CONFORMAL QUANTUM FIELD THEORIES AND INTEGRABLE SYSTEMS*

M. Karowski
Theoretical Physics, ETH-Hönggerberg
CH-8093 Zürich, Switzerland

ABSTRACT. The Yang-Baxter algebra method to solve integrable models is reviewed. Statistical systems at second order phase transition points should exhibit conformal invariance for long distances. Therefore they will yield conformal quantum field theories in the scaling limit. Their conformal properties can be analysed using finite size scaling behaviour. For integrable lattice models in two dimensions methods are proposed to calculate from the Bethe ansatz solution the "conformal anomaly" and scaling dimensions. As an application results for the q-state Potts model and modified six-vertex models are presented. It is conjectured that the latter exhaust the class of all unitary conformal quantum field theories with "central charge" $c < 1$.

1. Introduction.

I would like to report on results concerning conformal properties of integrable models in two dimensions. Some of the investigations rely on a collaboration with H. de Vega [1]. Here I will give only a summary of the main results and an introduction to the "Bethe ansatz" method which has been used. The detailed calculations can be found in ref. [2].

Conformal invariance of two dimensional quantum field theories has become a subject of main interest because of its connection to string theory. A string forms a two dimensional world sheet in space time. Reparameterization invariance of this surface causes the importance of conformal concepts in string theories. Statistical mechanical systems at second order phase transition points (i.e. critical systems) should possess

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conformal invariance[1] [3]. We shall use this fact to obtain information on conformal quantum field theories. It would be interesting to find **integrable** lattice models which yield these conformal quantum field theories in the scaling limit.

Since a direct calculation of the central charge of the Virasosro algebra and the conformal dimensions for specific models is usually not simple, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. This will be done for the six-vertex model (the critical eight-vertex model) and for some modified six-vertex models related to Potts models.

The theory of integrable systems has a long history. The technique known as **Yang-Baxter algebra** method, **algebraic Bethe ansatz** or **quantum inverse method** [4] has been applied to a large number of models. The spectrum has been calculated, the states have been constructed, and for some models also form factors and correlation functions have been obtained. The method can be applied to various kinds of models:

(i) quantum chains in one dimension, e.g. Heisenberg models [5]*

(ii) classical statistical models in two dimensions, e.g. Ising [6]* and vertex [7]* models

(iii) quantum field theories in $1+1$-dimensions [8]*:

bosonic models, e.g. nonlinear Schrödinger-, sine-Gordon-, σ- models

fermionic models, e.g. massive Thirring-, Gross-Neveu models.

2. Integrable Systems.

2.1 The six-vertex model. I like to explain the Yang-Baxter algebra method in terms of the six-vertex model, a model of type (ii), a classical statistical lattice model in two dimensions. Its partition function on an $N \times M$ periodic square lattice L is

$$Z = \sum_{conf.} \prod_{x \in L} \omega(x) \tag{1}$$

where the sum extends over all allowed "bond configurations". Each bond can accept one of two states characterized by $\alpha = 1, 2$ or arrows

$$\begin{aligned} \text{horizontal bonds} \quad & \alpha = 1,2 = \leftarrow , \ \rightarrow \\ \text{vertical bonds} \quad & \beta = 1,2 = \ \uparrow , \ \ \downarrow \ . \end{aligned}$$

[1] New results on representation theory of conformal invariance explain the old mysterious observation that critical exponents in two dimensions are usually rational numbers.
*and later works

The configuration is allowed if at each vertex x there are two incoming and two out going arrows. The vertex weights $\omega(x)$ are determined by the six bond configurations at the lattice site x. They assume the following values [4,9]

$$\begin{aligned} S_{11}^{11} &= \leftarrow\!\!\begin{smallmatrix}\uparrow\\ \uparrow\end{smallmatrix}\!\!\leftarrow = S_{22}^{22} = \rightarrow\!\!\begin{smallmatrix}\downarrow\\ \downarrow\end{smallmatrix}\!\!\rightarrow \;= a = \sin(\gamma-\theta) \\ S_{12}^{12} &= \leftarrow\!\!\begin{smallmatrix}\downarrow\\ \downarrow\end{smallmatrix}\!\!\leftarrow = S_{21}^{21} = \rightarrow\!\!\begin{smallmatrix}\uparrow\\ \uparrow\end{smallmatrix}\!\!\rightarrow \;= b = \sin\theta \\ S_{12}^{21} &= \rightarrow\!\!\begin{smallmatrix}\uparrow\\ \downarrow\end{smallmatrix}\!\!\leftarrow = S_{21}^{12} = \leftarrow\!\!\begin{smallmatrix}\downarrow\\ \uparrow\end{smallmatrix}\!\!\rightarrow \;= c = \sin\gamma \end{aligned} \tag{2}$$

