_2}\,\langle 0|\phi(x)|f(p_1)\,\bar{f}(p_2)\rangle^{\mathrm{in}}\;{}^{\mathrm{in}}\langle f(p_1)\,\bar{f}(p_2)|\phi(y)|0\rangle$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta_{12}\, g(\theta_{12})\, g(-\theta_{12})\, \Delta_+(x-y, (p_1+p_2)^2)\,, \tag{3.11}$$

where $(p_1+p_2)^2 = 2m^2(1+\operatorname{ch}\theta_{12})$,

$$\Delta_+(x, M^2) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \mathrm{e}^{-ikx} \theta(k_0)\, 2\pi\delta(k^2 - M^2)\,,$$

$$g(\theta_{12}) = \langle 0|\phi(0)|f(p_1)\bar{f}(p_2)\rangle^{\mathrm{in}}$$

$$= \frac{2\pi}{\beta} \frac{1}{\operatorname{ch}\frac{1}{2}\theta_{12}} F_-^{\mathrm{MT}}(\theta_{12}, \lambda)\,. \tag{3.12}$$

The last equation has been obtained from eqs. (3.7) and (3.9).

The function $g(\theta)$ has a pole at $\theta = ia = i\pi(1 - 1/\lambda)$ corresponding to the bound state b_1 (for $\lambda < 1$ in the unphysical sheet). This pole crosses the integration path, if λ changes from $\lambda < 1$ to $\lambda > 1$. Hence, we obtain by analytic continuation from (3.11) for $\lambda > 1$:

$$\langle\phi(x)\,\phi(y)\rangle^{\mathrm{f\bar{f}}} = \frac{1}{4\pi}\Big[\oint_{ia} + \oint_{-ia} + \int_{-\infty}^{\infty}\Big]\, \mathrm{d}\theta_{12} g(\theta_{12})\, g(-\theta_{12})\, \Delta_+(x-y, (p_1+p_2)^2)$$

$$= \frac{1}{2\pi} \oint_{ia} \mathrm{d}\theta\, g(\theta)\, g(-\theta)\, \Delta_+(x-y, m_1^2) + \dots\,. \tag{3.13}$$

Since the bound state b_1 can only be built up by an $\mathrm{f\bar{f}}$ pair [17], we can determine the sine-Gordon wave-function renormalization constant by comparing eqs. (3.10) and (3.13). After some calculation using eqs. (3.12), (3.5) and (3.6) it follows:

$$Z(\lambda) = \frac{1}{2\pi} \oint_{ia} \mathrm{d}\theta\, g(\theta)\, g(-\theta)$$

$$= \left(1 + \frac{1}{\lambda}\right)\left(\frac{2\lambda}{\pi} \sin\frac{\pi}{2\lambda}\right)^{-1} \exp\left[-\frac{1}{\pi}\int_0^{\pi/\lambda} \mathrm{d}x \frac{x}{\sin x}\right]. \tag{3.14}$$

For $\lambda \to 1$ the MTM becomes free and the bound state b_1 decays into an $\mathrm{f\bar{f}}$ pair. This fact is reflected by the relation

$$Z(1 + 2g/\pi) = \text{const.}\, g + \mathrm{O}(g^2)\,. \tag{3.15}$$

For $\lambda \to \infty$ the SG theory becomes free and

$$Z\left(\frac{8\pi}{\beta^2} - 1\right) = 1 - \left(\frac{\beta^2}{8\pi}\right)^2 \left(\tfrac{1}{2} - \tfrac{1}{24}\pi^2\right) + \mathrm{O}(\beta^6)\,. \tag{3.16}$$

For arbitrary λ with $1 \leqslant \lambda \leqslant \infty$ we have $0 \leqslant Z(\lambda) \leqslant 1$ which is a general consequence of positivity [20]. The first non-trivial contribution in SG perturbation theory is calculated in appendix A. The result, eq. (A.7), agrees with the proposed exact formula (3.14) or (3.16).

4. Form factors in the sine-Gordon theory

In this section we turn to some examples of exact form factors for the elementary bosons of the sine-Gordon theory (alias the lowest lying bound states in the Thirring model in the attractive region with mass $m_1 = 2m \sin(\pi/2\lambda)$). The necessary ingredient is the S-matrix element $S_{\mathrm{bb}}(\theta, \lambda)$ for the scattering of two such bosons. This is determined [17] as the residue at the appropriate poles in the (presumably) exact Zamolodchikov [19] S-matrix for the scattering of an initial four-particle state of two solitons and two antisolitons to be simply given by

$$S_{\mathrm{bb}}(\theta, \lambda) = \frac{\operatorname{sh}\theta + i\sin(\pi/\lambda)}{\operatorname{sh}\theta - i\sin(\pi/\lambda)} = -\exp 2\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\operatorname{ch} x(\frac{1}{2} - 1/\lambda)}{\operatorname{ch}\frac{1}{2}x}\,\operatorname{sh} x\,\frac{\theta}{i\pi}\,, \qquad (4.1)$$

a result which is in agreement with perturbation theory to order β^6 if the usual identification

$$\lambda = \frac{8\pi}{\beta^2} - 1$$

is made.

