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YANG-BAXTER ALGEBRA - INTEGRABLE SYSTEMS - CONFORMAL QUANTUM FIELD THEORIES*

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ABSTRACT. This series of lectures is based on investigations [1,2] of finite-size corrections for the six-vertex model by means of Bethe ansatz methods. In addition a review on applications of Yang-Baxter algebras and an introduction to the theory of integrable systems and the algebraic Bethe ansatz is presented. A Θ-vacuum like angle appearing in the RSOS-models is discussed. The continuum limit in the critical case of these statistical models is performed to obtain the "minimal models" of conformal quantum field theory.

TABLE OF CONTENTS:

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1. Introduction.

There exists a large number of fields in theoretical and mathematical physics where **Yang-Baxter Algebras** play a central role for solving problems. Some of them are:

(i) Quantum chains in 1-dimension.
E.g. Heisenberg models, as investigated by Bethe in 1931 [3]*. These models are solved by the "Bethe ansatz" which, in its algebraic formulation invented by Faddeev [4], is based on the Yang-Baxter algebra.

(ii) Classical statistical lattice models in 2-dimensions.
E.g. Ising- and vertex models, as investigated by Onsager in 1944 [5]* and Baxter in 1972 [6]*, respectively. These models are solved by means of the "star-triangle equations" which are equivalent to Yang-Baxter equations.

(iii) Quantum field theories in 1+1-dimensions.
E.g. the nonlinear Schrödinger field theory, as investigated by Yang in 1967 [7]*.

(iv) Soliton S-matrices in 1+1-dimensions.
The "factorization equations" [8]* which have been used to determine soliton S-matrices are Yang-Baxter equations.

Let me interrupt listing applications in order to introduce the Yang-Baxter algebra through this example, since its algebraic structure is quite obvious in terms of soliton S-matrices. Soliton scattering does not change the momenta of the particles but only quantum numbers are exchanged. The N-particle S-matrix is a product of two particle ones. In particular the three-particle scattering is given by two possible orders of the factors which have to give the same result

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}. \tag{1.1}$$

We define the two particle S-matrix elements (cf. fig. 1) by the action of S_{ik} on an N-particle state with momenta p_i and quantum numbers $\alpha_i = 1, \ldots, n$

$$S_{ik}|\ldots;p_i,\alpha_i;\ldots;p_k,\alpha_k;\ldots\rangle = |\ldots;p_i,\alpha_i';\ldots;p_k,\alpha_k';\ldots\rangle\ S_{\alpha_i\,\alpha_k}^{\alpha_i'\alpha_k'}(\vartheta_{ik}) \tag{1.2}$$

*and later works

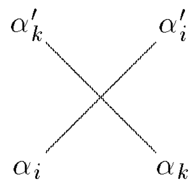


Fig. 1. *The two particle S-matrix*

where $p_i p_k = m_i m_k \cosh \vartheta_{ik}$. This represents S_{ik} as a matrix in the tensor product space $V_i \otimes V_k$ of two n-dimensional spaces.

The factorization or Yang-Baxter equation (1.1) reads more explicitly (cf. fig. 2)

$$S_{\alpha'\,\beta'}^{\alpha''\beta''}(\vartheta_{12})\; S_{\alpha\,\gamma'}^{\alpha'\gamma''}(\vartheta_{13})\; S_{\beta\,\gamma}^{\beta'\gamma'}(\vartheta_{23}) = S_{\beta'\,\gamma'}^{\beta''\gamma''}(\vartheta_{23})\; S_{\alpha'\,\gamma}^{\alpha''\gamma'}(\vartheta_{13})\; S_{\alpha\,\beta}^{\alpha'\beta'}(\vartheta_{12}). \quad (1.3)$$

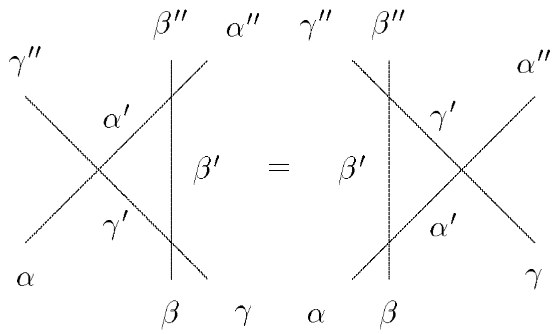


Fig. 2. *Two different orderings of a three-particle scattering process obtained by parallel shifting a line*

We can generalize eq. (1.1) as follows. For any solution of equation (1.1) we write the "structure relations"

$$S_{12} T_{13} T_{23} = T_{23} T_{13} S_{12} \quad (1.4)$$

of a **"Yang-Baxter algebra"** generated by the T's. The product in eq. (1.4) has to be understood analogously to eq. (1.3). However, the index γ now correspond to an abstract space V_3 and T_{i3} is an operator in the space $V_i \otimes V_3$.

(v) "Quantum groups". [9]
Sometimes (i.e. for the trigonometric case) the generators of a Yang-Baxter algebra $T(M^a, \vartheta, \gamma)$ may be written in terms of generators M^a and some complex

parameters ϑ and γ (cf. sections 2 and 3). In the limit

$$\vartheta \to i\infty \quad \text{and} \quad \gamma \to 0 \tag{1.5}$$

typically one finds that the Yang-Baxter relations (1.4) imply Lie-group commutation rules

$$[M^a, M^b] = f^{abc} M^c. \tag{1.6}$$

For nonvanishing γ one finds nonlinear commutation relations

$$[M^a, M^b] = f^{ab}(\gamma, M^c). \tag{1.7}$$

Interpreting eq. (1.6) as a Poisson algebra and γ as $\hbar$ one says eq. (1.7) describes the quantum version of the algebra defined by eq. (1.6). Or writing $q = e^{i\gamma}$ one says eq. (1.7) describes a "q-deformation" of the Lie-group. In section 3 we will consider an example of a quantum group.

(vi) Braids. [10]
Consider N points on a line in R^3 and the corresponding set of points on a parallel shifted line. A braid is given by set of nonintersecting monotonic curves connecting the points on one line with those on the other, see fig. 3. Two set of curves which can be continuously deformed into each other define the same braid.

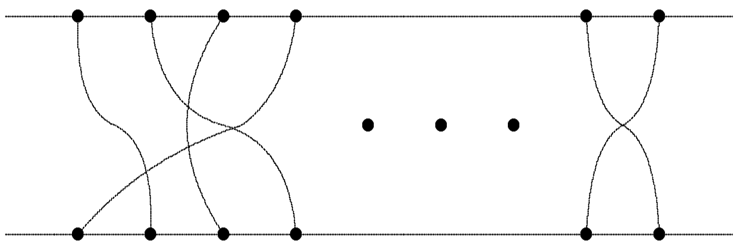
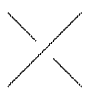


Fig. 3. *A braid given by nonintersecting curves in* R^3

Defining a product by putting two braids on top of each other they form a group. Artin has shown that a braid can by written as a product of generators of the form

 $= \tau$ and  $= \tau^{-1}$.