By this notation the weights have been arranged as a $4\times 4-$ matrix

$$\begin{matrix} & \delta & \\ \gamma - & | & - \alpha \\ & \beta & \end{matrix} = S_{\alpha\beta}^{\gamma\delta}(\theta,\gamma) = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}. \tag{3}$$

The parameterization in terms of the "spectral parameter" θ and the "coupling" γ has been introduced for later convenience (cf. eq. (11)). A trivial overall factor is set equal to one.

In order to express the partition function in terms of the matrix S we define the **monodromy matrix**

$$T_{\alpha\{\beta\}}^{\gamma\{\delta\}}(\theta,\gamma) = S_{\alpha_2\beta_1}^{\gamma\ \delta_1}(\theta,\gamma) S_{\alpha_3\beta_2}^{\alpha_2\delta_2}(\theta,\gamma)\ldots S_{\alpha\ \beta_N}^{\alpha_N\delta_N}(\theta,\gamma) = \ -\!\!\begin{smallmatrix}|\\ |\end{smallmatrix}\!\!-\ \begin{smallmatrix}|\\ |\end{smallmatrix}\!\!-\ldots-\ \begin{smallmatrix}|\\ |\end{smallmatrix}\!\!-\,. \tag{4}$$

On the r.h.s. we have a matrix product with respect to the "horizontal space" and a tensor product with respect to N "vertical spaces".

We define the **transfer matrix** as a trace in the horizontal space

$$\tau(\theta,\gamma) = \mathrm{tr}_h\ T(\theta,\gamma). \tag{5}$$

Then we can write the partition function (1) as a trace in the tensor product of the vertical spaces

$$Z = \mathrm{tr}_v\,\tau^M. \tag{6}$$

If we are able to solve the eigenvalue problem of the transfer matrix

$$\tau\psi = \lambda\psi \tag{7}$$

we can calculate the partition function in terms of the eigenvalues

$$Z = \sum_i \lambda_i^M. \tag{8}$$

In the thermodynamic limit only the maximal eigenvalue contribute to the free energy per lattice site

$$f = -kT\frac{1}{NM}\log Z \approx -kT\frac{1}{NM}\log \lambda_{max}^{M} \qquad \text{for } N, M \to \infty. \tag{10}$$

2.2 Yang-Baxter algebra. The model is integrable which means the eigenvalue problem (7) can be solved since the monodromy matrix (4) generates a Yang-Baxter algebra. This will be shown in the following. For fixed parameter γ the matrix of vertex weights S fulfil the Yang-Baxter equation

$$S_{\alpha'\,\beta'}^{\alpha''\beta''}(\theta-\theta')\; S_{\alpha\,\gamma'}^{\alpha'\gamma''}(\theta)\; S_{\beta\,\gamma}^{\beta'\gamma'}(\theta') = S_{\beta'\,\gamma'}^{\beta''\gamma''}(\theta')\; S_{\alpha'\,\gamma}^{\alpha''\gamma'}(\theta)\; S_{\alpha\,\beta}^{\alpha'\beta'}(\theta-\theta'). \tag{11}$$

This identity represents a set of relations for the vertex weights a, b and c which follow from the parameterization (2) and trigonometric identities. The Yang-Baxter equation (11) for the matrix S implies by means of iterations the Yang-Baxter relation for the monodromy matrix T

$$S_{\alpha'\,\beta'}^{\alpha''\beta''}(\theta-\theta')\; T_{\alpha\;\{\gamma'\}}^{\alpha'\{\gamma''\}}(\theta)\; T_{\beta\;\{\gamma\}}^{\beta'\{\gamma'\}}(\theta') = T_{\beta'\;\{\gamma'\}}^{\beta''\{\gamma''\}}(\theta')\; T_{\alpha'\;\{\gamma\}}^{\alpha''\{\gamma'\}}(\theta)\; S_{\alpha\,\beta}^{\alpha'\beta'}(\theta-\theta'). \tag{12}$$