We first consider the minimal form factor for a scalar operator $\mathcal{O}(x)$ connecting the two-particle state to the vacuum

$$\langle 0|\mathcal{O}(0)|b_1(p_1)\,b_1(p_2)\rangle^{\mathrm{in}} = F_{\mathrm{bb}}^{\mathrm{SG}}\,(\theta_{12}, \lambda)\,. \qquad (4.2)$$

Using the general methods described in sect. 2 together with the particular form for the S-matrix (4.1), we directly obtain

$$F_{\mathrm{bb}}^{\mathrm{SG,min}}(\theta, \lambda) = \operatorname{ch}\tfrac{1}{2}\hat{\theta}\exp 2\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\operatorname{ch} x(\frac{1}{2} - 1/\lambda)}{\operatorname{ch}\frac{1}{2}x}\,\frac{\sin^2(x\hat{\theta}/2\pi)}{\operatorname{sh} x}\,. \qquad (4.3)$$

As explained before, the full form factor of a specific operator $\mathcal{O}(x)$ depends crucially on further knowledge of the allowed poles. For example, with $\mathcal{O}(x) = :\phi^2:(x)$ we consider it probable that only the b_2 pole at $(p_1 + p_2)^2 = m_2^2 = 4m^2 \sin^2\pi/\lambda$

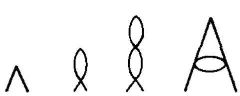

Fig. 3. Diagrams contributing to the sine-Gordon two-particle matrix element of the operator ϕ^2 up to order β^4.

[17] is present and, hence, postulate

$$F_{\mathrm{bb}}^{\mathrm{SG}}(\theta,\lambda) = -2ZZ_{(2)}K_{\mathrm{bb}}(\theta,\lambda)\,F_{\mathrm{bb}}^{\mathrm{SG,min}}(\theta,\lambda)\,,$$

$$K_{\mathrm{bb}}(\theta,\lambda) = \frac{1}{\operatorname{sh}\frac{1}{2}(\theta - i\pi/\lambda)\operatorname{sh}\frac{1}{2}(\theta + i\pi/\lambda)}\,. \tag{4.4}$$

The normalization constant $Z_{(2)}$ can be calculated by means of the asymptotic behaviour. Weinberg's power counting implies that in the limit of infinite momentum transfer the form factor tends to a constant given by the first diagram of fig. 3:

$$F_{\mathrm{bb}}^{\mathrm{SG}}(\theta,\lambda) \to 2Z \qquad \text{as } \hat{\theta} = i\pi - \theta \to \infty\,. \tag{4.5}$$

For $Z_{(2)}$ we obtain after some calculation

$$Z_{(2)} = Z^{-1}\left(1 + \frac{1}{\lambda}\right)\frac{\pi}{2\lambda}\cot\frac{\pi}{2\lambda}\,. \tag{4.6}$$

The diagrams contributing to the amputated three-point function are shown in fig. 3 and yield a contribution to the form factor

$$F_{\mathrm{bb}}^{\mathrm{SG}}(\theta,\lambda) =$$

$$2Z\left\{1 + \frac{\beta^2}{8\pi}\frac{\hat{\theta}}{\operatorname{sh}\theta} + \left(\frac{\beta^2}{8\pi}\right)^2\left[\frac{\hat{\theta}^2}{2\operatorname{sh}^2\theta} + \frac{\hat{\theta}}{\operatorname{sh}\theta} - \frac{\pi^2}{8\operatorname{sh}^2\frac{1}{2}\theta}\right] + O(\beta^6)\right\}\,. \tag{4.7}$$

It can be checked that this perturbation theoretic result is in complete agreement with the proposed exact expression (4.4) with $Z_{(2)}$ given by

$$Z_{(2)} = 1 + \frac{\beta^2}{8\pi} + \left(\frac{\beta^2}{8\pi}\right)^2\left(\tfrac{3}{2} - \tfrac{1}{2}\pi^2\right) + O(\beta^6)\,,$$

which agrees with eq. (4.6).

Furthermore, the leading threshold behaviour of the proposed exact form factor agrees with perturbation theory to all orders β^{2n}. To see this we note that the leading threshold behaviour in each order is given by the factor $\operatorname{sh}\frac{1}{2}\theta/\operatorname{sh}\frac{1}{2}(\theta - i\pi/\lambda)$ in (4.4), the remaining part being finite at threshold order by order. Hence, the leading threshold behaviour in nth order is given by

$$F_{\mathrm{bb}}^{\mathrm{SG}(n)}(\theta) \sim 2\left(\frac{\beta^2}{8\pi}\right)^n\left(\frac{i\pi}{\theta}\right)^n \quad \text{as } \theta \to 0\,, \tag{4.8}$$

which is the same as that of the chain diagram, fig. 4.

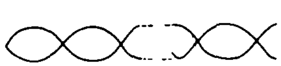

Fig. 4. Sine-Gordon chain diagram.