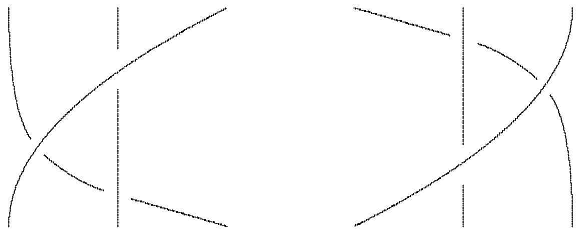


Fig. 4. *Reidemester move*

Deformations can be generated by the "Reidemeister move" of fig. 4.

Obviously, the generators of braids can be represented by Yang-Baxter matrices

$$\tau = S(\vartheta = i\infty) \quad \text{and} \quad \tau^{-1} = S(\vartheta = -i\infty) \tag{1.8}$$

since the identity of fig. 4 follows from the Yang-Baxter equations (1.3). One has to take the "spectral parameter" ϑ at infinity, because otherwise the sum rules of the ϑ's in eq. (1.3) cannot be fulfilled. Apparently, both, quantum groups and braids, correspond to Yang-Baxter algebras for discrete (infinite large) values of the spectral parameter.

(vii) Statistics matrix in 2-dimensional quantum field theories. [11]
Consider spacelike commutation rules of fields

$$\phi_A(x)\phi_B(x') = \phi_D(x')\phi_C(x)\ S_{AB}^{CD}(\epsilon(x_1 - x_1')). \tag{1.9}$$

In contrast to four dimensions the statistics is not necessarily bosonic or fermionic, but the statistics matrix S might be more general. Commuting three field $\phi_A\phi_B\phi_C$ in two different ways one finds that S fulfills braid- or Yang-Baxter commutation rules (1.3) for discrete ϑ's.

(viii) Conformal quantum field theories in 2-dimensions. [12]
A conformal field in two dimensions can be written as a sum of products of "light cone fields"

$$\phi(x) = \sum_i A_i(x+t)B_i(x-t) \tag{1.10}$$

Analogously to eq. (1.9) the light cone fields fulfil the commutation rule

$$A_i(u)A_j(u') = A_l(u')A_k(u)\ R_{ij}^{kl}(\epsilon(u-u')). \tag{1.11}$$

There exists a number of models (e.g. the "minimal" models considered below) with highly nontrivial braid representations R, whereas the statistics matrices of eq. (1.9) are usually much simpler.

In sections 2, 3, 4, 5 and 7 I will explain the Yang-Baxter algebra methods in terms of the six-vertex model, a model of type (ii). Section 6 contains a short introduction to conformal quantum field theories in two dimensions and section 7 applications of Bethe ansatz results to conformal models.

Conformal invariance of two dimensional quantum field theories has become a subject of main interest because of its connection to string theory. A string forms a two dimensional world sheet in space time. Reparametrization invariance of this surface causes the importance of conformal concepts in string theories. Statistical mechanical systems at second order phase transition points (i.e. critical systems) should possess conformal invariance[1]. We shall use this fact to obtain information on conformal quantum field theories. It would be interesting to find **integrable** lattice models which yield these conformal quantum field theories in the scaling limit.

Since a direct calculation of the central charge of the Virasosro algebra and the conformal dimensions for specific models is usually not simple, it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. This will be done for the six-vertex model (the critical eight-vertex model) and for some modified six-vertex models related to Potts- and RSOS-models.

2. Six-vertex model.

I like to explain the Yang-Baxter algebra method in terms of the six-vertex model, a classical statistical lattice model in two dimensions. Its partition function on an $N \times M$ periodic square lattice L is

$$Z^{6-vertex} = \sum_{conf.} \prod_{x \in L} S(x) \tag{2.1}$$

[1]New results on representation theory of conformal invariance explain the old mysterious observation that critical exponents in two dimensions are usually rational numbers.

where the sum extends over all allowed "bond configurations". Each bond can accept one of two states characterized by $\alpha = +, -$ or arrows

$$
\begin{aligned}
\text{horizontal bonds} \quad & \alpha = +, - = \leftarrow, \ \rightarrow \\
\text{vertical bonds} \quad & \beta = +, - = \ \uparrow, \ \downarrow \ .
\end{aligned}
$$

The configuration is allowed if at each vertex x there are two incoming and two outgoing arrows. The vertex weights $S(x)$ are determined by the six bond configurations at the lattice site x. They assume the following values

$$
\begin{aligned}
S^{++}_{++} &= \leftarrow\!\overset{\uparrow}{\underset{\uparrow}{\vert}}\!\leftarrow = S^{--}_{--} = \rightarrow\!\overset{\downarrow}{\underset{\downarrow}{\vert}}\!\rightarrow \ = a = \sin(\gamma - \vartheta) \\
S^{+-}_{+-} &= \leftarrow\!\overset{\downarrow}{\underset{\downarrow}{\vert}}\!\leftarrow = S^{-+}_{-+} = \rightarrow\!\overset{\uparrow}{\underset{\uparrow}{\vert}}\!\rightarrow \ = b = \sin\vartheta \\
S^{+-}_{-+} &= \leftarrow\!\overset{\downarrow}{\underset{\uparrow}{\vert}}\!\rightarrow = S^{-+}_{+-} = \rightarrow\!\overset{\uparrow}{\underset{\downarrow}{\vert}}\!\leftarrow \ = c = \sin\gamma.
\end{aligned}
\tag{2.2}
$$

By this notation the weights have been arranged as a $4 \times 4-$ matrix

$$
S^{\gamma\delta}_{\alpha\beta}(\vartheta,\gamma) = \ \gamma -\!\!\overset{\delta}{\underset{\beta}{\vert}}\!\!-\alpha = \begin{pmatrix} \leftarrow\!\overset{\uparrow}{\underset{\uparrow}{\vert}}\!\leftarrow & & & \\ & \leftarrow\!\overset{\downarrow}{\underset{\downarrow}{\vert}}\!\leftarrow & \leftarrow\!\overset{\downarrow}{\underset{\uparrow}{\vert}}\!\rightarrow & \\ & \rightarrow\!\overset{\uparrow}{\underset{\downarrow}{\vert}}\!\leftarrow & \rightarrow\!\overset{\uparrow}{\underset{\uparrow}{\vert}}\!\rightarrow & \\ & & & \rightarrow\!\overset{\downarrow}{\underset{\downarrow}{\vert}}\!\rightarrow \end{pmatrix} = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}.
\tag{2.3}
$$

The parameterization in terms of the "spectral parameter" ϑ and the "coupling" γ has been introduced for later convenience (cf. eq. (3.1)). A trivial overall factor is set equal to one.

