CONFINEMENT IN TWO-DIMENSIONAL MODELS WITH FACTORIZATION

M. KAROWSKI, V. KURAK [1] and B. SCHROER
Institut für Theoretische Physik, Freie Universität Berlin, D-1000 Berlin 33, Arnimallee 3–5, Germany

Received 3 November 1978

We show that the particle triplet of the O(3) σ-model can be obtained from the SU(2) bound state model by letting the (quark) mass of the fundamental particle approach infinity.

By now one has a rather rich supply of two-dimensional field theories with an infinite number of conservation laws. The massive Thirring model [1], the $O(n)$ nonlinear σ-model [2], the Gross–Neveu model [3], and their supersymmetric generalizations [4] have been studied by various authors, and there can be no serious doubt that in due time one will obtain all the generalized form factors and, hence, the correlation functions [5]. For the Gross-Neveu model in which the Lagrangians contain $O(n)$ fields ($n > 2$), there are more fundamental objects [6] which transform with respect to the universal (two-fold) covering spin (n)-group. In this case a lagrangian formulation of the internal spin (n) particles as fundamental fields has not yet been constructed, however the form of the factorizing kink S-matrix was recently suggested [6]. This state of affairs leads to the question whether certain models without such particles in the covering representation, for example the $O(n)$ σ-model studied by Zamolodchikov, can be interpreted as a "confinement" of these particles. A suggestion in favour for such an interpretation comes from a recent paper of Lüscher [7,8] who showed that the CP^{n-1} σ-models, i.e. the σ-models for the adjoint representation of $SU(n)$, have a lagrangian formulation in terms of a gauge theory. The long-range nature of the (composite) gauge field leads to θ-vacua and confinement of the fundamental $SU(n)$ field, so that only the adjoint representation particles survive. The O(3) σ-model is a special case in this class of models. We demonstrate in the following that all factorizing U(2) S-matrices with a meson bound state yield the O(3) σ-model S-matrix as the quark mass becomes infinitely heavy. In this limit the adjoint representation decouples completely other states.

There are several classes of U(2) S-matrices. The simplest one is the class II [9] using the language of Berg, Karowski, Weisz, and Kurak. The S-matrix in that class has the form (we take Bose-statistics)

$$S|\alpha(\theta_1)\beta(\theta_2)\rangle^{\rm in} = |\alpha(\theta_1)\beta(\theta_2)\rangle^{\rm in} u_1(\theta_{12}) + |\beta(\theta_1)\alpha(\theta_2)\rangle^{\rm in} u_2(\theta_{12}) \,, \tag{1a}$$

$$S|\alpha(\theta_1)\bar\beta(\theta_2)\rangle^{\rm in} = |\alpha(\theta_1)\bar\beta(\theta_2)\rangle^{\rm in} t_1(\theta_{12}) + \delta_{\alpha\beta}\sum_\gamma |\gamma(\theta_1)\bar\gamma(\theta_2)\rangle^{\rm in} t_2(\theta_{12}) \,, \tag{1b}$$

θ_i = rapidity. The t_2 and u_1 are related to t_1 as follows:

$$t_2(\varphi) = \frac{1}{\varphi - 1}\, t_1(\varphi)\,, \quad u_1(\varphi) = t_1(1-\varphi)\,, \quad u_2(\varphi) = -\frac{1}{\varphi} t_1(1-\varphi)\,, \tag{2a}$$

$$t_1^{\rm min}(\varphi) = [\Gamma(1/2+\varphi/2)\,\Gamma(1-\varphi/2)]/[\Gamma(1/2-\varphi/2)\,\Gamma(1+\varphi/2)]\,. \tag{2b}$$

Here we traded the rapidity for the more convenient $\varphi = \theta/\mathrm{i}\pi$.

[1] Supported by DFG contract Az 160/5.

The t_1 which defines our model is not the minimal one, but rather has a pole at say $\varphi = \alpha, 0 < \alpha < 1$

$$t_1(\varphi) = t_1^{\min}(\varphi)\,[\sin\pi\varphi + \sin\pi\alpha]/[\sin\pi\varphi - \sin\pi\alpha]\,. \tag{3}$$

This pole appears in the triplet t_1 as well as in the singlet $t_1 + 2t_2$ amplitude.

