INTEGRABLE SYSTEMS AND CONFORMAL INVARIANCE[1]

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ABSTRACT. Statistical systems at second order phase transition points should exhibit conformal invariance for long distances. Therefore they will yield conformal quantum field theories in the scaling limit. Their conformal properties can be analysed using finite size scaling behaviour. For integrable lattice models in two dimensions methods are proposed to calculate from the Bethe ansatz solution the "conformal anomaly" and scaling dimensions. As an application results for the q-state Potts model and modified six-vertex models are presented. It is conjectured that the latter exhaust the class of all unitary conformal quantum field theories with "central charge" $c < 1$.

1. Conformal Invariance.

Conformal invariance of two dimensional quantum field theories has become a subject of main interest because of its connection to string theory. I would like to report on results obtained for two-dimensional integrable models. Some of the investigations rely on a collaboration with H. de Vega [1]. Here I will explain only the main results which are obtained by the "Bethe ansatz" method. The detailed calculations can be found in ref. [2]. Statistical mechanical systems at second order phase transition points (i.e. critical systems) should possess conformal invariance [3]. We shall use this fact to obtain information on conformal quantum field theories.

In two dimensions conformal invariance is a rather strong restriction since the transformation group is infinite dimensional, in contrast to higher dimensions. It is related to the Virasoro algebra [4]

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}c(n^3-n)\delta_{n,-m} \tag{1}$$

[1] To be published in 'Differential Geometrical Methods in Theoretical Physics', ed. K. Bleuler, Reidel Pub. Co.

$$(c = \text{"central charge"})$$

for which a well developed theory exists [5]. Belavin, Polyakov and Zamolodchikov [6] classified conformal quantum field theories in terms of the central charge

$$c = 1 - \frac{6}{\nu(\nu-1)} \tag{2}$$

where ν is a rational number if there exists only a finite number of "primary fields", i.e. fields transforming under conformal transformations

$$z = x + iy \to w(z)$$

$$\bar{z} = x - iy \to \bar{w}(\bar{z}) \tag{3}$$

like

$$A(z,\bar{z}) \to \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} A(w,\bar{w}). \tag{4}$$

The conformal dimensions can assume values given by the Kac-formula [5]

$$\Delta, \bar{\Delta} = h_{pq} = \frac{(\nu p - (\nu-1)q)^2 - 1}{4\nu(\nu-1)} \tag{5}$$

$$1 \leq q \leq p < \nu - 1 \qquad p, q = \text{integer}.$$

Friedan, Qiu and Shenker [7] have shown that unitarity restricts the number ν to

$$\nu = integer \geq 2 \quad \text{or} \quad c \geq 1. \tag{6}$$

It would be interesting to find integrable lattice models which yield these conformal quantum field theories in the scaling limit. Therefore it is useful to calculate the central charge and the conformal dimensions for integrable critical statistical models. This can be done by analysing finite size effects.

2. Finite Size Behaviour.

For a statistical system on a periodic $M \times N$ square lattice one introduces a transfer matrix τ (defined on an N-chain). The partition function can be written as

$$Z = \mathrm{tr}\,\tau^M = \sum_i \lambda_i^M \tag{7}$$

where the sum extends over all eigenvalues of the transfer matrix

$$\tau\psi = \lambda\psi. \tag{8}$$

Cardy [8] has shown that conformal invariance implies for the maximal eigenvalue

$$\lambda_{max} \approx \exp(-Nf + \frac{1}{N}\frac{\pi}{6}c) \qquad \text{for} \quad N \to \infty \tag{9}$$

where f is the free energy per site (in the thermodynamic limit) and c is the central charge of the Virasoro algebra (cf. eq. (1)).

Introducing in addition the "crossed" transfer matrix $\hat{\tau}$ (defined on an M-chain) we obtain for $1 \ll N \ll M$ two expressions for the partition function

$$Z \approx \exp(-NMf) \sum_i (\hat{\lambda}_i / \hat{\lambda}_{max})^N \tag{10a}$$

$$Z \approx \exp(-NMf + \frac{M}{N}\frac{\pi}{6}c) \tag{10b}$$

where only the "low energy excitations" with $\hat{\lambda}_i \approx \hat{\lambda}_{max}$ contribute to the sum in eq. (10a). Hence two methods to investigate the partition function on a strip $1 \ll N \ll M \to \infty$ are availble:

(a) one considers the transfer matrix $\hat{\tau}$ for an infinite chain ($M \to \infty$) and calculates the large N corrections to the sum in eq. (10a) or

(b) one considers the transfer matrix τ on an N-sites chain and calculates the large N corrections to the maximal eigenvalue λ_{max} acording to eq. (10b).