In order to express the partition function in terms of the matrix S we define the **monodromy matrix**

$$
\begin{aligned}
T^{\gamma\{\delta\}}_{\alpha\{\beta\}}(\vartheta,\gamma) &= S^{\gamma\ \delta_1}_{\alpha_2\beta_1}(\vartheta,\gamma) S^{\alpha_2\delta_2}_{\alpha_3\beta_2}(\vartheta,\gamma) \ldots S^{\alpha_N\delta_N}_{\alpha\ \ \beta_N}(\vartheta,\gamma) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
\gamma -\!\!\overset{\{\delta\}}{\underset{\{\beta\}}{\Vert}}\!\!-\alpha &= \ -\!\vert\!- \vert\!- \ldots - \vert\!- \ = \begin{pmatrix} \leftarrow\!\Vert\!\leftarrow & \leftarrow\!\Vert\!\rightarrow \\ \rightarrow\!\Vert\!\leftarrow & \rightarrow\!\Vert\!\rightarrow \end{pmatrix}.
\end{aligned}
\tag{2.4}
$$

We have a matrix product with respect to the "horizontal space" and a tensor product with respect to N "vertical spaces".

We define the **transfer matrix** as a trace in the horizontal space

$$
\tau(\vartheta,\gamma) = \mathrm{tr}_h \ T(\vartheta,\gamma).
\tag{2.5}
$$

Then we can write the partition function (2.1) as a trace in the tensor product of the vertical spaces

$$Z = \mathrm{tr}_v\, \tau^M. \tag{2.6}$$

If we are able to solve the eigenvalue problem of the transfer matrix

$$\tau\psi = \lambda\psi \tag{2.7}$$

we can calculate the partition function in terms of the eigenvalues

$$Z = \sum_i \lambda_i^M. \tag{2.8}$$

In the thermodynamic limit only the maximal eigenvalue contribute to the free energy per lattice site

$$f = -kT\frac{1}{NM}\log Z \approx -kT\frac{1}{NM}\log\lambda_{max}^M \qquad \text{for } N, M \to \infty. \tag{2.10}$$

In the following we will solve the eigenvalue equation (2.7).

3. Yang-Baxter algebra.

The model is integrable which means the eigenvalue problem (2.7) can be solved since the monodromy matrix (2.4) generates a Yang-Baxter algebra. This can be shown as follows.

Proposition:

For fixed parameter γ the matrix of vertex weights S fulfills the Yang-Baxter equation

$$S_{\alpha'\,\beta'}^{\alpha''\beta''}(\vartheta-\vartheta')\; S_{\alpha\,\gamma'}^{\alpha'\gamma''}(\vartheta)\; S_{\beta\,\gamma}^{\beta'\gamma'}(\vartheta') = S_{\beta'\,\gamma'}^{\beta''\gamma''}(\vartheta')\; S_{\alpha'\,\gamma}^{\alpha''\gamma'}(\vartheta)\; S_{\alpha\,\beta}^{\alpha'\beta'}(\vartheta-\vartheta'). \tag{3.1}$$

Sums have to be taken over the internal indices $(\alpha', \beta', \gamma')$. This identity represents a set of relations for the vertex weights a, b and c.

Proof:

As an example take the external indices $(\alpha'', \beta'', \gamma'') = (+, +, -)$ and $(\alpha, \beta, \gamma) = (-, +, +)$, then eq. (3.1) means

$$S_{++}^{++}S_{-+}^{+-}S_{++}^{++} = S_{+-}^{+-}S_{-+}^{+-}S_{++}^{-+} + S_{-+}^{+-}S_{++}^{++}S_{-+}^{+-} \quad \text{or} \quad aca = bcb + cac.$$

The arguments on the r.h.s. and the l.h.s. have to be taken as in eq. (3.1). The last equation follows from the parameterization (2.2) and trigonometric identities.

Corollary:

The Yang-Baxter equation (3.1) for the matrix S implies by means of iterations the Yang-Baxter relation for the monodromy matrix T

$$S_{\alpha'\,\beta'}^{\alpha''\beta''}(\vartheta-\vartheta')\,T_{\alpha\;\{\gamma'\}}^{\alpha'\{\gamma''\}}(\vartheta)\,T_{\beta\;\{\gamma\}}^{\beta'\{\gamma'\}}(\vartheta')=T_{\beta'\;\{\gamma'\}}^{\beta''\{\gamma''\}}(\vartheta')\,T_{\alpha'\;\{\gamma\}}^{\alpha''\{\gamma'\}}(\vartheta)\,S_{\alpha\,\beta}^{\alpha'\beta'}(\vartheta-\vartheta'). \tag{3.2}$$

This will be the basic formula in the following. It reflects the integrability of the model. Note that it implies the commutativity of the transfer matrix for different spectral parameters

$$[\tau(\vartheta),\tau(\vartheta')]=0 \tag{3.3}$$

which represents (an infinite set of) conservation laws.

$$[c_n,c_m]=0 \quad , \quad \text{where} \quad \log\tau=-\sum c_n\vartheta^n. \tag{3.4}$$

The lowest operators are related to the momentum and the Heisenberg model Hamiltonian

$$c_0=iP \quad , \qquad c_1=H_{xxz}. \tag{3.5}$$

We write the monodromy matrix T defined by eq. (2.4) explicitly as a matrix in the two dimensional horizontal space

$$T=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.6}$$

where A, B, C and D are operators in the tensor product of the vertical spaces. From the Yang-Baxter relation (3.2) we obtain the commutation rules

$$\begin{aligned}
[A(\vartheta),A(\vartheta')]=&[B(\vartheta),B(\vartheta')]=[C(\vartheta),C(\vartheta')]=[D(\vartheta),D(\vartheta')]=0\\
A(\vartheta)B(\vartheta')=&\frac{a(\vartheta'-\vartheta)}{b(\vartheta'-\vartheta)}B(\vartheta')A(\vartheta)-\frac{c(\vartheta'-\vartheta)}{b(\vartheta'-\vartheta)}B(\vartheta)A(\vartheta')\\
D(\vartheta)B(\vartheta')=&\frac{a(\vartheta-\vartheta')}{b(\vartheta-\vartheta')}B(\vartheta')D(\vartheta)-\frac{c(\vartheta-\vartheta')}{b(\vartheta-\vartheta')}B(\vartheta)D(\vartheta')\\
[C(\vartheta),B(\vartheta')]=&\frac{c(\vartheta-\vartheta')}{b(\vartheta-\vartheta')}\big(A(\vartheta')D(\vartheta)-A(\vartheta)D(\vartheta')\big)\\
[D(\vartheta),A(\vartheta')]=&\frac{c(\vartheta-\vartheta')}{b(\vartheta-\vartheta')}\big(B(\vartheta')C(\vartheta)-B(\vartheta)C(\vartheta')\big).
\end{aligned} \tag{3.7}$$

At the end of this section let me show how to find a **"quantum group"** corresponding to the Lie-group $SU(2)$ from the Yang-Baxter algebra defined by eqs. (2.2), (3.1) and (3.2) in the limit $\vartheta \to \pm i\infty$. This procedure is typical for "trigonometric" solutions of the Yang-Baxter equations (3.1).

