Field theories in 1+1-Dimensions with soliton behaviour: form factors and Green's functions*

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Abstract

For a number of field theoretic models with soliton behaviour the on-shell solutions have been obtained in the last few years. In this paper it is shown how to calculate off-shell quantities like form factors and Green's functions by means of the knows S-matrices, general properties of quantum field theory and minimality assumptions.

1 Introduction

Various 1+1 dimensional field theoretic models with soliton behaviour have been investigated recently, e.g. the sine-Gordon alias the massive Thirring model [1], the $O(N)$ nonlinear σ-model and the Gross-Neveu model [2], the chiral $SU(N)$ model [3] and a $Z(N)$-Ising model in the scaling limit [4]. The later will be used in this paper to demonstrate the bootstrap program for the construction of off-shell quantities like form factors and Green's functions from the known exact S-matrix, general properties of quantum field theory and some minimality assumptions. A $Z(N)$-Ising model in the scaling limit is defined by taking the correlation functions of a $Z(N)$-Ising model on a 2-dimensional lattice in the limit

$$\lim_{T\to T_c} |1-T/T_c|^{-n\beta} \langle\, \sigma_{R_1}\cdots\sigma_{R_n}\,\rangle = S(r_1,\ldots,r_n)$$

with $R_i \to \infty$ and $r_i = R_i|1-T/T_c|^{\nu}$ fixed. By analytic continuation one obtains from the Schwinger functions the Green's functions in Minkowski space. For $N > 2$ there exist several $Z(N)$-invariant nearest neighbor interactions. Let us make the assumption that one of these interactions enforces soliton behaviour for the model in the scaling limit.

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*Talk presented at International Colloquium on Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory, Les Houches, September 1979

†Supported by Deutsche Forschungsgemeinschaft, contract Schr 160/5 and Schr 415.

For field theories with soliton behaviour the dynamics is governed by an infinite number of higher conservation law. In particular for scattering processes they imply the absence of particle production and factorization of the n-particle S-matrix [5]

$$S^{(n)}(p_1,\dots,p_n)=\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}S^{(2)}(p_i,p_j) \tag{1}$$

where the two-particle S-matrix $S^{(2)}(p_i,p_j)$ applied to an n-particle state act on the particles with momenta p_i and p_j

$$S^{(2)}(p_i,p_j)\;|\dots,\alpha_i(p_i),\dots,\alpha_j(p_j),\dots\rangle^{in}=|\dots,\alpha_i'(p_i),\dots,\alpha_j'(p_j),\dots\rangle^{in}{}_{\alpha_i'\alpha_j'}S_{\alpha_i\alpha_j}(p_i,p_j)\,. \tag{2}$$

The α's label the kinds of particles. Together with unitarity, crossing and minimality assumptions this property allows the exact calculation of the S-matrix [1, 2, 3, 4]. Let us repeat briefly the arguments for the $Z(N)$-Ising model with soliton behaviour [4]. Assume the existence of an elementary boson b_1 corresponding to the order variable $\sigma(x)$ and the existence of a two-particle bound state $b_2=(b_1b_1)$. Then [6] we get the series of bound states b_k with masses

$$m_k=m_1\frac{\sin ak}{\sin a}\,. \tag{3}$$

Since the order variable σ assumes the values – the N-th roots of one – we have

$$\sigma^\dagger=\sigma^{N-1} \tag{4}$$

which suggests the assumption that the anti-particle $\bar b_1$ is identical to the bound state b_{N-1}. Then from $m_1=m_{N-1}$ we get for the parameter a the value

$$a=\pi/N\,. \tag{5}$$

Minimality means that the transmission amplitude for the scattering of two particles b_1 has only the pole corresponding to the bound state b_2. This implies uniquely [4]

$$S_{11}(\theta)=\frac{\sinh\frac12(\theta+2\pi i/N)}{\sinh\frac12(\theta-2\pi i/N)} \tag{6}$$

where θ is the rapidity difference defined by

$$p_1p_2=m^2\cosh\theta\,. \tag{7}$$

From eq. (6) we obtain for the scattering of the bound state b_j and b_k

$$S_{jk}(\theta)=\exp 4\int_0^\infty\frac{dx}{x}\frac{\cosh x(1-|j-k|/N)-\cosh x(1-(j+k)/N)}{\sinh x\tanh(x/N)}\sinh x\frac{\theta}{i\pi}\,. \tag{8}$$

This formula is consistent with crossing, the above assumption $\bar{b}_1 = b_{N-1}$ and more general even with $\bar{b}_k = b_{N-k}$ since

$$S_{j\bar{k}}(\theta) = S_{jk}(i\pi - \theta) = S_{j\,N-k}(\theta)\,. \tag{9}$$

This model is simple because of the absence of backward scattering which is a consequence of $\bar{b}_1 = b_{N-1}$ [4].