By these proceedures one obtains the central charge of the model.

In addition the conformal dimensions of operators of the model can be determined if one looks for excited states with energy $E = -\Re \log \lambda$ and momentum $P = -\Im \log \lambda$. As is argued in ref. [6] one has for $N \to \infty$

$$E_i - E_0 \approx \frac{2\pi}{N} d_i \tag{11}$$

$$P_i - P_0 \approx \frac{2\pi}{N} s_i$$

where $d_i = \Delta + \bar{\Delta} \quad (s_i = \Delta - \bar{\Delta})$ is the "scaling dimension" ("spin") of the operator A associated to the excitation ψ_i, i.e. $\langle 0|A|\psi_i\rangle \neq 0$.

Since a direct calculation of the central charge and the conformal dimensions for specific models is usually not simple, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. This will be done in the following for the six-vertex model (the critical 8-vertex model) and for some modified six-vertex models related to Potts models and other models of interest. Method (a) has been developed in ref. [1], method (b) is applied in sect. 4 and ref. [2].

3. Integrable Models.

By means of the technique known as Yang-Baxter algebra method, algebraic Bethe ansatz or quantum invers method [9] the eigenvalue problem eq. (8) of the transfer matrix is solved. The eigenstates appear as "n-pseudoparticle" states, where the momenta p_j of the pseudoparticles have to fulfil the "Bethe ansatz equations"

$$\exp(iNp_j)\prod_{\substack{i=1\\ i\neq j}}^{n}\exp(\Phi_{ji})=1 \tag{12}$$
$$j=1,\ldots,n \qquad 0\leq n\leq N/2.$$

These eqations express the condition that the total phase factor around one period N has to be equal to one. This factor consists of the free part $\exp(iNp_j)$ and the scattering parts with respect to all other pseudoparticles $\exp(i\Phi_{ji})$.

For the six-vertex model (cf. [9,10]) parametrized by

$$\omega_1=\omega_2:\omega_3=\omega_4:\omega_5=\omega_6=\sin(\gamma-\theta):\sin\theta:\sin\gamma \tag{13}$$

the "phase shifts" $\Phi_{ji}=\Phi(q_j-q_i)$ are given by

$$e^{ip(q)}=\frac{\sinh\frac{1}{2}(q-i\gamma)}{\sinh\frac{1}{2}(q+i\gamma)} \qquad e^{i\Phi(q)}=\frac{\sinh\frac{1}{2}(q+2i\gamma)}{\sinh\frac{1}{2}(q-2i\gamma)}. \tag{14}$$

There exist detailed investigations of the Bethe ansatz equations (12) (see e.g. ref. [11]). The analysis simplifies for the discrete values of the "coupling"

$$\gamma=\pi/\nu \qquad \nu=2,3,4,\ldots. \tag{15}$$

We are interested in these values. So we assume eq. (15) in the following.

Any set of roots of eq. (12) $\{q_1,\ldots,q_n\}$ defines an eigenstate and an eigenvalue $\lambda(\theta,\gamma)=\lambda_A+\lambda_D$ of the transfer matrix $\tau(\theta,\gamma)$

$$\log\lambda_A(\theta,\gamma)=N\omega_1-i\sum_{i=1}^{n}\left(p\left(q_i+2i\theta\right)-\pi\right) \tag{16}$$
$$\lambda_D=(\omega_1/\omega_3)^N\lambda_A^*(\gamma-\theta^*,\gamma).$$

Taking the logarithm of eq. (12)

$$2\pi I_j=Np(q_j)+\sum_{i=1}^{n}\Phi(q_j-q_i) \tag{17}$$

we observe that sets of roots $\{q_1, \ldots, q_n\}$ are given by sets of numbers $\{I_1, \ldots, I_n\} \subset \mathbf{Z} + \frac{1}{2}$ [2]. For the "ground state" (corresponding to the maximal eigenvalue λ_{max}) the number of real roots is maximal. This means $n = \frac{1}{2}N$ (N even) and the numbers I_j accept all possible values in an interval $I_- \leq I_j \leq I_+$. For "excited states" there may be "holes" in the sequence of the I_j. In addition there may appear complex roots q: the "k-strings" (i.e. "bound states" of k pseudoparticles)