First we can obviously write the matrix of Boltzmann weights of the six-vertex model given by eqs. (2.2 and 2.3) in terms of spin-$\frac{1}{2}$ matrices s_3 and $s_\pm = s_1 \pm i s_2$ ($s_i = \frac{1}{2}\times$ the Pauli matrices) as follows

$$S = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} = \begin{pmatrix} \sin(\vartheta' + s_3\gamma') & s_- \sin\gamma' \\ s_+ \sin\gamma' & \sin(\vartheta' - s_3\gamma') \end{pmatrix} \tag{3.8}$$

where $\vartheta' = \vartheta + \gamma'/2$ and $\gamma' = \pi - \gamma$. One can show that the Yang-Baxter equation (3.1) for the matrix S implies the $SU(2)$ structure relations of the spin matrices

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \quad \Rightarrow \quad [s_3, s_\pm] = \pm s_\pm \ , \quad [s_+, s_-] = 2s_3. \tag{3.9}$$

Expanding the monodromy matrix (2.4) for $\vartheta' \to i\infty$ we can write, analogously to eq. (3.8)

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \propto \begin{pmatrix} \sin(\vartheta' + J_3\gamma') & J_- \sin\gamma' \\ J'_+ \sin\gamma' & \sin(\vartheta' - J_3\gamma') \end{pmatrix}. \tag{3.10}$$

For $\vartheta' \to -i\infty$ this relations holds with $J_\pm \leftrightarrow J'_\pm$ where $J'_\pm(\gamma') = J_\pm(-\gamma') = \big(J_\mp(-\gamma'^*)\big)^+$. From the Yang-Baxter relations (3.2) for T we find using eqs. (3.7) the commutation rules for the J's

$$S_{12}T_{13}T_{23} = T_{23}T_{13}S_{12} \quad \Rightarrow \quad [J_3, J_\pm] = \pm J_\pm \ , \quad [J_+, J_-] = \frac{\sin(\gamma 2J_3)}{\sin\gamma}. \tag{3.11}$$

These commutation rules describe a deformation or the "quantum version" of $SU(2)$. One can write formally:

$$\text{Yang-Baxter algebra} \quad \xrightarrow{\vartheta \to i\infty} \quad \text{quantum group} \quad \xrightarrow{\gamma \to 0} \quad \text{Lie-group.}$$

4. Algebraic Bethe ansatz.

4.1 Bethe ansatz equations.

The resolution of the six-vertex model has been traced back to the eigenvalue problem of the transfer matrix

$$\tau(\vartheta)\psi = (A(\vartheta) + D(\vartheta))\psi = \lambda\psi \tag{4.1}$$

which will be solved by the **algebraic Bethe ansatz** method [4]. This method is quite analogous to the algebraic procedure used in quantum mechanics to solve eigenvalue problems (as e.g. for the angular momentum) or more general to find irreducible representations of Lie-algebras. The Yang-Baxter relations (3.2) correspond to the Lie-algebra structure relations. The diagonal elements A and D of T play the rôle of the "weights". The off diagonal elements B and C are creation and annihilation operators of "pseudo-particles", respectively.

The "pseudo-vacuum" ϕ is defined by

$$C(\vartheta)\ \phi = \rightarrow\!\underset{|}{\overset{|}{}}\!- \ldots -\!\underset{|}{\overset{|}{}}\!\leftarrow\ = 0 \qquad \forall\ \vartheta\ . \tag{4.2}$$

From eqs. (2.2 and 2.4) we find the solution of this equation

$$\phi^{\{\beta\}} = \prod_{i=1}^{N} \delta_{\beta_i +} = \uparrow \ldots \uparrow\ . \tag{4.3}$$

It is an eigenstate of A and D, fulfilling

$$\begin{aligned} A(\vartheta)\ \phi &= \ \leftarrow\!\underset{\uparrow}{\overset{\uparrow}{}}\!\leftarrow \ldots \leftarrow\!\underset{\uparrow}{\overset{\uparrow}{}}\!\leftarrow\ = a^N(\vartheta)\ \phi \\ D(\vartheta)\ \phi &= \ \rightarrow\!\underset{\uparrow}{\overset{\uparrow}{}}\!\rightarrow \ldots \rightarrow\!\underset{\uparrow}{\overset{\uparrow}{}}\!\rightarrow\ = b^N(\vartheta)\ \phi\ . \end{aligned} \tag{4.4}$$

However, applying B to ϕ one gets something new

$$B(\vartheta)\ \phi = \leftarrow\!\underset{\uparrow}{\overset{|}{}}\!- \ldots -\!\underset{\uparrow}{\overset{|}{}}\!\rightarrow\ \neq 0, \phi. \tag{4.5}$$

This fact motivates us to write an eigenstate of the transfer matrix as an **algebraic Bethe ansatz**

$$\psi = B(\vartheta_1) \ldots B(\vartheta_n)\ \phi \tag{4.6}$$

where the parameters ϑ_i are to be determined.

We follow the general strategy [4] of the algebraic Bethe ansatz method to solve eq. (4.1). We apply $A(\vartheta)$ and $D(\vartheta)$ to ψ and push them through all the B's using the commutation rules (3.7). From the first terms on the r.h.s. of eq. (3.7) we get the "wanted" contributions

$$(A(\vartheta) + D(\vartheta))\ \psi = (\lambda_A(\vartheta) + \lambda_D(\vartheta))\ \psi\ +\ \text{"unwanted terms"} \qquad (4.7)$$

where from eqs. (3.7 and 4.4) we find

$$\lambda_A(\vartheta) = a^N(\vartheta) \prod_{i=1}^{n} \frac{a(\vartheta_i - \vartheta)}{b(\vartheta_i - \vartheta)} \qquad \lambda_D(\vartheta) = b^N(\vartheta) \prod_{i=1}^{n} \frac{a(\vartheta - \vartheta_i)}{b(\vartheta - \vartheta_i)}. \qquad (4.8)$$