Following the derivation of the breather-breather scattering in the massive Thirring model [10], we write (suppressing the rapidity arguments which are assumed to have the canonical ordering $\theta_1 > \theta_2 > \theta_3 > \theta_4$)

$$S|\alpha\bar{\beta}\gamma\bar{\delta}\rangle = S_{23}S_{13}S_{24}S_{14}S_{12}S_{34}|\alpha\bar{\beta}\gamma\bar{\delta}\rangle\,. \tag{4}$$

The two-particle S-matrix for scattering of bound states ("π^i and η-mesons") is [11]

$$S(\theta) = (P_{12}^{\pi} + P_{12}^{\eta}/r)(P_{34}^{\pi} + P_{34}^{\eta}/r)S_{23}S_{13}S_{24}S_{14}(P_{12}^{\pi} + rP_{12}^{\eta})(P_{34}^{\pi} + rP_{34}^{\eta})\,, \tag{5}$$

where P_{ik}^{π} and P_{ik}^{η} are the projectors on the singlet and triplet states, respectively, of momenta p_i and p_k. The number r is given by the ratio of the residue of the singlet and triplet amplitudes

$$r = [(-t_1 + 2t_2)/t_1]^{1/2}|_{\varphi=\alpha} = [(1+\alpha)/(1-\alpha)]^{1/2}\,. \tag{5a}$$

The rapidities of the constituent are determined by $\theta_1 - \theta_2 = \theta_3 - \theta_4 = \mathrm{i}\pi\alpha$ and $\theta_1 + \theta_2 - \theta_3 - \theta_4 = \theta$ ($\theta_1 + \theta_2$, $\theta_3 + \theta_4$ real).

From this expression one may now compute

$$\langle\pi^k\pi^l|S|\pi^i\pi^j\rangle\,,\quad \langle\eta\eta|S|\eta\eta\rangle\,,\quad \langle\eta\eta|S|\pi^i\pi^j\rangle\,, \tag{6}$$

with

$$|\pi^i\rangle = \tfrac{1}{2}\,[|\alpha(\theta_1)\bar{\beta}(\theta_2)\rangle - |\bar{\beta}(\theta_1)\alpha(\theta_2)\rangle]\,\tau_{\alpha\beta}^i\,,\qquad |\eta\rangle = \tfrac{1}{2}\,[|\alpha(\theta_1)\bar{\alpha}(\theta_2)\rangle + |\bar{\alpha}(\theta_1)\alpha(\theta_2)\rangle]\,. \tag{7}$$

The last amplitude contains a factor r^{-2}. Since r approaches infinity for $\alpha \to 1$, the triplet-triplet scattering decouples from the singlet states, i.e. the former sub-S-matrix becomes unitary. Note that $\alpha \to 1$ means that $m_{\text{quark}}/m_{\text{meson}} \to \infty$, a limit in which the U(2)-S-matrix looses its wave packet, i.e. Hilbert-space interpretation, however the new triplet-triplet S-matrix retains it. We obtain

$$\begin{aligned}4\langle\pi^k\pi^l|S|\pi^i\pi^j\rangle &= t_1u_1u_1t_1\,\mathrm{tr}\,(\tau^i\tau^k)\,\mathrm{tr}\,(\tau^j\tau^l) + t_1u_2u_2t_1\,\mathrm{tr}\,(\tau^j\tau^k)\,\mathrm{tr}\,(\tau^i\tau^l) + t_2u_1u_1t_2\,\mathrm{tr}\,(\tau^j\tau^k)\,\mathrm{tr}\,(\tau^i\tau^l)\\ &+ t_2u_1u_1t_1\,\mathrm{tr}\,(\tau^l\tau^j\tau^i\tau^k) + t_1u_2u_1t_1\,\mathrm{tr}\,(\tau^k\tau^i\tau^l\tau^j) + t_1u_1u_2t_1\,\mathrm{tr}\,(\tau^k\tau^j\tau^i\tau^l)\\ &+ t_2u_2u_2t_1\,\mathrm{tr}\,(\tau^k\tau^l\tau^i\tau^j) + t_1u_1u_1t_2\,\mathrm{tr}\,(\tau^k\tau^i\tau^j\tau^l) + t_2u_2u_1t_2\,\mathrm{tr}\,(\tau^k\tau^l\tau^i\tau^j)\\ &+ t_2u_1u_2t_2\,\mathrm{tr}\,(\tau^k\tau^l\tau^i\tau^j) + t_1u_2u_2t_2\,\mathrm{tr}\,(\tau^k\tau^l\tau^i\tau^j) + 2t_2u_2u_2t_2\,\mathrm{tr}\,(\tau^k\tau^l\tau^i\tau^j)\,.\end{aligned}$$