This will be the basic formula in the following. It reflects the integrability of the model. Note that it implies the commutativity of the transfer matrix for different spectral parameters

$$[\tau(\theta), \tau(\theta')] = 0 \tag{13}$$

which represents (an infinite number of) conservation laws. We write the monodromy matrix T defined by eq. (4) explicitly as a matrix in the two dimensional horizontal space

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{14}$$

where A, B, C and D are operators in the tensor product of the vertical spaces. From the Yang-Baxter relation (12) we obtain the commutation rules

$$\begin{aligned} A(\theta)B(\theta') &= \frac{a(\theta'-\theta)}{b(\theta'-\theta)}B(\theta')A(\theta) - \frac{c(\theta'-\theta)}{b(\theta'-\theta)}B(\theta)A(\theta') \\ D(\theta)B(\theta') &= \frac{a(\theta-\theta')}{b(\theta-\theta')}B(\theta')D(\theta) - \frac{c(\theta-\theta')}{b(\theta-\theta')}B(\theta)D(\theta') \end{aligned} \tag{15}$$

We do not need the other formulas in the following.

2.3 Algebraic Bethe ansatz. The resolution of the six-vertex model has been traced back to the eigenvalue problem of the transfer matrix

$$\tau(\theta)\psi = (A(\theta) + D(\theta))\psi = \lambda\psi \tag{16}$$

which will be solved by the **algebraic Bethe ansatz** method. This method is quite analogous to the algebraic procedure used in quantum mechanics to solve eigenvalue problems (as e.g. for the angular momentum) or more general to find irreducible representations of Lie-algebras. The Yang-Baxter relations (12) correspond to the Lie-algebra structure relations. The diagonal elements A and D of T play the rôle of the "weights". The off diagonal elements B and C are creation and annihilation operators of "pseudoparticles", respectively.

The "pseudovacuum" ϕ is defined by

$$C(\theta)\ \phi = \rightarrow\Big|-\ldots-\Big|\leftarrow = 0 \qquad \forall\ \theta\ . \tag{17}$$

From eqs. (2, 4 and 14) we find the solution of this equation

$$\phi^{\{\beta\}} = \prod_{i=1}^{N} \delta_{\beta_i 1} = \uparrow \ldots \uparrow\ . \tag{18}$$

It is an eigenstate of A and D, fulfilling

$$\begin{aligned} A(\theta)\ \phi &= \ \leftarrow\overset{\uparrow}{\uparrow}\leftarrow \ldots \leftarrow\overset{\uparrow}{\uparrow}\leftarrow = a^N(\theta)\ \phi \\ D(\theta)\ \phi &= \ \rightarrow\overset{\uparrow}{\uparrow}\rightarrow \ldots \rightarrow\overset{\uparrow}{\uparrow}\rightarrow = b^N(\theta)\ \phi\ . \end{aligned} \tag{19}$$

However, applying B to ϕ one gets something new

$$B(\theta)\ \phi = \leftarrow\Big|_{\uparrow}-\ldots-\Big|_{\uparrow}\rightarrow \neq 0, \phi. \tag{20}$$

This fact motivates us to write an eigenstate of the transfer matrix as an **algebraic Bethe ansatz**

$$\psi = B(\theta_1)\ldots B(\theta_n)\ \phi \tag{21}$$

where the parameters θ_i are to be determined.

We follow the general strategy [4] of the algebraic Bethe ansatz method to solve eq. (16). We apply $A(\theta)$ and $D(\theta)$ to ψ and push them through all the B's using the commutation rules (15). From the first terms on the r.h.s. of eq. (15) we get the "wanted" contributions

$$(A(\theta) + D(\theta))\ \psi = (\lambda_A(\theta) + \lambda_D(\theta))\ \psi\ +\ \text{"unwanted terms"} \tag{22}$$

where from eqs. (15 and 19) we find

$$\lambda_A(\theta) = a^N(\theta) \prod_{i=1}^{n} \frac{a(\theta_i - \theta)}{b(\theta_i - \theta)} \qquad \lambda_D(\theta) = b^N(\theta) \prod_{i=1}^{n} \frac{a(\theta - \theta_i)}{b(\theta - \theta_i)}. \tag{23}$$