Our next example is the matrix element of the field between three lowest lying boson state and the vacuum

$$\langle 0| - (\Box + m_1^2)\, \phi(0) |b_1(p_1)\, b_1(p_2)\, b_1(p_3)\rangle^{\mathrm{in}} = F^{\mathrm{SG}}_{\mathrm{bbb}}(\theta_{12}, \theta_{13}, \theta_{23})\,. \tag{4.9}$$

We propose the exact $F^{\mathrm{SG}}_{\mathrm{bbb}}$ to be given by

$$F^{\mathrm{SG}}_{\mathrm{bbb}}(\theta_{12}, \theta_{13}, \theta_{23}) = \frac{8\pi}{\lambda} m_1^2\, Z_{(3)} Z^{3/2} \prod_{1 \leqslant i < j \leqslant 3} K_{\mathrm{bb}}(\theta_{ij}, \lambda)\, F^{\mathrm{SG\ min}}_{\mathrm{bb}}(\theta_{ij}, \lambda)\,. \tag{4.10}$$

Note that we suggest no other explicit factors corresponding to higher bound states to be present. However, there is a pole at $(p_1 + p_2 + p_3)^2 = m_3^2$ on submanifolds where a subenergy $(p_i + p_j)^2$ is on the m_2^2 mass shell. This is analogous to the situation in the three-particle S-matrix element [21]. By crossing one of the particles and using the LSZ formalism (4.10) should reproduce the 2-2 T-matrix element. Indeed it can be checked that

$$\frac{1}{4m_1^2 \operatorname{sh}\theta}\, \frac{1}{iZ^{1/2}}\, F^{\mathrm{SG}}_{\mathrm{bbb}}(\theta, i\pi, i\pi - \theta) = S_{\mathrm{bb}}(\theta) - 1\,, \tag{4.11}$$

provided that

$$Z_{(3)} = \frac{(1 + 1/\lambda)^2 (\lambda/\pi) \sin(\pi/\lambda) \cos^4(\pi/2\lambda)}{Z^3((2\lambda/\pi)\sin(\pi/2\lambda))^2} = 1 + \frac{2}{\lambda} + \frac{1}{\lambda^2}\left(\tfrac{5}{2} - \tfrac{17}{24}\pi^2\right) + \ldots\,. \tag{4.12}$$

The perturbation theory checks up to one-loop are easily performed, the diagrams being simply given by fig. 5. However, the two-loop diagrams in fig. 6 are more involved to analyse. We will not give all the details here but assert that (4.10) is in agreement with perturbation theory to order β^6 (see appendix A).

Finally, we can calculate the electromagnetic transition form factor for bound states b_1 and b_2,

$$\langle b_2(p_1)| j_\mu(0) |b_1(p_2)\rangle = i\epsilon_{\mu\nu}(p_1 - p_2)^\nu\, F_{\mathrm{b_2 b_1}}(i\pi - \theta_{12})\,, \tag{4.13}$$

by taking the residue of eq. (4.9) at $\theta_{12} = i\pi/\lambda$. We obtain

$$F_{\mathrm{b_2 b_1}}(\theta) = Z_{(2,1)}\, K_{\mathrm{b_2 b_1}}(\theta, \lambda)\, F^{\mathrm{SG,min}}_{\mathrm{b_2 b_1}}(\theta, \lambda)\,, \tag{4.14}$$

Fig. 5. Sine-Gordon four-point vertex function diagrams up to order β^4.

Fig. 6. Sine-Gordon four-point vertex function diagrams in order β^6.

where the pole corresponding to the bound state b_3 is given by

$$K_{b_2 b_1} = \frac{1}{\operatorname{sh} \frac{1}{2}(\theta - 3\pi i/2\lambda) \operatorname{sh} \frac{1}{2}(\theta + 3\pi i/2\lambda)}, \tag{4.15}$$

$$F^{\mathrm{SG,min}}_{b_2 b_1}(\theta, \lambda) = \exp 4 \int_0^\infty \frac{dx}{x} \frac{\operatorname{ch} x(\frac{1}{2} - 1/\lambda) \operatorname{ch}(x/2\lambda)}{\operatorname{ch} \frac{1}{2}x} \frac{\sin^2(x\hat{\theta}/2\pi)}{\operatorname{sh} x}. \tag{4.16}$$

The function $F^{\mathrm{SG,min}}_{b_2 b_1}$ is, of course, the minimal solution of Watson's eqs. (2.13) with [17]

$$S_{b_2 b_1}(\theta, \lambda) = \frac{\operatorname{sh}\theta + i \sin(3\pi/2\lambda)}{\operatorname{sh}\theta - i\sin(3\pi/2\lambda)} \frac{\operatorname{sh}\theta + i\sin(\pi/2\lambda)}{\operatorname{sh}\theta - i\sin(\pi/2\lambda)}. \tag{4.17}$$

In the limit of infinite momentum transfer the transition form factor tends to a constant.