The "unwanted terms" arise from the second terms in eqs. (3.7). They contain a $B(\vartheta)$ and so they can never give a vector proportional to ψ. The unwanted terms vanish if the **Bethe ansatz equations** hold

$$\left(\frac{b(\vartheta_j)}{a(\vartheta_j)}\right)^N \prod_{i=1}^{n} \frac{a(\vartheta_j - \vartheta_i)}{b(\vartheta_j - \vartheta_i)} \frac{b(\vartheta_i - \vartheta_j)}{a(\vartheta_i - \vartheta_j)} = -1 \qquad j = 1, \ldots, n. \qquad (4.9)$$

Obviously these equations are necessary for ψ to be an eigenstate of τ. This is because the eigenvalue

$$\lambda(\vartheta) = \lambda_A(\vartheta) + \lambda_D(\vartheta) \qquad (4.10)$$

cannot have poles at $\vartheta = \vartheta_i$ as eqs. (2.2 and 4.8) would suspect. The cancellation condition of these poles is equivalent to the Bethe ansatz equations (4.9). It can be shown [4] that they also imply the cancellation of the unwanted terms. We have transformed the eigenvalue problem of the transfer matrix (4.1) to sets of transcendental equations (4.9) for the parameters $\vartheta_1, \ldots, \vartheta_n$, where $n = 0, \ldots, \frac{1}{2}N$ (Neven).

Solutions of Bethe ansatz equations are available for small or very large N and n. We are interested in the latter case. I will explain some more technical details. Introducing a "rapidity" q, a "momentum" p and a "scattering phase shift" Φ for the pseudo-particles by

$$\vartheta = \tfrac{1}{2}(iq + \gamma), \qquad e^{ip(q)} = \frac{\sinh \frac{1}{2}(q - i\gamma)}{\sinh \frac{1}{2}(q + i\gamma)}, \qquad e^{i\Phi(q)} = \frac{\sinh \frac{1}{2}(q + 2i\gamma)}{\sinh \frac{1}{2}(q - 2i\gamma)}. \quad (4.11)$$

we can write the Bethe ansatz equations (4.9) as

$$\begin{gathered} e^{iNp(q_j)} \prod_{i=1}^{n} e^{\Phi(q_j - q_i)} = -1 \\ j = 1, \ldots, n \qquad 0 \leq n \leq N/2. \end{gathered} \qquad (4.12)$$

The "physical" interpretation of this equations is, that the total phase factor around one period $1, \ldots, N$ has to be equal to one. This phase factor consists of the free part $\exp(iNp(q_j))$ and the scattering parts with respect to all other pseudo-particles $\exp(i\Phi(q_j - q_i))$. Each set of roots of eq. (4.12) $\{q_1, \ldots, q_n\}$ defines an eigenstate of the transfer matrix by eq. (4.6) and an eigenvalue λ given by eqs. (4.8, 4.10 and 4.11)

$$\lambda_A(\vartheta, \gamma) = a^N \prod_{i=1}^{n} e^{-e(q_i)} \tag{4.13}$$
$$e(q) = i(p(q + 2i\vartheta) - \pi).$$

We will consider the region $0 < \vartheta < \gamma/2$ where $|\lambda_D| < |\lambda_A|$. So λ_D will not contribute to the partition function for large M.

There exist detailed investigations of the Bethe ansatz equations (4.12) (see e.g. ref. [13]). The analysis simplifies for the discrete values of the "coupling"

$$\gamma = \pi/\nu \qquad \nu = 2, 3, 4, \ldots. \tag{4.14}$$

We are interested in these values. So we assume eq. (4.14) in the following.

4.2 Solutions of Bethe ansatz equations: real roots.

If we define the phase function

$$z(q) = Np(q) + \sum_{i=1}^{n} \Phi(q - q_i). \tag{4.15}$$

the logarithm of the Bethe ansatz equations (4.12) yields

$$2\pi I_j = z(q_j), \quad I_j \in \mathbf{Z} + \tfrac{1}{2}, \quad j = 1, \ldots, n. \tag{4.16}$$

We observe that eigenstates of the transfer matrix τ i.e. sets of roots $\{q_1, \ldots, q_n\}$ are given by sets of numbers $\{I_1, \ldots, I_n\}$, where $I_1 < I_2 \ldots < I_n$. If $I_{j+1} - I_j > 1$ we say: there are $I_{j+1} - I_j - 1$ holes between I_j and I_{j+1}.

We introduce the density of real roots

$$\rho(q) = 2\pi \sum_{i=1}^{n} \delta(q - q_i) \tag{4.17}$$

and the "density of holes"

$$\varphi(q) = \frac{dz(q)}{dq} - \rho(q). \tag{4.18}$$

The latter name is motivated by the fact that

$$\int_Q^{Q'} \frac{dq}{2\pi}\varphi(q) = I(Q') - I(Q) - \#\mathrm{roots} \in [Q, Q']$$

is the number of possible minus the number of actual roots in an interval $[Q, Q']$. But note that only in the limit $N, n \to \infty$, where the roots become dense on the real line, the function φ is of the form

$$\varphi(q) = 2\pi \sum_{holes} \delta(q - q_h). \tag{4.19}$$

Taking the derivative d/dq of eq. (4.15) we obtain with eqs. (4.17 and 4.18)

$$z' = \rho + \varphi = Np' + \Phi' * \rho \tag{4.20}$$

where the convolution is defined by $(f * g)(q) = \int (dq/2\pi) f(q - q')g(q')$. Hence the density of roots of the Bethe ansatz equations (corresponding to eigenstates $\psi(q_1, \ldots, q_n)$ of the transfer matrix) can be written as follows

$$\rho = \rho_0 - R * \varphi \tag{4.21}$$

where $R = (1 - \Phi')^{-1}$ and $\rho_0 = N\, R * p'$ can be calculated by means of the Fourier transformation $f(q) = \int dx e^{-iqx} \tilde{f}(x)$

$$\tilde{R}(x) = \frac{\sinh \pi x}{2\cosh \gamma x\, \sinh(\pi - \gamma)x} \qquad \tilde{p'}(x) = \frac{\sinh(\pi - \gamma)x}{\sinh \pi x} \tag{4.22}$$

$$\tilde{\rho}_0(x) = \frac{N}{2\cosh \gamma x}. \tag{4.23}$$

The eigenvalue of the transfer matrix is given by

$$\begin{aligned} \log \lambda_A(\vartheta, \gamma) &= N \log a - \int \frac{dq}{2\pi}\rho(q)e(q) \\ &= N \log a - \int dx \tilde{\rho}(-x)\tilde{e}(x). \end{aligned} \tag{4.24}$$

In the thermodynamic limit $N \to \infty$ the problem is solved by eqs. (4.21-4.24) since φ is of the form (4.19).