The "unwanted terms" arise from the second terms in eqs. (15). They contain a $B(\theta)$ and so they can never give a vector proportional to ψ. The unwanted terms vanish if the **Bethe ansatz equations** hold

$$\left(\frac{b(\theta_j)}{a(\theta_j)}\right)^N \prod_{i=1}^{n} \frac{a(\theta_j - \theta_i)}{b(\theta_j - \theta_i)} \frac{b(\theta_i - \theta_j)}{a(\theta_i - \theta_j)} = -1 \qquad j = 1, \ldots, n. \tag{24}$$

Obviously these equations are necessary for ψ to be an eigenstate of τ. This is because the eigenvalue

$$\lambda(\theta) = \lambda_A(\theta) + \lambda_D(\theta) \tag{25}$$

cannot have poles at $\theta = \theta_i$ as eqs. (2 and 23) would suspect. The cancelation condition of these poles is equivalent to the Bethe ansatz equations (24). It can be shown [4] that they also imply the cancelation of the unwanted terms. We have transformed the eigenvalue problem of the transfer matrix (16) to sets of transcendental equations (24) for the parameters $\theta_1, \ldots, \theta_n,\ n = 0, \ldots, \frac{1}{2}N$ (Neven).

Solutions of Bethe ansatz equations are available for small or very large N and n. We are interested in the latter case. I will explain some more technical details. Introducing a "rapidity" q, a "momentum" p and a "scattering phase shift" Φ for the pseudoparticles by

$$\theta = \tfrac{1}{2}(iq + \gamma) \qquad e^{ip(q)} = \frac{\sinh \frac{1}{2}(q - i\gamma)}{\sinh \frac{1}{2}(q + i\gamma)} \qquad e^{i\Phi(q)} = \frac{\sinh \frac{1}{2}(q + 2i\gamma)}{\sinh \frac{1}{2}(q - 2i\gamma)}. \tag{26}$$

we can write the Bethe ansatz equations (24) as

$$\begin{gathered} \exp(iNp(q_j)) \prod_{i=1}^{n} \exp(\Phi(q_j - q_i)) = -1 \\ j = 1, \ldots, n \qquad 0 \leq n \leq N/2. \end{gathered} \tag{27}$$

The "physical" interpretation of this equations is, that the total phase factor around one period $1, \ldots, N$ has to be equal to one. This phase factor consists of the free part $\exp(iNp(q_j))$ and the scattering parts with respect to all other pseudoparticles

$\exp(i\Phi(q_j - q_i))$. Each set of roots of eq. (27) $\{q_1, \ldots, q_n\}$ defines an eigenstate of the transfer matrix by eq. (21) and an eigenvalue λ given by eqs. (23, 25 and 26)

$$\begin{aligned} \log \lambda_A(\theta, \gamma) &= N \log a - \sum_{i=1}^{n} e(q_i) \\ e(q) &= i(p(q + 2i\theta) - \pi) \end{aligned} \tag{28}$$

We will consider the region $0 < \theta < \gamma/2$ where $|\lambda_D| < |\lambda_A|$. So λ_D will not contribute to the partition function for large M.

There exist detailed investigations of the Bethe ansatz equations (27) (see e.g. ref. [10]). The analysis simplifies for the discrete values of the "coupling"

$$\gamma = \pi/\nu \qquad \nu = 2, 3, 4, \ldots . \tag{29}$$

We are interested in these values. So we assume eq. (29) in the following.

2.4 Solutions of Bethe ansatz equations: real roots. If we define the phase function

$$z(q) = Np(q) + \sum_{i=1}^{n} \Phi(q - q_i). \tag{30}$$

the logarithm of eq. (27) yields

$$2\pi I_j = z(q_j) \qquad j = 1, \ldots, n. \tag{31}$$

We observe that sets of roots $\{q_1, \ldots, q_n\}$ are given by sets of numbers $\{I_1, \ldots, I_n\} \subset \mathbf{Z} + \frac{1}{2} \ \cap \ \frac{1}{2\pi}]z(-\infty), z(\infty)[$ where eqs. (26 and 30) yield

$$\begin{aligned} z(-\infty) &= 2\pi n + 2\gamma\Sigma \\ z(\infty) &= 2\pi N - 2\gamma\Sigma \end{aligned} \tag{32}$$

and $\Sigma = \frac{1}{2}N - n$ is the "spin" of the Bethe state.