The arguments are from the left to the right: $\varphi - 1, \varphi, \varphi, \varphi + 1$. For the coefficient of $\delta_{ik}\delta_{jl}$ we obtain (using eq. (2))

$$t_1(\varphi-1)\,t_1(1-\varphi)\,t_1(1-\varphi)\,t_1(\varphi+1)\,\tfrac{1}{2}\left[2 - \frac{1}{2-\varphi} - \frac{1}{\varphi} - \frac{1}{2-\varphi}\frac{1}{\varphi^2} - \frac{1}{\varphi^3} + \frac{1}{2-\varphi}\frac{1}{\varphi^3}\right]. \tag{9}$$

With [9]

$$t_1(1-\varphi)\,t_1(\varphi-1) = 1\,,\qquad t_1(1-\varphi)\,t_1(1+\varphi) = \varphi^2/(\varphi^2-1)\,, \tag{10}$$

we obtain the Zamolodchikov's [2] coefficient

$$\sigma_2 = -\frac{\varphi}{2-\varphi}\frac{\varphi-1}{\varphi+1}\,. \tag{11}$$

Similarly the coefficient of $\delta_{jk}\delta_{ik}$ turns out to be

$$t_1(\varphi-1)t_1(1-\varphi)t_1(1-\varphi)t_1(\varphi+1)\tfrac{1}{2}\left[\frac{2}{\varphi^2}+\frac{1}{2-\varphi}-\frac{3}{\varphi}+\frac{1}{2-\varphi}\frac{1}{\varphi^2}+\frac{1}{\varphi^3}-\frac{2}{2-\varphi}\frac{1}{\varphi^3}\right] \tag{12}$$

$$=-(2/\varphi)\,\sigma_2\,, \tag{13}$$

which checks with the known result [2]. The coefficient of $\delta_{ij}\delta_{kl}$ is related to eq. (13) by crossing symmetry.

This S-matrix approach does not shine any light on the possible lagrangian field theory of our non-confining $\alpha < 1$ model from which we started our discussion. It is believed that the $\alpha \to 1$ limit of the unknown model is the confining CP^1-model studied by Lüscher. This model is a special case of the CP^{n-1} model which is formally equivalent to the "idempotent" σ-model studied in its classical version by Eichenherr [12]. It has been recently demonstrated by Berg and Weisz [13] that the only factorizing type of adjoint SU(n) representation σ-model reduces to the O(n^2-1) σ-model. In a subsequent paper we will return to the confinement mechanism for SU(n) $n \geqslant 3$. Since Lagrangian's CP^{n-1} model at least on a superficial level seems to be different from the O(n^2-1) model, one may indulge in the interesting speculation that the former model, which has infinitely many conservation laws, is the first example of a non-factorizing dynamical system.

We are very thankful to Bernd Berg and Peter Weisz for many illuminating discussions. We are also indebted to Klaus Rothe.

References

[1] A.B. Zamolodchikov, Pisma v JETP 25 (1977) 499;
M. Karowski, H.J. Thun, T.T. Truong and P. Weisz, Phys. Lett. 67B (1977) 321.
[2] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. 133B (1975) 525.
[3] A.B. Zamolodchikov and Al.B. Zamolodchikov, Phys. Lett. 72B (1978) 481.
[4] E. Witten and R. Shankar, Harvard preprint HUTP 77/AD76 (1977);
See also P. Di Vecchia and S. Ferrara, Nucl. Phys. 130B (1977) 93.
[5] For the Ising model this construction was shown to lead to the known field theory of this model, B. Berg, M. Karowski and P. Weisz, FU preprint 78/16.
[6] E. Witten and R. Shankar, Yale Univ. preprint 5/78.
[7] M. Lüscher, preprint NBI-HE-78-9.
[8] A. D'Adda, M. Lüscher and P. di Vecchia, Copenhagen preprint August 1978.
[9] B. Berg, M. Karowski, V. Kurak and P. Weisz, Nucl. Phys. 134B (1978) 125.
[10] M. Karowski and H.J. Thun, Nucl. Phys. B130 (1977) 295;
A.B. Zamolodchikov, Moscow preprint ITEP-12 (1977).
[11] M. Karowski, to be published.
[12] H. Eichenherr, Freiburg preprint.
[13] B. Berg and P. Weisz, FU-Berlin preprint August 78/18.