(i) *Ground state:* The ground state corresponds to the maximal eigenvalue of the transfer matrix. Since $\Re e(q) < 0$ for real q all possible real roots are occupied. This means that for the ground state a "Dirac-sea" of real roots is completely filled. Eqs. (4.22-4.24) imply

$$\rho(q) = \rho_0(q) = \frac{N\pi}{2\gamma}\left(\cosh\frac{\pi q}{2\gamma}\right)^{-1} \tag{4.25}$$

$$\log\lambda_A(\vartheta,\gamma) = \log\lambda_{A,0}(\vartheta,\gamma) = N\log a + N\int_{-\infty}^{\infty}\frac{dx}{x}\frac{\sinh(\pi-\gamma)x\,\sinh 2\vartheta x}{2\cosh\gamma x\,\sinh\pi x}. \tag{4.26}$$

(ii) *Holes:* For excitations given by holes at q_h we find from eqs. (4.19, 4.21, 4.22 and 4.24)

$$\rho(q) = \rho_0(q) - \sum_{holes} R(q - q_h) \tag{4.27}$$

$$\lambda_A(\vartheta,\gamma) = \lambda_{A,0}(\vartheta,\gamma)\prod_{holes}\tan\frac{\pi}{\gamma}\left(iq_h + \tfrac{1}{4}(\gamma - 2\vartheta)\right). \tag{4.28}$$

4.3 Solutions of Bethe ansatz equations: complex roots.

For "excited states" there may be, in addition to holes, complex roots q: the "k-strings" (one could say "bound states" of k pseudo-particles)

$$q = \text{real} + i\gamma(-k+1, -k+3, \ldots, k-1) \qquad 1 < k \le \nu - 1 \tag{4.29}$$

and the "anti-pseudo-particle"

$$q = \text{real} + i\pi. \tag{4.30}$$

The density of the real roots and the eigenvalue λ, if complex roots are present, can be computed analogously to eqs. (4.27 and 4.28) (see e.g. ref. [1]).

A typical set of roots of Bethe ansatz equations corresponding to an excited state is shown in fig. 5.

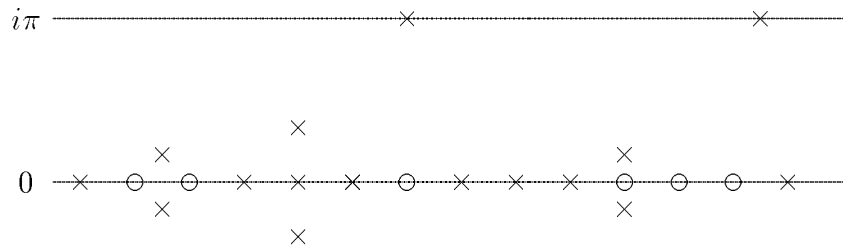


Fig. 5. *A typical set of roots of Bethe ansatz equations in the complex q-plane, including real roots, holes ∘, k-strings and "anti-pseudo-particles"*

5. Eight-vertex model - (R)SOS-model - "Θ-angle".

The "Solid On Solid"- models, closely related to the six-vertex model, are very interesting in the context of conformal quantum field theoretic models (see below). Baxter [6] introduced these models in order to solve a generalization of the six-vertex model, the eight-vertex model which is in general non-critical. Since we will at the end construct conformal quantum field theories as the scaling limit of lattice models, we are interested in the critical case. However, the transformation between the SOS-model and the eight-vertex model is singular for the critical case.[2] Therefore let me introduce both models away from the critical point.

5.1 Eight-vertex model.

The partition function of the eight-vertex model on an $N \times M$ periodic square lattice L is

$$Z^{8-vertex} = \sum_{conf.} \prod_{x \in L} S(x) \tag{5.1}$$

where the sum extends over all allowed "bond configurations". Each bond can accept one of two states as for the six-vertex model. The configuration is allowed if at each vertex x there are an even number of incoming and out going arrows. The vertex weights $S(x)$ are determined by the eight bond configurations at the

[2]In fact the situation is more complicated: There exits of course the limit which gives the six-vertex model of sect. 2, 3 and 4. But we are looking for the limit which yields the critical RSOS-model.

lattice site x. They assume the following values

$$
\begin{aligned}
S^{++}_{++} &= \leftarrow\!\begin{smallmatrix}\uparrow\\ \uparrow\end{smallmatrix}\!\leftarrow = S^{--}_{--} = \rightarrow\!\begin{smallmatrix}\downarrow\\ \downarrow\end{smallmatrix}\!\rightarrow \;= a = \;\mathrm{sn}(\gamma-\vartheta)\\
S^{+-}_{+-} &= \leftarrow\!\begin{smallmatrix}\downarrow\\ \downarrow\end{smallmatrix}\!\leftarrow = S^{-+}_{-+} = \rightarrow\!\begin{smallmatrix}\uparrow\\ \uparrow\end{smallmatrix}\!\rightarrow \;= b = \;\mathrm{sn}\vartheta\\
S^{+-}_{-+} &= \leftarrow\!\begin{smallmatrix}\downarrow\\ \uparrow\end{smallmatrix}\!\rightarrow = S^{-+}_{+-} = \rightarrow\!\begin{smallmatrix}\uparrow\\ \downarrow\end{smallmatrix}\!\leftarrow \;= c = \;\mathrm{sn}\gamma\\
S^{++}_{++} &= \leftarrow\!\begin{smallmatrix}\uparrow\\ \downarrow\end{smallmatrix}\!\rightarrow = S^{--}_{--} = \rightarrow\!\begin{smallmatrix}\downarrow\\ \uparrow\end{smallmatrix}\!\leftarrow \;= d = k\;\mathrm{sn}\gamma\;\mathrm{sn}\vartheta\;\mathrm{sn}(\gamma-\vartheta)
\end{aligned}
\tag{5.2}
$$

Where sn is the Jacobian elliptic functions of modulus k and periods $4K$ and $2iK'$. In the critical limit ($k \to 0, K \to \frac{1}{2}\pi, K' \to \infty$ and $\mathrm{sn} \to \sin$) we recover eqs. (2.2) for the six-vertex model. For fixed γ and k the vertex weights fulfil the Yang-Baxter equation (3.1).

5.2 SOS-model.

The partition function of the SOS-model on an $N \times M$ periodic square lattice L is

$$
Z^{SOS} = \sum_{conf.} \prod_{x \in L} R(x). \tag{5.1a}
$$

A "height" l equal to an integer number is assigned to each plaquette of the lattice, such that heights of neighbor plaquettes differ by ± 1. The Boltzmann weights

$$
\begin{array}{ccc} i & | & k \\ - & R & - \\ j & | & l \end{array} \quad = \quad R(i,j;k,l) \tag{5.3}
$$

corresponding to the six possible configurations at a vertex x

$$
\begin{array}{c|c} l & l-1 \\ \hline l+1 & l \end{array} \quad
\begin{array}{c|c} l & l+1 \\ \hline l-1 & l \end{array} \quad
\begin{array}{c|c} l-1 & l \\ \hline l & l+1 \end{array} \quad
\begin{array}{c|c} l+1 & l \\ \hline l & l-1 \end{array} \quad
\begin{array}{c|c} l-1 & l \\ \hline l & l-1 \end{array} \quad
\begin{array}{c|c} l+1 & l \\ \hline l & l+1 \end{array}
$$

can be found in Baxter's papers [6].