We introduce the density of roots

$$\rho(q) = 2\pi \sum_{i=1}^{n} \delta(q - q_i) \tag{33}$$

and the "density of holes"

$$\varphi(q) = \frac{dz(q)}{dq} - \rho(q). \tag{34}$$

The latter name is motivated by the fact that

$$\int_Q^{Q'} \frac{dq}{2\pi}\varphi(q) = I(Q') - I(Q) - \#\text{roots} \in [Q, Q']$$

is the number of possible minus the number of actual roots in an interval $[Q, Q']$. But note that only in the limit $N, n \to \infty$, where the roots become dense on the real line, the function φ is of the form

$$\varphi(q) = 2\pi \sum_{holes} \delta(q - q_h). \tag{35}$$

Taking the derivative d/dq of eq. (30) we obtain with eqs. (33 and 34)

$$z' = \rho + \varphi = Np' + \Phi' * \rho \tag{36}$$

where the convolution is defined by $(f*g)(q) = \int (dq/2\pi) f(q-q')g(q')$. Hence the density of roots of the Bethe ansatz equations (corresponding to eigenstates $\psi(q_1, \ldots, q_n)$ of the transfer matrix) can be written in terms of the "density of holes"

$$\rho = \rho_0 - R * \varphi \tag{37}$$

where $R = (1 - \Phi')^{-1}$ and $\rho_0 = N\, R * p'$ can be calculated by means of the Fourier transformation $f(q) = \int dx e^{-iqx} \tilde{f}(x)$

$$\tilde{R}(x) = \frac{\sinh \pi x}{2 \cosh \gamma x \, \sinh(\pi - \gamma)x} \qquad \tilde{p'}(x) = \frac{\sinh(\pi - \gamma)x}{\sinh \pi x} \tag{38}$$

$$\tilde{\rho}_0(x) = \frac{N}{2 \cosh \gamma x}. \tag{39}$$

The eigenvalue of the transfer matrix is given by

$$\begin{aligned} \log \lambda_A(\theta, \gamma) &= N \log a - \int \frac{dq}{2\pi} \rho(q) e(q) \\ &= N \log a - \int dx \tilde{\rho}(-x) \tilde{e}(x). \end{aligned} \tag{40}$$

In the thermodynamic limit $N \to \infty$ the problem is solved by eqs. (37-40) since φ is of the form (35).

(i) Ground state: There is a "Dirac-sea" of real roots. Since $\Re e(q) < 0$ for real q the eigenvalue is maximal if all possible real roots are occupied. From eq. (32) follows $n = \frac{1}{2}N$ and $\Sigma = 0$ which means that the ground state is a singlet. Eqs. (38-40) imply

$$\rho(q) = \rho_0(q) = \frac{N\pi}{2\gamma} \left(\cosh \frac{\pi q}{2\gamma} \right)^{-1} \tag{41}$$

$$\log \lambda_A(\theta,\gamma) = \log \lambda_{A,0}(\theta,\gamma) = N \log a + N \int_{-\infty}^{\infty} \frac{dx}{x} \frac{\sinh(\pi-\gamma)x \, \sinh 2\theta x}{2\cosh \gamma x \, \sinh \pi x}. \tag{42}$$

(ii) Holes: For excitations given by holes at q_h we find from eqs. (35, 37, 38 and 40)

$$\rho(q) = \rho_0(q) - \sum_{holes} R(q - q_h) \tag{43}$$

$$\lambda_A(\theta,\gamma) = \lambda_{A,0}(\theta,\gamma) \prod_{holes} \tan \frac{\pi}{\gamma} \left(i q_h + \tfrac{1}{4}(\gamma - 2\theta) \right). \tag{44}$$

2.5 Solutions of Bethe ansatz equations: complex roots. For "excited states" there may be, in addition to holes, complex roots q: the "k-strings" (one could say "bound states" of k pseudoparticles)

$$q = \text{real} + i\gamma(-k+1, -k+3, \ldots, k-1) \qquad 1 < k \leq \nu - 1 \tag{45}$$

and the "antipseudoparticle"

$$q = \text{real} + i\pi. \tag{46}$$

The density of the real roots and the eigenvalue λ, if complex roots are present, can be computed analogously to eqs. (43 and 44) (see e.g. refs. [10,1]).