$$q = \text{real} + i\gamma(-k+1, -k+3, \ldots, k-1) \qquad 1 < k \leq \nu - 1 \tag{18a}$$

and the "anti-pseudoparticle"

$$q = \text{real} + i\pi. \tag{18b}$$

If there are only finitely many holes, in the limit $N \to \infty$ the roots become dense on the whole real line and the "last roots" near to $\pm\infty$ behave like

$$q_\pm \approx \pm\tfrac{1}{2}\nu \log N\,. \tag{19}$$

4. Results for Six-Vetex and Potts Models.

The Potts models are a generalizations of the Ising model where the spin at each site can accept q values instead of two. For $q = 2, 3, 4$ the models posses a critical point which we consider in the following. The partition function on a cylinder [10] can be written in terms of a six-vertex model with a "seam" from the bottom to the top of the cylinder. This seam carries extra Boltzmann weights which produce extra terms $-i\gamma m$ and $2\gamma m$ with $m = 1$ on the r.h.s. of eqs. (16) and (17), respectively. The number q ($q = 2, 3, 4$) is related to the six-vertex coupling $\gamma = \pi/\nu$ ($\nu = 4, 6, \infty$) by

$$q = 4\cos^2\gamma \tag{20}$$

In the following the partition function Z_m and other formulas apply for $m = 0$ to the ordinary six-vertex model and for $m = 1$ to the Potts model

$$Z_0 = Z^{6-vertex} \qquad Z_1 = Z^{Potts}. \tag{21}$$

In refs. [12,13,1,2] techniques were developed to calculate finite size corrections for a large chain of length N. Using these methods we find [1,2] for the "ground state"

$$\log\lambda_{max} \approx Nf + \frac{1}{N}\frac{\pi}{6}\left(1 - \frac{6m^2}{\nu(\nu-1)}\right)\sin\nu\theta. \tag{22}$$

[2] The $\frac{1}{2}$ is due to $\Phi(0) = \pi$.

The vertex weights (13) fulfil $\omega_1 = \omega_2 = \omega_3 = \omega_4$ for $\nu\theta = \pi/2$ where we expect isotropy and conformal invariance in the scaling limit. Comparing eqs. (22) and (9) we conclude that the central charges of the Virasoro algebra of the six-vertex and the q-state Potts models are

$$c^{6-vertex} = 1 \tag{23a}$$

$$c^{Potts} = 1 - \frac{6}{\nu(\nu-1)}. \tag{23b}$$

As allready mentioned in sect. 3 the ground state correspond to the maximal set of real roots $q_1, \ldots, q_{N/2}$. Excitations possess "holes" and complex roots. We obtain [2] for these states taking eqs. (11) into account[3]

$$\log \lambda_i - \log \lambda_{max} \approx -\frac{2\pi}{N}\{(\Delta + \bar{\Delta})\sin \nu\theta + i(\Delta - \bar{\Delta})\cos \nu\theta\}. \tag{24}$$

The general expressions for the conformal dimensions Δ and $\bar{\Delta}$ are rather complicated. But the primary fields correspond to low energy exitations with a small number of holes $h_\pm$ at the ends of the real roots "near to $\pm\infty$". In addition there may be a number $n_\pm^{(k)}$ of k-strings "near to $\pm\infty$".[4] For these configurations of roots we obtain [2]

$$\Delta = \frac{\left(\nu\left(h_+ - 2\sum_k n_+^{(k)}(k-1)\right) - \Sigma + m\right)^2 - m^2}{4\nu(\nu-1)} + \text{integer} \tag{25a}$$

$$\bar{\Delta} = \frac{\left(\nu\left(h_- - 2\sum_k n_-^{(k)}(k-1)\right) - \Sigma - m\right)^2 - m^2}{4\nu(\nu-1)} + \text{integer} \tag{25b}$$

where $\Sigma = \frac{1}{2}h - \sum_k n^{(k)}(k-1)$ is the "spin"[5] of the Bethe state and h ($n^{(k)}$) the total number of holes (k-strings). We have $\Sigma = 0$ since we consider fixed boundary conditions at the top and the bottom of the cylinder. It turns out [2] that we get minimal values for Δ and $\bar{\Delta}$ (i.e. the integers on the r.h.s. of eqs. (25) vanish) if there are no holes inside $[q_-, q_+]$, $n_+^{(k)} = 0$ for all k, $n_-^{(k)} = 1$ for only one k and the number of holes have to fulfil $h_+ = h_- = k-1$. For the six-vertex model we obtain the conformal dimensions

[3]Note that the imaginary term correspond to the momentum since $\tau(\theta = 0) = \exp(-iP)$.