Baxter proved that the eight-vertex and the SOS-model can be related by means of two matrices X and Y:

$$
\sum_{\alpha\beta} S^{\gamma\delta}_{\alpha\beta}\; X^{\alpha}(k,l)\; Y^{\beta}(j,l) = \sum_{i} X^{\gamma}(i,k)\; Y^{\delta}(i,j)\; R(i,j;k,l). \tag{5.4}
$$

We do not need the explicit expressions for X and Y, they can be found in refs. [6] and [4]. It follows that the transfer matrices of the models fulfil

$$
\tau^{8-vert.\,\{\delta\}}_{\quad\;\;\{\beta\}}\; Y^{\{\beta\}}_{\otimes}(\{l\}) = Y^{\{\delta\}}_{\otimes}(\{l'\})\; T^{SOS}(\{l'\};\{l\}) \tag{5.5}
$$

where

$$
\begin{aligned}
Y_{\otimes}^{\{\beta\}}(\{l\}) &= Y^{\beta_1}(l_0,l_1)\; Y^{\beta_2}(l_1,l_2)\ldots Y^{\beta_N}(l_{N-1},l_0),\\
T^{SOS}(\{l'\};\{l\}) &= R(l'_0,l_0;l'_1,l_1)\ldots R(l'_{N-1},l_{N-1};l'_0,l_0)
\end{aligned}
\tag{5.6}
$$

and $\{l\} = (l_0, l_1, \ldots, l_{N-1}, l_0)$ is a periodic ($l_0 = l_N$) and admissible ($|l_i - l_{i+1}| = 1$) set of integers. From eqs. (5.4) and (3.1) follow the Yang-Baxter relations for the matrix R: $\quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

For generic parameters γ the SOS-model is an "unrestricted" one, that means the heights l assume all values. Andrews, Baxter and Forrester [14] found that for a discrete set of the parameter

$$\gamma = 2K/\nu, \quad \nu = 3, 4, 5, \ldots \tag{5.7}$$

the SOS-model splits into an "unrestricted" and a "restricted" one the RSOS-model

$$\begin{cases} SOS & -\infty < l < \infty \\ RSOS & 0 < l < \nu. \end{cases} \tag{5.8}$$

In ref. [14] was shown that there are four different regimes of the parameters where the RSOS-model has different "ground state" structures (cf. fig. 6)

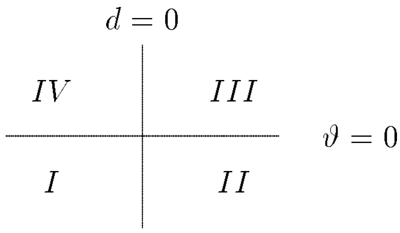


Fig. 6. *The four different regimes of the RSOS-model.*

We are interested in regime III near to the critical line $d = 0$. There are $\nu - 2$ "ground states" labeled by $\delta = 1, \ldots, \nu - 2$. An order parameter corresponding to these states can be expressed in terms of the heights at two neighbor points

$$\delta(x) = \tfrac{1}{2}(l(x) + l(x+1) - 1). \tag{5.9}$$

Andrews, Baxter and Forrester calculated the critical exponent α and the probability

$$P_l^{(\delta)} = \langle \delta_{l,l(x)} \rangle^{(\delta)} \tag{5.10}$$

that at a point x the height l is assumed for boundary conditions corresponding to δ. Using this result Huse [15] showed that the one-point functions of the fields

$$\phi_m(x) = \sin\left(\pi\frac{\delta(x)}{\nu-1}\right)\sin\left(\pi m\frac{\delta(x)}{\nu-1}\right), \qquad m = 1,\ldots,\nu-2 \tag{5.11}$$

behave near to the critical point ($d = 0$) like

$$\langle\phi_m\rangle \approx (t - t_{crit.})^{\beta_m} \tag{5.12}$$

where the exponents β_m correspond to the scaling dimensions of the fields $\phi_m(x)$ (using a scaling relation)

$$d_m = \frac{2\beta_m}{2-\alpha} = \frac{m^2-1}{2\nu(\nu-1)}. \tag{5.13}$$

These are just the scaling dimensions of some fields appearing in the "minimal models" of conformal quantum field theories investigated by Belavin, Polyakov and Zamolodchikov [16] (see sect. 6). This result suggests that in the scaling limit (where the lattice constant goes to zero) the RSOS-models in regime III at a critical point should give a realization of the "minimal" conformal quantum field theories.

5.3 "Θ-angle".

The Bethe ansatz in the simple version of section 4 does not work for the eight-vertex model. The "pseudo-vacuum" ϕ defined by $C\phi = 0$ does not exist because of the d-vertex. Baxter invented the transformation to the SOS-model in order to construct a generalized Bethe ansatz. I will follow here the formulation of Takhtadzhan and Faddeev [4].

Introduce $\tilde{X}$ as an inverse of X

$$\sum_\alpha \tilde{X}_\alpha(i,k)\; X^\alpha(j,k) = \delta_{ij}\;, \qquad \sum_i X^{\alpha'}(i,k)\; \tilde{X}_\alpha(i,k) = \delta_{\alpha'\alpha}. \tag{5.14}$$

The eight vertex model monodomy matrix is defined by the same expression as for the six-vertex case eq. (2.4). We can transform this monodromy matrix by means of $\tilde{X}$ and X

$$T^{\{\delta\}}_{\{\beta\}}(i,j;k,l) = \tilde{X}_\gamma(i,j)\; T^{\gamma\{\delta\}}_{\alpha\{\beta\}}\; X^\alpha(k,l) \tag{5.15}$$

(and obtain a hybrid of eight-vertex and SOS-monodromy matrix). Generalized Bethe ansatz states for the eight-vertex model (analogously to eq. (4.6)) are

$$\psi_l(\vartheta_1,\ldots,\vartheta_n) = B_{l+1,l-1}(\vartheta_1)\ldots B_{l+n,l-n}(\vartheta_n)\ \phi_{l+n,l-n}\ . \tag{5.16}$$