3. Conformal Invariance.

3.1 Virasoro algebra. In two dimensions conformal invariance is a rather strong restriction since the transformation group is infinite dimensional, in contrast to higher dimensions. It is related to the Virasoro algebra [11]

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \tfrac{1}{12} c (n^3 - n)\delta_{n,-m} \\ c &= \text{"central charge"} \end{aligned} \tag{47}$$

for which a well developed theory exists [12]. Belavin, Polyakov and Zamolodchikov [13] classified conformal quantum field theories in terms of the central charge

$$c = 1 - \frac{6}{\nu(\nu-1)} \tag{48}$$

where ν is a rational number if there exists only a finite number of "primary fields", i.e. fields transforming under conformal transformations

$$\begin{aligned} z = x + iy &\to w(z) \\ \bar{z} = x - iy &\to \bar{w}(\bar{z}) \end{aligned} \tag{49}$$

like

$$A(z,\bar{z}) \to \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} A(w,\bar{w}). \tag{50}$$

The conformal dimensions can assume values given by the Kac-formula [12]

$$\begin{aligned} &\Delta,\bar{\Delta} = h_{pq} = \frac{(\nu p-(\nu-1)q)^2-1}{4\nu(\nu-1)} \\ &1 \le q \le p < \nu-1 \qquad p,q = \text{integer}. \end{aligned} \tag{51}$$

Friedan, Qiu and Shenker [14] have shown that unitarity restricts the number ν to

$$\nu = \text{integer} \ge 2 \quad \text{or} \quad c \ge 1. \tag{52}$$

It would be interesting to find integrable lattice models which yield in the scaling limit conformal quantum field theories corresponding to these unitary representations of the Virasoro algebra. Therefore it is useful to calculate the central charge and the conformal dimensions for integrable critical statistical models. This can be done by analysing finite size effects.

3.2 Finite size behaviour. For a statistical system on a periodic $N \times M$ square lattice one introduces a transfer matrix τ as explained in sect. 2. The partition function can be written as

$$Z = \operatorname{tr} \tau^M = \sum_i \lambda_i^M$$

where the sum extends over all eigenvalues of the transfer matrix. Cardy [15] has shown that conformal invariance implies for the maximal eigenvalue

$$\lambda_{max} \approx \exp(-Nf + \frac{1}{N}\frac{\pi}{6}c) \qquad \text{for} \quad N \to \infty \tag{53}$$

where f is the free energy per site (in the thermodynamic limit) and c is the central charge of the Virasoro algebra (cf. eq. (47)).

Introducing in addition the "crossed" transfer matrix $\hat{\tau}$ (defined on an M-chain) we obtain for $1 \ll N \ll M$ two expressions for the partition function

$$Z \approx \exp(-NMf) \sum_i (\hat{\lambda}_i/\hat{\lambda}_{max})^N \tag{54a}$$

$$Z \approx \exp(-NMf + \frac{M}{N}\frac{\pi}{6}c) \tag{54b}$$

where only the "low energy excitations" with $\hat{\lambda}_i \approx \hat{\lambda}_{max}$ contribute to the sum in eq. (54a). Hence two methods to investigate the partition function on a strip $1 \ll N \ll M \to \infty$ are available:

(a) one considers the transfer matrix $\hat{\tau}$ for an infinite chain ($M \to \infty$) and calculates the large N corrections to the sum in eq. (54a) or

(b) one considers the transfer matrix τ on an N-sites chain and calculates the large N corrections to the maximal eigenvalue λ_{max} according to eq. (54b).

By both of these procedures one obtains the central charge of the model.

In addition the conformal dimensions of operators of the model can be determined if one looks for excited states with energy $E = -\Re \log \lambda$ and momentum $P = -\Im \log \lambda$. As is argued in ref. [15] one has for $N \to \infty$

$$\begin{aligned} E_i - E_0 &\approx \frac{2\pi}{N} d_i \\ P_i - P_0 &\approx \frac{2\pi}{N} s_i \end{aligned} \tag{55}$$

where $d_i = \Delta + \bar{\Delta}$ ($s_i = \Delta - \bar{\Delta}$) is the "scaling dimension" ("spin") of the operator A associated to the excitation ψ_i, i.e. $\langle 0|A|\psi_i\rangle \neq 0$.

Since a direct calculation of the central charge and the conformal dimensions for specific models is usually not simple, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. This will be done in the following for the six-vertex model and for some modified six-vertex models related to Potts models and other models of interest. Method (a) has been developed in ref. [1], method (b) is applied in sect. 4 and ref. [2].