5. Form factors in the non-linear σ-model and Gross-Neveu model

In this section we study two-particle form factors in the O(N) non-linear σ-model and the SU(N) (or more precise O($2N$)) Gross-Neveu model. (For the GN model we use O($2N$) notations, i.e. we combine the SU(N) fermions f_α and their antifermions $\overline{f}_\alpha(\alpha = 1, ..., N)$ to $2N$ self-conjugate fermions f_i.) Both models have recently been shown to possess infinite sets of conserved charges which at the quantum level imply absence of particle production and factorization. Assuming the situations to be so, Zamolodchikov and Zamolodchikov [12] previously analysed the constraints of factorisation on the 2-2 S-matrix elements and further incorporating the qualitative information on the spectrum of the models from semiclassical analyses proposed exact $2 \to 2$ S-matrix elements for both models. For the non-linear O(N) σ-model

$${}_{kl}S^{\mathrm{NLS}}_{ij}(\theta, N) = {}_{kl}S^{\min}_{ij}(\theta, N), \tag{5.1}$$

and for the O($2N$) Gross-Neveu model

$${}_{kl}S^{\mathrm{GN}}_{ij}(\theta, 2N) = \frac{\operatorname{sh}\theta + i\sin\pi/(N-1)}{\operatorname{sh}\theta - i\sin\pi/(N-1)} {}_{kl}S^{\min}_{ij}(\theta, 2N), \tag{5.2}$$

where $S^{\min}(\theta, N)$ is the unique minimal O(N)-symmetric S-matrix which is consistent with factorization. The three eigenvalues of $S^{\min}$ belonging to the scalar, symmetric traceless, and antisymmetric tensor representations, respectively, are

$$(S^{\min}_0, S^{\min}_+, S^{\min}_-) = \left(\frac{\theta + i\pi}{\theta - i\pi}, \frac{\theta - \dfrac{2\pi i}{N-2}}{\theta + \dfrac{2\pi i}{N-2}}, 1\right) S^{\min}_-(\theta, N), \tag{5.3}$$

where

$$S_-^{\min}(\theta, N) = \exp 2 \int_0^\infty \frac{dx}{x} \frac{e^{-2x/(N-2)} - 1}{e^x + 1} \operatorname{sh} x \frac{\theta}{i\pi} . \tag{5.4}$$

These expressions have been checked in perturbation theory to order $1/N^2$ [12,15] and we thus have some confidence in their validity.

We first consider the form factor of the O(N) current

$$J_{ij}^\mu = n_i \partial^\mu n_j - n_j \partial^\mu n_i \tag{5.5}$$

in the non-linear σ-model

$$\langle 0|J_{ij}^\mu(0)|b_k(p_1)\, b_2(p_2)\rangle^{\text{in}} = i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})(p_1 - p_2)^\mu F_-^{\text{NLS}}(\theta, N) . \tag{5.6}$$

The phase shift in the channel corresponding to the relevant O(N) representation (antisymmetric tensor of rank two) is given by eq. (5.4). Hence, using the general methods described in sect. 2 we postulate the current form factor F_-^{NLS} to be precisely the minimum one

$$F_-^{\text{NLS}}(\theta, N) = \exp 2 \int_0^\infty \frac{dx}{x} \frac{e^{-2x/(N-2)} - 1}{e^x + 1} \frac{\sin^2(x\hat{\theta}/2\pi)}{\operatorname{sh} x} . \tag{5.7}$$

We have checked that this expression agrees with perturbation theory in the NLS model to order $1/N$ (we relegate the details to appendix B). We note that the form factor tends to zero as a power of the logarithm of $-t$, the momentum transfer, as $t \to -\infty$

$$F_-^{\text{NLS}}(\theta, N) \sim \hat{\theta}^{-1/(N-2)} \qquad \text{as } \hat{\theta} = i\pi - \theta \to \infty . \tag{5.8}$$

This is to be anticipated from the asymptotic freedom of the model.

Considerations of the form factor of the O($2N$) current $J_{ij}^\mu = \bar{\psi}_i \gamma^\mu \psi_j$ in the Gross-Neveu model are analogous. The only important difference being that we must take into account the presence of bound-state poles. Defining the form factor

$$\langle 0|J_{ij}^\mu(0)|f_k(p_1) f_l(p_2))\rangle^{\text{in}} = i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\, \bar{v}(p_1)\, \gamma^\mu u(p_2)\, F_-^{\text{GN}}(\theta, 2N) , \tag{5.9}$$

we make the usual minimality assumptions and propose

$$F_-^{\text{GN}}(\theta, 2N) = \cos^2 \frac{\pi}{2(N-1)} K_{\text{bb}}(\theta, N-1)\, F_{\text{bb}}^{\text{SG,min}}(\theta, N-1)\, F_-^{\text{NLS}}(\theta, 2N) , \tag{5.10}$$

where $K_{\text{bb}}(\theta, \lambda)\, F_{\text{bb}}^{\text{SG, min}}(\theta, \lambda)$ was introduced in the discussion of the sine-Gordon theory form factors (4.4). Again we have checked agreement of (5.10) in perturbation theory to order $1/N$ (see appendix B). The form factor $F_-^{\text{GN}}(\theta, 2N)$ also decreases logarithmically for large momentum transfer

$$F_-^{\text{GN}}(\theta, 2N) \sim \hat{\theta}^{-1/(2N-2)} \qquad \text{as } \hat{\theta} = i\pi - \theta \to \infty . \tag{5.11}$$

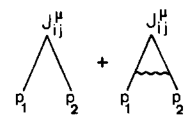

Fig. 7. Diagrams contributing to the O(N) current form factor in the non-linear σ- and Gross-Neveu models up to order N^{-1}.