For $N = 2$ the model is much more simple since there is only one kind of particles $b = \bar{b}$ and the two-particle S-matrix is

$$S^{(2)} = -1\,. \tag{10}$$

Because of this simplification the bootstrap program has been carried though for this model [8].

2 Watson's Theorem

For form factors, i.e. matrix elements of local operators, we derive a set of equations which follow from general principles of quantum field theory, 'maximal analyticity', and the S-matrix factorization. For simplicity we first consider the case where we have only one kind of bosons in the model and a hermitian operator $\mathcal{O}(x)$. If we define the function $F(\theta)$ (for θ c.f. eq. (7)) by

$$F(\theta) = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \,\rangle^{in}$$

it follows from CPT-invariance that $F(-\theta) = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \,\rangle^{out}$, unitarity and factorization that $F(\theta) = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \,\rangle^{out} S(\theta)$, and crossing that $F(i\pi - \theta) = \langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2 \rangle^{in}$. Hence we have Watson's equations:

$$\begin{array}{rcl} F(\theta) & = & F(-\theta)\, S(\theta) \\ F(i\pi - \theta) & = & F(i\pi + \theta)\,. \end{array} \tag{11}$$

The corresponding equations for the general case where the α's denote the different kinds of particles read [7]

$$\begin{array}{rcl} F_{\alpha_1 \ldots \alpha_n}(\theta_{ij}, i\pi - \theta_{rs}, \theta_{kl}) & = & {}^{out}\langle\, \alpha_1(p_1), \ldots, \alpha_m(p_m) \,|\, \mathcal{O}(0) \,|\, \ldots, \alpha_n(p_n) \,\rangle^{in} \\ & = & {}_{\alpha_1 \ldots \alpha_m} S^{(m)}_{\alpha'_1 \ldots \alpha'_m}(\theta_{ij}) F_{\alpha'_1 \ldots \alpha'_n}(-\theta_{ij}, i\pi + \theta_{rs}, -\theta_{kl})\, {}_{\alpha'_{m+1} \ldots \alpha'_n} S^{(n-m)}_{\alpha_{m+1} \ldots \alpha_n}(\theta_{kl}) \end{array} \tag{12}$$

where $1 \le i < j \le m$, $1 \le r \le m < s \le n$, $m < k < l \le n$ and $0 \le m \le n$. Solutions of these equations are only known for simple cases.

3 Construction of form factors

The construction of form factors has been performed for models with arbitrary number of kinds of particles for the case $n = 2$. Hence one particle expectation values of local operators, the usual form factors are known for various models [7]. Absence of backward scattering enables us to make proposals for more general form factors with $n > 2$, e.g. for the sine-Gordon [7] and the $Z(N)$-Ising model. Furthermore arbitrary n-particle matrix elements are known for the $Z(2)$-Ising model in the scaling limit [8].

There are two distinct problems in this procedure:

a) The "matrix problem": The set of equations (12) has not been solved for the general case. For $n = 2$ we diagonalize the two-particle S-matrix and get for each eigenvalue the simple equation (11).

Theorem [7]
An analytic function $F(\theta)$ fulfilling eqs. (11) is (up to a normalization) uniquely determined by the positions of poles (and zeros) at $\theta = ia_k$ in the physical strip $0 < \Im\theta < \pi$:

$$F(\theta) = K(\theta)\, F^{min}(\theta) \tag{13}$$

where

$$K(\theta) = const \left(\prod_k \sinh\frac{1}{2}(\theta - ia_k)\, \sinh\frac{1}{2}(\theta + ia_k)\right)^{-1} \tag{13a}$$

is a solution of eqs. (11) with $S = 1$ and

$$F^{min}(\theta) = \exp\frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{dz}{\sinh\frac{1}{2}(z-\theta)}\,\frac{\cosh\frac{1}{2}\theta}{\cosh\frac{1}{2}z}\,\ln S(z)\,. \tag{13b}$$

For vanishing reflection $S^{(n)}$ is diagonal and a solution of the general equation (12) is

$$F_{\alpha_1\ldots\alpha_n}(\theta_{12}\ldots) = K_{\alpha_1\ldots\alpha_n}(\theta_{12}\ldots)\prod_{i<j}F^{min}_{\alpha_i\alpha_j}(\theta_{ij}) \tag{14}$$

where the function K is a solution of eqs. (12) with $S = 1$, this means it has no branch cuts in terms of the variables p_ip_j.

b) The "pole problem": The poles in the physical region of the function

$$F_{\alpha_1\ldots\alpha_n}(\theta_{12}\ldots) = \langle\, 0\,|\,\mathcal{O}(0)\,|\,\alpha_1(p_1),\ldots,\alpha_n(p_n)\,\rangle^{in}$$

which are contained in the function K, are determined by one-particle states in all sub-channels $\alpha_{i_1}, \ldots, \alpha_{i_j}$. If we make the minimality assumption that there are no further so-called redundant poles and no zeros in the physical region, we can make proposals for exact form factors, which can be checked by perturbation theory.