4. Results for Six-Vertex and Potts Models.

The Potts models are a generalizations of the Ising model where the spin at each site can accept q values instead of two. For $q = 2, 3, 4$ the models possess a critical point which we consider in the following. The partition function on a cylinder [9] can be written in terms of a six-vertex model with a "seam" from the bottom to the top of the cylinder. This seam carries extra Boltzmann weights which produce an extra term $-i\gamma m$ on the r.h.s. of eq. (28) and extra terms $2\gamma m$ on the r.h.s. of eqs. (30 and 32) with $m = 1$[2]. The number q ($q = 2, 3, 4$) is related to the six-vertex coupling $\gamma = \pi/\nu$ ($\nu = 4, 6, \infty$) by

$$q = 4\cos^2 \gamma. \tag{56}$$

In the following the partition function Z_m and other formulas apply for $m = 0$ to the ordinary six-vertex model and for $m = 1$ to the Potts model

$$Z_0 = Z^{6-vertex} \qquad Z_1 = Z^{Potts}. \tag{57}$$

[2]The variable m has been introduced for later convenience.

4.1 Ground state. In refs. [16,17,1,2] techniques were developed to calculate finite size corrections for a large chain of length N. Using these methods we find [1,2] for the "ground state"

$$\log \lambda_{max} \approx -Nf + \frac{1}{N}\frac{\pi}{6}\left(1 - \frac{6m^2}{\nu(\nu-1)}\right)\sin\nu\theta. \tag{58}$$

The vertex weights (2) fulfil $a = b$ for $\nu\theta = \pi/2$ where we expect isotropy and conformal invariance in the scaling limit. Comparing eqs. (58) and (53) we conclude that the central charges of the Virasoro algebra of the six-vertex and the q-state Potts models are

$$c^{6-vertex} = 1 \tag{59a}$$

$$c^{Potts} = 1 - \frac{6}{\nu(\nu-1)}. \tag{59b}$$

4.2 Excitations. As already mentioned in sect. 2 the ground state correspond to the maximal set of real roots $q_1, \ldots, q_{N/2}$. Excitations possess "holes" and complex roots. We obtain [2] for these states taking eqs. (55) into account[3]

$$\log \lambda_i - \log \lambda_{max} \approx -\frac{2\pi}{N}\{(\Delta + \bar{\Delta})\sin\nu\theta + i(\Delta - \bar{\Delta})\cos\nu\theta\}. \tag{60}$$

The general expressions for the conformal dimensions Δ and $\bar{\Delta}$ are rather complicated. But the primary fields correspond to low energy excitations with a small number of holes $h_\pm$ "near to the last roots" $q_\pm$ which approach $\pm\infty$ for $N \to \infty$. In addition there may be a number $n_\pm^{(k)}$ of k-strings "near to $q_\pm$"[4]. For these configurations of roots we obtain [2]

$$\Delta = \frac{\left(\nu\left(h_+ - 2\sum_k n_+^{(k)}(k-1)\right) - \Sigma + m\right)^2 - m^2}{4\nu(\nu-1)} + \text{integer} \tag{61a}$$

$$\bar{\Delta} = \frac{\left(\nu\left(h_- - 2\sum_k n_-^{(k)}(k-1)\right) - \Sigma - m\right)^2 - m^2}{4\nu(\nu-1)} + \text{integer} \tag{61b}$$

where $\Sigma = \frac{1}{2}h - \sum_k n^{(k)}(k-1)$ is the "spin"[5] of the Bethe state and h ($n^{(k)}$) is the total number of holes (k-strings). We have $\Sigma = 0$ since we consider fixed boundary conditions

[3] Note that the imaginary term correspond to the momentum since $\tau(\theta = 0) = \exp(-iP)$.

[4] The k-strings which do not approach $\pm\infty$ like the last real roots $q_\pm$ as $N \to \infty$ will not affect Δ and $\bar{\Delta}$.