[4]The k-strings which do not approach $\pm\infty$ like the last real roots $q_\pm$ as $N \to \infty$ will not affect Δ and $\bar{\Delta}$.

[5]Σ must not be confused with the "spin" s of the field operator (cf. eq. (11)).

from eqs. (25) with $m = 0$. For the q-state Potts model the extra weights on the seam imply [2] that for the minimal Δ and $\bar{\Delta}$ only the k-strings with

$$k = 2, 3, \ldots, \nu - m - 1 \tag{26}$$

contribute (i.e. for $m = 1$ the $\nu - 1$-string and the "anti pseudoparticle" do not appear).

Comparing eqs. (25) for $m = 1$ and $\Sigma = 0$ with the Kac-formula (5) we find for the Potts model the conformal dimensions

$$\Delta = \bar{\Delta} = h_{k,1} \qquad k = 1, 2, \ldots, \nu - 2. \tag{27}$$

These correspond to the "energy operators" of the Potts model (cf. [14,8,15]). The equivalence of the Potts model and a six-vertex model with "seam" for $m = 1$ holds for the partition functions (cf. eq. (21)) but not for the operator content. The order parameter of the Potts model, for example, appear in a six-vertex model with a "seam" for $m = \nu/2$ (cf. [15]).

If we consider a partition function of a generalized model

$$Z = Z_1 + \sum Z_m \tag{28}$$

where the sum extends over some values of $m \in [2, \nu - 1]$ we obtain the dimensions corresponding to Bethe states with a k-string

$$\Delta = \bar{\Delta} = h_{k+m-1,m} \qquad k = 1, \ldots, \nu - m - 1 \quad 1 \leq m \leq \nu - 1. \tag{29}$$

Note that the ground state always belongs to Z_1. The "magnetization" operators of the $q = 3$-state Potts model ($\nu = 6$) correspond to $m = \nu/2 = 3$ (cf. [15]). Their dimensions are therefore (in agreement with numerical estimates [15] and conjectures [14,8])

$$\Delta = \bar{\Delta} = h_{k+2,3} \qquad k = 1, 2. \tag{30}$$

The $1/q$-para fermion operators [16] correspond to "boundary conditions" $m = \nu/q = 2$ for $q = 2, 3$ [16,15]. The Bethe states have spin $\Sigma = 1$ and a symmetric hole configuration $h_+ = h_- = 1$. For $\Sigma \neq 0$ the integers on the r.h.s. of eqs. (25) only vanish, if there are no higher strings. Hence we obtain from eq. (25) the scaling dimensions and spin

$$d = \Delta + \bar{\Delta} = h_{21} + h_{23} = \frac{\nu - 1}{2\nu} + \frac{\nu^2 - q2}{2q^2\nu(\nu - 1)} \tag{31}$$

$$s = \Delta - \bar{\Delta} = h_{21} - h_{23} = 1/q$$

in agreement with ref. [16,15]. Finaly let us look for the "chiral" operators of the 3-state Potts model (cf. [14,8]). Their conformal dimensions are given by the "boundary conditions" $m = \nu/2 = 3$ and Bethe states with $\Sigma = 2$ or four holes. For $h_+ = 0, 1, 2, 3$ and $h_- = 4 - h_+$ one finds from eq.(25) respctively

$$\Delta = h_{11}, h_{21}, h_{31}, h_{41} = 0, \tfrac{2}{5}, \tfrac{7}{5}, 3$$

$$\bar{\Delta} = h_{15}, h_{25}, h_{35}, h_{45} = 3, \tfrac{7}{5}, \tfrac{2}{5}, 0.$$

At the end let us consider a partition function given by eq (28) where the sum extends over all $m = 2, \ldots, \nu - 1$. The dimension formula (29) obviously describes the dimensional spectrum of the RSOS models [17], in the language of ref. [18] the $(A_\nu, A_{\nu-1})$ series of conformal quantum field theories. For this interpretation of eq. (28) the number m is not connected to extra weights on a "seam" or to "boundary conditions", but it is related [2] to a "Θ-vaccum" like angle appearing in the construction of Bethe states [9]. The RSOS models of ref. [17] can be generalized [19] to models corresponding to all modular invariant conformal quantum field theories with $c < 1$ [18]. All these models are related to modified six-vertex models. Results for these models will be published elsewhere [20].

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