Finally, we calculate the form factor of the isoscalar operator $\Sigma_i \overline{\psi}_i \psi_i(x)$ defined by

$$\langle 0 | \overline{\psi}\psi(0) | f_i(p_1) f_j(p_2) \rangle^{\text{in}} = \delta_{ij} \overline{v}(p_1) u(p_2) F_0^{\text{GN}}(\theta, 2N) , \tag{5.12}$$

with the normalization $F_0^{\text{GN}}(i\pi) = 1$. One easily sees from eqs. (5.2,3) that

$$F_0^{\text{GN}}(\theta, 2N) = 2 \frac{\text{th}\frac{1}{2}\hat{\theta}}{\hat{\theta}} F_-^{\text{GN}}(\theta, 2N) \tag{5.13}$$

satisfies Watson's equations with S_0^{GN} and has the right pole structure, since the scalar and antisymmetric tensor channel contains bound states at the same mass [22]. Perturbative checks of the proposed scalar form factor eq. (5.13) are much more complicated, since in contrast to eq. (5.10) not only one but 12 diagrams contribute to the order $1/N$ (see appendix B). For large momentum transfer the scalar form factor tends also to zero logarithmically:

$$F_0^{\text{GN}}(\theta, 2N) \sim \hat{\theta}^{-1/(2N-2)-1} \qquad \text{as } \hat{\theta} = i\pi - \theta \to \infty . \tag{5.14}$$

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Appendix A

The sine-Gordon theory is described by the Lagrangian

$$\mathcal{L}^{\text{SG}} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{\alpha}{\beta^2} (\cos \beta\phi - 1) . \tag{A.1}$$

In such a theory all ultraviolet divergences are removed by normal ordering of the interaction Hamiltonian in the interaction picture. This will in the following always be understood, with respect to the mass parameter $\alpha^{1/2}$.

The ϕ self-energy defined in the ϕ propagator

$$D_\phi(k^2) = \frac{i}{k^2 - \alpha - \Pi(k^2)} \tag{A.2}$$

is to lowest order given by the two-loop diagram in fig. 7:

$$\Pi(k^2) = -\alpha\left(\frac{\beta^2}{8\pi}\right)^2 \frac{2}{3\pi^2} K\left(\frac{k^2}{\alpha}\right) + O(\beta^6), \tag{A.3}$$

where

$$K(p^2) = \int d^2k_1 d^2k_2 \,[(k_1^2 - 1)(k_2^2 - 1)((p - k_1 - k_2)^2 - 1)]^{-1} . \tag{A.4}$$

The physical mass of the elementary SG boson (alias the lowest lying soliton-anti-solition bound state) and the (finite) wave function renormalization constant Z defined in

$$D_\phi(k^2) \approx \frac{iZ}{k^2 - m_1^2} \tag{A.5}$$

are then given (using $K(1) = \frac{1}{4}\pi^4$),

$$m_1^2 = \alpha + \Pi(m_1^2) = \alpha\left[1 - \left(\frac{\beta^2}{8\pi}\right)^2 \tfrac{1}{6}\pi^2 + O(\beta^6)\right] \tag{A.6}$$

and (using $K'(1) = \frac{1}{4}\pi^2(3 - \frac{1}{4}\pi^2)$),

$$\begin{aligned} Z^{-1} &= 1 - \Pi'(m_1^2) \\ &= 1 + \left(\frac{\beta^2}{8\pi}\right)^2 (\tfrac{1}{2} - \tfrac{1}{24}\pi^2) + O(\beta^6) . \end{aligned} \tag{A.7}$$

For the calculation of the diagrams of figs. 3, 4, and 5 the following identities have been used (with $p_i^2 = 1$ and $p^2 = (p_1 + p_2)^2 = -4\,\mathrm{sh}^2\,\frac{1}{2}\hat\theta$)

$$\int d^2k \frac{1}{k^2 - 1}\frac{1}{(k - p)^2 - 1} = i\pi\frac{\hat\theta}{\mathrm{sh}\,\hat\theta}, \tag{A.8}$$

$$\begin{aligned} &\int d^2k_1 d^2k_2 \frac{1}{(k_1 + p_2)^2 - 1}\frac{1}{(k_1 - k_2)^2 - 1}\frac{1}{(k_1 - p_1)^2 - 1}\frac{1}{k_2^2 - 1} \\ &= \tfrac{1}{4}\pi^2\left(\frac{\hat\theta^2}{\mathrm{sh}^2\hat\theta} - \frac{2\hat\theta}{\mathrm{sh}\,\hat\theta} - \frac{\pi^2}{4\,\mathrm{ch}^2\frac{1}{2}\hat\theta}\right), \end{aligned} \tag{A.9}$$

$$\begin{aligned} &\int d^2k_1 d^2k_2 \frac{1}{k_1^2 - 1}\frac{1}{k_2^2 - 1}\frac{1}{(p - k_1 - k_2)^2 - 1}\Bigg\{1 + \frac{1}{p^2 - 1} \\ &+ \left[\frac{1}{(k_1 - p_1 - p_2)^2 - 1} + \frac{1}{(k_2 - p_2 - p_3)^2 - 1} + \frac{1}{(p_1 - k_1 - k_2)^2 - 1}\right] \\ &+ [p_1 \leftrightarrow p_2] + [p_1 \leftrightarrow p_3]\Bigg\} = \tfrac{1}{4}\pi^4\frac{1}{p^2 - 1} - \tfrac{3}{2}\pi^2\left[\frac{\hat\theta_{12}}{\mathrm{sh}\,\hat\theta_{12}}\frac{\hat\theta_{23}}{\mathrm{sh}\,\hat\theta_{23}}\right. \end{aligned}$$