4 Examples

a) The $Z(N)$-Ising model in the scaling limit with soliton behaviour

The minimal form factor function corresponding to the external legs b_j and b_k for the $Z(N)$-Ising model follows from eqs. (8,13b):

$$F_{jk}^{min}(i\pi - \theta) = \exp \int_0^\infty \frac{dx}{x} \frac{\cosh x(1 - \frac{|j-k|}{N}) - \cosh x(1 - \frac{j+k}{N})}{\sinh^2 x \tanh(x/N)} \left(1 - \cosh x \frac{\theta}{i\pi}\right) . \tag{15}$$

Since there is no backward scattering in this model, the general form factor is given by eq. (14). The determination of K as a rational function of $p_i p_j$ is in general a complicated problem. Let us consider in more detail the case $N = 3$. Then we have two particles $b_1, \; b_2 = \bar{b}_1$ with mass m and the two-particle S-matrix

$$S_{11}(\theta) = S_{12}(i\pi - \theta) = \frac{\sinh \frac{1}{2}(\theta + 2\pi i/3)}{\sinh \frac{1}{2}(\theta - 2\pi i/3)} \tag{16}$$

The minimal form factor functions are

$$\begin{aligned} F_{11}^{min}(i\pi - \theta) &= \exp \int_0^\infty \frac{dx}{x} \frac{\sinh(x/3)}{\sinh^2 x} \left(1 - \cosh x \frac{\theta}{i\pi}\right) , \\ F_{12}^{min}(i\pi - \theta) &= \exp \int_0^\infty \frac{dx}{x} \frac{\sinh(2x/3)}{\sinh^2 x} \left(1 - \cosh x \frac{\theta}{i\pi}\right) . \end{aligned} \tag{17}$$

The matrix elements of the order variable $\sigma(x)$ must have poles corresponding to the bound state $b_2 = \bar{b}_1$ of two particles b_1. Hence we make the proposals:

$$\langle\, 0 \,|\, \sigma(0) \,|\, b_2(p_1), b_2(p_2) \,\rangle^{in} = const \frac{1}{(p_1 + p_2)^2 - m^2} \, F_{11}^{min}(\theta_{12}) \tag{18a}$$

$$\langle\, 0 \,|\, \sigma(0) \,|\, b_1(p_1), b_1(p_2), b_2(p_3) \,\rangle^{in} = \frac{const'}{(p_1 + p_2 + p_3)^2 - m^2} \frac{(p_1 + p_2)^2}{(p_1 + p_2)^2 - m^2} \prod_{i<j} F_{\alpha_i \alpha_j}^{min}(\theta_{ij}) \tag{18b}$$

$$\begin{aligned} &\langle\, 0 \,|\, \sigma(0) \,|\, b_1(p_1), b_1(p_2), b_1(p_3), b_1(p_4) \,\rangle^{in} \\ &= const'' \Big((p_1 + p_2 + p_3 + p_4)^2 - m^2 \Big) \prod_{i<j} \frac{F_{\alpha_i \alpha_j}^{min}(\theta_{ij})}{(p_i + p_j)^2 - m^2} \end{aligned} \tag{18c}$$

The numerators in eqs. (18b,c) prevent reflection in two particle scattering and particle production $2 \to 3$, respectively. The proposals (18) fulfill the consistency conditions

$$\operatorname*{Res}_{\theta_{12}=2\pi i/3} \langle\, 0 \,|\, \sigma(0) \,|\, b_1(p_1), b_1(p_2), b_2(p_3) \,\rangle^{in} = const \langle\, 0 \,|\, \mathcal{O}(0) \,|\, b_2(p_1+p_2), b_2(p_3) \,\rangle^{in} \tag{19a}$$

$$\begin{aligned} &\operatorname*{Res}_{\theta_{12}=2\pi i/3} \langle\, 0 \,|\, \sigma(0) \,|\, b_1(p_1), b_1(p_2), b_1(p_3), b_1(p_4) \,\rangle^{in} \\ &\quad = \; const \langle\, 0 \,|\, \mathcal{O}(0) \,|\, b_2(p_1+p_2), b_1(p_3), b_1(p_4) \,\rangle^{in} \end{aligned} \tag{19b}$$

Furthermore from the four point function (18b) one gets the two-particle S-matrix eq. (16):

$$\begin{aligned} S_{11}(\theta_{12}) \;&=\; 1 + \frac{1}{4m^2 i \sinh\theta_{12}} \\ &\times \Big[\Big((p_1+p_2+p_3)^2 - m^2 \Big) \langle\, b_1(p_2) \,|\, \sigma(0) \,|\, b_1(p_1), b_1(p_2) \,\rangle^{in} \Big]_{(p_1+p_2+p_3)^2=m^2} . \end{aligned} \tag{20}$$