[5] Σ must not be confused with the "spin" s of the field operator (cf. eq. (55)).

at the top and the bottom of the cylinder. It turns out [2] that we get minimal values for Δ and $\bar{\Delta}$ (i.e. the integers on the r.h.s. of eqs. (61) vanish) if some conditions are fulfilled: There are no holes inside $[q_-, q_+]$, $n_+^{(k)} = 0$ for all k, $n_-^{(k)} = 1$ for only one k and the number of holes have to fulfil $h_+ = h_- = k - 1$. For the six-vertex model we obtain the conformal dimensions from eqs. (61) with $m = 0$. For the q-state Potts model the extra weights on the seam imply [2] that for the minimal Δ and $\bar{\Delta}$ only the k-strings with

$$k = 2, 3, \ldots, \nu - m - 1 \tag{62}$$

contribute (i.e. for $m = 1$ the $\nu - 1$-string and the "antipseudoparticle" do not appear).

Comparing eqs. (61) for $m = 1$ and $\Sigma = 0$ with the Kac-formula (51) we find for the Potts model the conformal dimensions

$$\Delta = \bar{\Delta} = h_{k,1} \qquad k = 1, 2, \ldots, \nu - 2. \tag{63}$$

These correspond to the "energy operators" of the Potts model (cf. [18, 15, 19]). The equivalence of the Potts model and a six-vertex model with "seam" for $m = 1$ holds for the partition functions (cf. eq. (57)) but not for the operator content. The order parameter of the Potts model, for example, appears in a six-vertex model with a "seam" for $m = \nu/2$ (cf. [15]).

If we consider a partition function of a generalized model

$$Z = Z_1 + \sum Z_m \tag{64}$$

where the sum extends over some values of $m \in [2, \nu - 1]$ we obtain the dimensions corresponding to Bethe states with a k-string

$$\Delta = \bar{\Delta} = h_{k+m-1,m} \qquad k = 1, \ldots, \nu - m - 1 \quad 1 \leq m \leq \nu - 1. \tag{65}$$

Note that the ground state always belongs to Z_1. The "magnetization" operators of the $q = 3$-state Potts model ($\nu = 6$) correspond to $m = \nu/2 = 3$ (cf. [19]). Their dimensions are therefore (in agreement with numerical estimates [19] and conjectures [18,15])

$$\Delta = \bar{\Delta} = h_{k+2,3} \qquad k = 1, 2. \tag{66}$$

The $1/q$-para fermion operators [20] correspond to "boundary conditions" $m = \nu/q = 2$ for $q = 2, 3$ [20,19]. The Bethe states have spin $\Sigma = 1$ and a symmetric hole configuration $h_+ = h_- = 1$. For $\Sigma \neq 0$ the integers on the r.h.s. of eqs. (61)

only vanish, if there are no higher strings. Hence we obtain from eq. (61) the scaling dimensions and spin

$$\begin{aligned} d &= \Delta + \bar{\Delta} = h_{21} + h_{23} = \frac{\nu - 1}{2\nu} + \frac{\nu^2 - q^2}{2q^2\,\nu(\nu-1)} \\ s &= \Delta - \bar{\Delta} = h_{21} - h_{23} = 1/q \end{aligned} \tag{67}$$

in agreement with ref. [20,19]. Finally let us look for the "chiral" operators of the 3-state Potts model (cf. [18,15]). Their conformal dimensions are given by the "boundary conditions" $m = \nu/2 = 3$ and Bethe states with $\Sigma = 2$ or four holes. For $h_+ = 0, 1, 2, 3$ and $h_- = 4 - h_+$ one finds from eq. (61), respectively

$$\Delta = h_{11}, h_{21}, h_{31}, h_{41} = 0, \tfrac{2}{5}, \tfrac{7}{5}, 3$$

$$\bar{\Delta} = h_{15}, h_{25}, h_{35}, h_{45} = 3, \tfrac{7}{5}, \tfrac{2}{5}, 0.$$

At the end let us consider a partition function given by eq (64) where the sum extends over all $m = 2, \ldots, \nu - 1$. The dimension formula (65) obviously describes the dimensional spectrum of the RSOS models [21], in the language of ref. [22] the $(A_\nu, A_{\nu-1})$ series of conformal quantum field theories. For this interpretation of eq. (64) the number m is not connected to extra weights on a "seam" or to "boundary conditions", but it is related [2] to a "Θ-vaccum" like angle appearing in the construction of Bethe states [4]. The RSOS models of ref. [21] can be generalized [23] to models corresponding to all modular invariant conformal quantum field theories with $c < 1$ [22]. All these models are related to modified six-vertex models. Results for these models will be published elsewhere [24].

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