$$+\frac{\hat{\theta}_{23}}{\operatorname{sh}\hat{\theta}_{23}}\frac{\hat{\theta}_{31}}{\operatorname{sh}\hat{\theta}_{31}}+\frac{\hat{\theta}_{31}}{\operatorname{sh}\hat{\theta}_{31}}\frac{\hat{\theta}_{12}}{\operatorname{sh}\hat{\theta}_{12}}\Bigg]\,. \tag{A.10}$$

The expression of the left-hand side of eq. (A.10) is (up to factors) the contribution from the last three diagrams in fig. 6. Note that the bracket in the integrand is just the sum of trees occurring in scattering $\{p_1p_2p_3\} \to \{k_1k_2k_3\}$ the principal part of which vanishes when all k's are on-shell. We have not completely succeeded in demonstrating (A.10) although we believe in its validity. Indeed, we have shown (a) the equality on-shell (as stated above the exact S_{bb}) is in agreement with perturbation theory to order β^6, (b) the discontinuities over all three two-particle cuts taken simultaneously are equal on either side of eq. (A.10) (i.e. $\theta_{ij} \to -\theta_{ij}$), and finally (c) we have checked the relation at a non-trivial p^2 off-shell unphysical point, namely at $p = 0$, $\theta_{ij} = \frac{2}{3}\pi i$.

Appendix B

In this appendix we check the proposed exact form factors in the nonlinear σ- and Gross-Neveu models in $1/N$ expansion. Fig. 7 shows the diagrams contributing to F_-^{NLS} and F_-^{GN} in order N^0 and N^{-1} which give (besides isospin matrices)

$$i(p_1-p_2)_\mu F_-^{\mathrm{NLS}} = i(p_1-p_2)_\mu + i\int\frac{\mathrm{d}^2k}{(2\pi)^2}D^{\mathrm{NLS}}(k^2)$$

$$\times\left[\frac{(p_1-p_2-2k)_\mu}{((k-p_1)^2-m^2)((k+p_2)^2-m^2)} - \text{subtr.}\right] + \dots\,, \tag{B.1a}$$

$$i\bar{v}(p_2)\gamma^\mu u(p_1)F_-^{\mathrm{GN}} = i\bar{v}(p_2)\gamma^\mu u(p_1)$$

$$+\,i\int\frac{\mathrm{d}^2k}{(2\pi)^2}D^{\mathrm{GN}}(k^2)\left[\bar{v}(p_2)\frac{1}{\not{p}_2+\not{k}-m}\gamma^\mu\frac{1}{\not{p}_1-\not{k}-m}u(p_1) - \text{subtr.}\right]. \tag{B.1b}$$

We use spectral representation of the propagators $D(k)$ [23] (with $k^2 = -4m^2\operatorname{sh}^2\frac{1}{2}\kappa$ and $M^2 = 4m^2\operatorname{ch}^2\frac{1}{2}\phi$)

$$D^{\mathrm{NLS}}(k^2) = \frac{8\pi i}{N}m^2\frac{\operatorname{sh}\kappa}{\kappa} = \frac{8\pi i}{N}m^2\left(1-\frac{k^2}{2\pi i}\int_{4m^2}^{\infty}\mathrm{d}M^2\frac{\rho^{\mathrm{NLS}}(M^2)}{M^2(k^2-M^2)}\right), \tag{B.2a}$$

$$D^{\mathrm{GN}}(k^2) = -\frac{2\pi i}{N}\frac{\operatorname{th}\frac{1}{2}\kappa}{\kappa} = \frac{1}{N}\int_{4m^2}^{\infty}\mathrm{d}M^2\frac{\rho^{\mathrm{GN}}(M^2)}{k^2-M^2}\,, \tag{B.2b}$$

where

$$\rho^{\mathrm{NLS}}(M^2) = -2\pi i\frac{\operatorname{sh}\phi}{\phi^2+\pi^2}\,, \tag{B.3a}$$

$$\rho^{\mathrm{GN}}(M^2) = 2\pi i \frac{\coth\frac{1}{2}\phi}{\phi^2 + \pi^2}\,. \tag{B.3b}$$