We have, up to now, no other possibilities to check these results, because we do not know, as mentioned in the introduction, which interaction is that one which enforces the soliton behaviour for the $Z(3)$-Ising model in the scaling limit.

b) The $Z(2)$-Ising model

For the $Z(2)$-Ising model in the scaling limit the problem of calculating arbitrary matrix elements of the order variable σ

$$\langle\, 0 \,|\, \sigma(0) \,|\, p_1, \ldots, p_n \,\rangle^{in} = F^{(n)}(\theta_{12}, \ldots) \tag{21}$$

has been solved completely [8]. The minimal solutions of Watson's equations (11) with $S^{(2)} = -1$ is obtained from eq. (15).

$$F^{min}(\theta) = -i \sinh\frac{\theta}{2} . \tag{22}$$

Since there is no reflection we can use formula (14). The function $K^{(n)}$ contains the poles from one-particle states in all sub-channels. Because there is only one kind of boson b with mass m, we obtain for $n = 3$

$$K^{(3)} = \frac{const}{(p_1+p_2+p_3)^2 - m^2} = \frac{Z^{(3)}}{\cosh\frac{1}{2}\theta_{12}\, \cosh\frac{1}{2}\theta_{13}\, \cosh\frac{1}{2}\theta_{23}} . \tag{23}$$

This proposal fulfills the consistency condition that $F^{(3)}$ with the leg corresponding to $\sigma(x)$ on-shell, as in eq. (20), has to reproduce the correct S-matrix $S^{(2)}$. From this condition we also obtain the constant to be $Z^{(3)} = 2$.

For $n \geq 5$ the arguments are just slightly more involved. The function $K^{(n)}$ must have poles at $(p_i + p_j + p_k)^2 = m^2$ where any set of particles i, j, k is combined to give a one-particle state. However, the residues of "higher poles" where more than three particles build up a one-particle state must vanish, since the presence of such poles would be in contradiction to the absence of particle production. Hence the minimality hypothesis implies that $K^{(n)}$ has the form

$$K^{(n)} = \frac{const\left(\prod_{i<j}(p_i+p_j)^2\right)^{\frac{n-3}{2}}}{\prod_{i<j<k}\left((p_i+p_j+p_k)^2-m^2\right)} = \frac{Z^{(n)}}{\prod_{i<j}\cosh\frac{1}{2}\theta_{ij}} \tag{24}$$

The numerator is necessary in order to modify the severe singularities of the denominator at $p_i + p_j = 0$ into simple poles. The constant $Z^{(n)}$ can again be determined to be $Z^{(n)} = 2^{(n-2)/2}$ by crossing and taking the off-shell leg $\sigma(x)$ on-shell. Finally we obtain

$$\langle\, 0 \,|\, \sigma(0) \,|\, p_1, \ldots, p_n \,\rangle^{in} = (2i)^{\frac{n-1}{2}} \prod_{i<j} \tanh\frac{\theta_{ij}}{2} \tag{25}$$

for odd n. The matrix elements vanish for even n because of $Z(2)$-invariance.

5 Green's functions

Till now the problem of constructing Green's functions has been solved only for the simplest field theory with soliton behaviour, the $Z(2)$-Ising model in the scaling limit $T \to T_c$ [8]. Having deduced all matrix elements of the field $\sigma(x)$, we can immediately write down expressions for the Green's functions. For example the two-point function is

$$\tau(p) = \int d^2x\, e^{ipx}\,\langle\, T\,\sigma(x)\,\sigma(0)\,\rangle = i\int d\kappa^2 \frac{\rho(\kappa^2)}{p^2-\kappa^2+i\epsilon} \tag{26}$$

with the spectral function ρ given by

$$\rho(p^2) = \sum_{n\ odd} \frac{1}{n!}\int\frac{d\theta_1}{4\pi}\cdots\int\frac{d\theta_1}{4\pi}\, 2\pi\,\delta^{(2)}(p-\sum_i p_i) \prod_{1\le i<i\le n} \tanh^2\frac{\theta_{ij}}{2}\,. \tag{27}$$

There are similar formulae for the n-point Green's functions which are all in agreement with those derived in [9] by different methods. This agreement supports the validity of the minimality hypothesis and gives some confidence in the feasibility of the bootstrap program. The determination of the correlation functions for one of the more involved soliton field theories remains, however, a challenging open problem.

Acknowledgment:

This talk is based on collaborations at the Institut für Theoretische Physik, FU-Berlin, with B. Berg, V. Kurak, B. Schroer, H.J. Thun, T.T. Troung and P. Weisz. I would like to thank them for many discussions.

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