The k integration can now be performed and we obtain (with $p_1 p_2 = m^2 \operatorname{ch}\theta$ and $\hat{\theta} = i\pi - \theta$)

$$F_-^{\mathrm{NLS}}(\theta, N) = 1 + \frac{1}{N}\int_0^\infty d\phi \frac{\operatorname{sh}^2\frac{1}{2}\phi}{\phi^2+\pi^2}\left\{\frac{\phi\coth\frac{1}{2}\phi - \hat{\theta}\coth\frac{1}{2}\hat{\theta}}{\operatorname{ch}^2\frac{1}{2}\phi - \operatorname{ch}^2\frac{1}{2}\hat{\theta}} - (\hat{\theta}\to 0)\right\} + \mathrm{O}(N^{-2})$$

$$= 1 + \frac{1}{N}\left\{1 - \tfrac{1}{2}\hat{\theta}(\coth\tfrac{1}{2}\hat{\theta} - \operatorname{th}\tfrac{1}{2}\hat{\theta}) - \tfrac{1}{2}\psi\left(\frac{1}{2} + \frac{\hat{\theta}}{2\pi i}\right) - \tfrac{1}{2}\psi\left(\frac{1}{2} - \frac{\hat{\theta}}{2\pi i}\right) + \psi(\tfrac{1}{2})\right\}$$
$$+ \mathrm{O}(N^{-2})\,, \tag{B.4a}$$

$$F_-^{\mathrm{GN}}(\theta, 2N) - 1 = \tfrac{1}{2}(F_-^{\mathrm{NLS}}(\theta, 2N) - 1) + \frac{1}{N}\left(\frac{\hat{\theta}}{\operatorname{sh}\hat{\theta}} - 1\right) + \mathrm{O}(N^{-2})\,, \tag{B.4b}$$

where $\psi(z) = (\ln\Gamma(z))'$.

These perturbative results agree with the proposed exact expressions, eqs. (5.7) and (5.10).

The $1/N$ expansion for F_- is simple because a loop vanishes which has one antisymmetric J_{ij}^{μ} vertex. This is no longer true for F_0^{GN}. The diagrams contributing to F_0^{GN} in order N^0 are drawn in fig. 8. A corresponding equality holds in any order. This reflects the fact that the field equation for the auxiliary field $\sigma(x)$ introduced by $\mathcal{L}^{\mathrm{GN}} \to \mathcal{L}^{\mathrm{GN}} - \frac{1}{2}(\sigma/g + g\bar{\psi}\psi)^2$ [8] reads

$$\sigma(x) + g^2\bar{\psi}\psi(x) = 0\,. \tag{B.5}$$

Thus we consider the matrix element $\langle 0|\sigma(0)|f_i(p_1)\, f_j(p_2)\rangle^{\mathrm{in}}$ in order N^0, N^{-1} (*cf.*

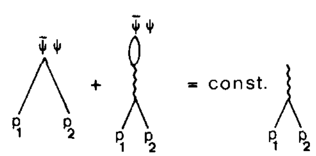

Fig. 8. Diagrams contributing to the scalar form factor in the Gross-Neveu model in order N^0.

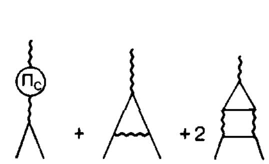

Fig. 9. Diagrams contributing to the scalar form factor in the Gross-Neveu model in order N^{-1}.

figs. 8 and 9) and obtain, with the normalization $F_0^{\rm GN}(i\pi) = 1$,

$$F_0^{\rm GN}(\theta, 2N) = {\rm const}\, D^{\rm GN}(p_1 + p_2)^2) + O(N^{-1})$$

$$= \frac{2}{\hat\theta}\,{\rm th}\tfrac{1}{2}\hat\theta \Bigg\{ 1 + \Pi_c(p_1 + p_2)\, D^{\rm GN}(p_1 + p_2)$$

$$+ \int \frac{d^2 l}{(2\pi)^2} D^{\rm GN}(l) \Bigg[\frac{4m^2 + l^2 - 8m^2 \frac{(p_1 - p_2)\, l}{(p_1 - p_2)^2}}{(l^2 - 2lp_1)(l^2 + 2lp_2)} - {\rm subtr.} - 4m D^{\rm GN}(p_1 + p_2 - l)$$

$$\times \frac{1 - \frac{(p_1 - p_2)\, l}{(p_1 - p_2)^2}}{l^2 - 2lp_1} N \int \frac{d^2 q}{(2\pi)^2} {\rm tr}\Big(\frac{1}{\not q - m} \frac{1}{\not q + \not k - m} \frac{1}{\not q + \not l - m} \Big) \Bigg] \Bigg\} + O(N^{-2}), \tag{B.6}$$

in agreement with eq. (5.13) in order N^0. There are two independent methods to calculate the σ self-energy $\Pi_c(k)$ contributing to $F_0^{\rm GN}$ in order N^{-1}. The direct calculation of the diagrams is rather complicated [23]. An alternative method uses the factorization constraints which relate the three independent two-particle scattering amplitudes $\sigma_1, \sigma_2, \sigma_3$ for the processes

$$f_i(p_1) + f_i(p_2) \to f_j(p_1) + f_j(p_2),$$

$$f_i(p_1) + f_j(p_2) \to f_i(p_1) + f_j(p_2),$$

$$f_i(p_1) + f_j(p_2) \to f_j(p_1) + f_i(p_2),$$

respectively. They are related to the S-matrix eigenvalues, eq. (5.3), by

$$S_0 = N\sigma_1 + \sigma_2 + \sigma_3,$$

$$S_\pm = \sigma_2 \pm \sigma_3. \tag{B.7}$$

The relations from crossing and factorization are [12]

$$\sigma_1(\theta, 2N) = \sigma_3(\hat\theta, 2N) = -\frac{i\pi}{N-1} \frac{\sigma_2(\hat\theta, 2N)}{\hat\theta}. \tag{B.8}$$

The r.h.s. can easily be calculated up to order N^{-1} to give (since only the tree dia-

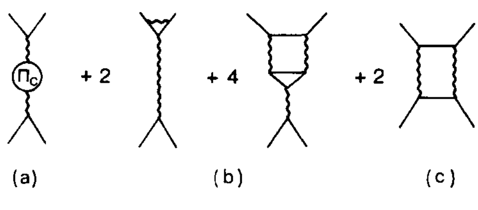

Fig. 10. Diagrams contributing to the scattering amplitude σ_1 in the Gross-Neveu model in order N^{-2}.

gram contributes to σ_2)

$$\sigma_1(\theta, 2N) = -\frac{1}{N}\frac{i\pi}{\hat{\theta}} - \frac{1}{N^2}\frac{i\pi}{\hat{\theta}}\left(1 + \frac{i\pi}{\operatorname{sh}\theta}\right) + O(N^{-3}) . \tag{B.9}$$

On the other hand, σ_1 in order N^{-1} is given by the diagrams of fig. 10.

The vertex graphs and the box graph can be calculated (with $k = p_1 + p_2$)

$$\sigma_1^{(b)} 4m^2 \operatorname{sh}\theta = -2\bar{u}(p_2)\, v(p_1)\, D(k) \int \frac{d^2 l}{(2\pi)^2} D(l) \left[\bar{v}(p_1) \frac{1}{\not l - \not p_1 - m} \frac{1}{\not l + \not p_2 - m}\right.$$

$$\times u(p_2) - \text{subtr.}$$

$$\left. - 4D(k-l)\,\bar{v}(p_1) \frac{1}{\not l - \not p_1 - m} u(p_2)\, N \int \frac{d^2 q}{(2\pi)^2} \operatorname{tr}\left(\frac{1}{\not q - m}\frac{1}{\not q + \not l - m}\frac{1}{\not q + \not k - m}\right)\right] , \tag{B.11}$$

$$\sigma_1^{(c)} 4m^2 \operatorname{sh}\theta = -2 \int \frac{d^2 l}{(2\pi)^2} D(l)\, D(k-l)\, \bar{v}(p_1) \frac{1}{\not p_2 - \not l - m} u(p_2)$$

$$\times \left[\bar{u}(p_2) \frac{1}{\not l - \not p_1 - m} v(p_1) + (p_1 \leftrightarrow p_2)\right] . \tag{B.11}$$

Thus we obtain for the finite part of the Π_c self-energy after some calculations from (B.9, 10, 11)

$$\Pi_c(k) = D^{-1}(k) \left\{ \frac{1}{N}\frac{i\pi}{\operatorname{sh}\theta} - 2\int \frac{d^2 l}{(2\pi)^2} D(l) \left[\frac{4m^2 + l^2 - 8m^2 \dfrac{(p_1 - p_2) l}{(p_1 - p_2)^2}}{(l^2 - 2lp_1)(l^2 + 2lp_2)} - \text{subtr.}\right]\right.$$

$$+ 2m^2 D^{-1}(k) \int \frac{d^2 l}{(2\pi)^2} D(l)\, D(k-l) \left[\frac{1 - \dfrac{(p_1 - p_2) l}{(p_1 - p_2)^2}}{l^2 - 2lp_1} + (p_1 \leftrightarrow p_2)\right]^2$$

$$\left. - 16m^2 \frac{d^2 l}{(2\pi)^2} D(l) \frac{1 - \dfrac{(p_1 + p_2) l}{(p_1 + p_2)^2}}{(l^2 - 2lp_1)(l^2 - 2lp_2)} + \text{const.}\right\} \tag{B.12}$$

Inserting eq. (B.12) into eq. (B.6) we obtain

$$F_0^{\mathrm{GN}}(\theta, 2N) = \frac{2}{\hat{\theta}} \operatorname{th}\tfrac{1}{2}\hat{\theta} \left[1 + \int \frac{d^2 l}{(2\pi)^2} D^{\mathrm{GN}}(l) \left(\frac{4m^2 - l^2}{(l^2 - 2lp_1)(l^2 + 2lp_2)} - \text{subtr.}\right)\right]$$

$$+ O(N^{-2}) , \tag{B.13}$$

using the identity

$$\frac{1}{N}\frac{i\pi}{\operatorname{sh}\theta} - 8m^2\int\frac{d^2l}{(2\pi)^2}D(l)\,\frac{1-\dfrac{(p_1+p_2)l}{(p_1+p_2)^2}}{(l^2-2lp_1)(l^2-2lp_2)} = 8m^2\int\frac{d^2l}{(2\pi)^2}D(l)$$

$$\times\frac{1-\dfrac{(p_1-p_2)l}{(p_1-p_2)^2}}{(l^2-2lp_1)(l^2+2lp_2)}. \tag{B.14}$$

Comparing eqs. (B.13) and (B.1b) we observe agreement of F_0^{GN} up to order N^{-1} with the proposed exact scalar form factor, eq. (5.13).

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