The transformed B's and the generalized "pseudo-vacuum" are defined by

$$\begin{aligned} B_{i,k} &= T(i-1,i;k+1,k) \\ \phi^{\{\beta\}}_{l_0,l_N} &= Y^{\beta_1}(l_0,l_1)\ldots Y^{\beta_N}(l_{N-1},l_N), \quad l_{i+1}=l_i-1. \end{aligned} \tag{5.17}$$

In contrast to the six-vertex Bethe ansatz the number of B-operators is not arbitrary. In eq. (5.16) we have $l_0 = l+n$ and $l_N = l-n$. Therefore $l_{i+1} = l_i - 1$ in eq. (5.17) implies

$$n = N/2. \tag{5.18}$$

Because of eqs. (5.14 and 5.15), the transfer matrix of the eight-vertex model can be expressed in terms of the transformed monodomy matrices independently of l

$$A_{ll} + D_{ll} = T(l-1,l;l-1,l) + T(l+1,l;l+1,l) = \tau^{8-vert.}. \tag{5.19}$$

Analogously to the six-vertex case (cf. eq. (4.7)) one can show

$$(A_{ll}+D_{ll})\ \psi_l = \lambda_A\ \psi_{l-1} + \lambda_D\ \psi_{l+1} \tag{5.20}$$

Hence, the eigenstates of the eight-vertex model have the following form

$$\psi_\Theta(\vartheta_1,\ldots,\vartheta_n) = \sum_{l=-\infty}^{\infty} e^{-i\Theta l}\ \psi_l(\vartheta_1,\ldots,\vartheta_n) \tag{5.21}$$

$$\tau^{8-vert.}\ \psi_\Theta = \lambda\ \psi_\Theta \tag{5.22}$$

and the eigenvalues are given by

$$\lambda = e^{-i\Theta}\ \lambda_A + e^{i\Theta}\ \lambda_D. \tag{5.23}$$

The Bethe ansatz equations, which are equivalent to the vanishing of the "unwanted terms" read (analogously to eqs. (4.11 and 4.12))

$$e^{iNp(q_j)} \prod_{i=1}^{n} e^{i\Phi(q_j-q_i)}\ e^{i2\Theta} = -1, \qquad j=1,\ldots,n=N/2 \tag{5.24}$$

where

$$\vartheta = \tfrac{1}{2}(iq+\gamma), \qquad e^{ip(q)} = \frac{\mathrm{h}(\frac{i}{2}(q-i\gamma))}{\mathrm{h}(\frac{i}{2}(q+i\gamma))}, \qquad e^{i\Phi(q)} = \frac{\mathrm{h}(\frac{i}{2}(q+2i\gamma))}{\mathrm{h}(\frac{i}{2}(q-2i\gamma))}. \tag{5.25}$$

The function $h(x) = H(x)\Theta(x)$, given by the Jacobi theta-functions of modulus k, approach $\sin(x)$ in the critical limit $k \to 0$. We recover the Bethe ansatz equations (4.12) for the six-vertex model. However, we lost the freedom having $n \neq N/2$, but, instead, obtained the new parameter Θ. For generic values of Θ the sum in eq. (5.20) will vanish. So Θ has to be determined together with the solution of the Bethe ansatz equations from the condition $\psi_\Theta \neq 0$.

For the special values of the parameter $\gamma = 2K/\nu$, leading to the RSOS-models (cf. eq. (5.7)), the problem simplifies. Everything is periodic with respect to the heights l with period ν. So the admissible values are (see ref. [4])

$$\Theta = m\ \pi/\nu, \qquad m = 0,1,\dots,\nu-1. \tag{5.26}$$

The sum in eq. (2.20) can be restricted to one period $l = 0,1,\dots,\nu-1$ and the condition (5.18) has to be replaced by $n = N/2 \bmod \nu$.

5.4 Critical RSOS-model.

Now let us turn to the SOS-model. All eigenvalues of the eight-vertex transfer matrix are also eigenvalues of the SOS-transfer matrix, because both are related by eq. (5.5) $\tau^{8-vert.}Y_\otimes = Y_\otimes T^{SOS}$.[3] We specialize the model now in two steps:

(a) $\gamma = 2K/\nu, \quad \nu = 3,4,5,\dots$

As mentioned above, for these special values of the parameter γ the SOS-model splits into an "unrestricted" and a "restricted" one the RSOS-model where also the values of Θ are restricted[4]

$$\begin{cases} SOS & -\infty < l < \infty \quad \Theta = m\ \pi/\nu \quad m = 0,1,\dots,\nu-1 \\ RSOS & \quad 0 < l < \nu \quad \Theta = m\ \pi/\nu \quad m = 1,\dots,\nu-1. \end{cases} \tag{5.27}$$

In the following we will concentrate on the RSOS-model.

(b) the critical case $d \to 0,\ k \to 0,\ K \to \pi/2$

---

[3] The inverse is also true, at least in the limit $N \to \infty$ in the "Fock space" of the finite-energy excitations.

[4] I have no simple proof for the fact that $\Theta = 0$ is forbidden for the RSOS-model (cf. section 7 and for example ref. [17]).

In this limit momentum and scattering phase shift of the pseudo-particles defined by eq. (5.25) approach the six-vertex expressions (4.11). Therefore the Bethe ansatz equations of both models eqs. (4.12) and (5.24) coincide, with two exceptions:

(i) for the RSOS case there is the extra constant phase shift 2Θ and

(ii) the "Bethe-spin"

$$\Sigma = N/2 - n \tag{5.28}$$

can assume only the values $\Sigma = 0 \bmod \nu$, whereas for the six-vertex model it can assume all values $\Sigma = 0, 1, \ldots, N/2$.

So we may write formally for the eigenstate of the models taking eqs. (4.12), (5.24) and (5.27) into account

$$\{\mathrm{RSOS}\} = \sum_{m=1}^{\nu-1} \{\text{6-vertex}(\Theta = m\gamma, \Sigma = 0 \bmod \nu)\}. \tag{5.29}$$

Here we have introduced a modified six-vertex model depending on Θ and Σ. This model is defined on a cylinder of height M and circumference N. We will be interested in the limit $M \gg N \to \infty$. The boundary conditions are periodic around and fixed at the bottom and the top of the cylinder. The numbers of up-minus down-arrows at the bottom and the top are 2Σ. This means there is a total "charge flow" along the cylinder of 2Σ (note that for the six-vertex model we have "charge" conservation). We introduce a "seam" from the bottom to the top of the cylinder and write

$$2\hat{\Sigma} = \text{number of left- minus right-arrows on the seam}\ . \tag{5.30}$$

Then for each six-vertex configuration on the cylinder $2\hat{\Sigma}$ is a well defined topological number equal to the total "charge flow" around the cylinder. The modified six-vertex model in eq. (5.29) is defined by the partition function

$$Z^{6-vert.}(\Theta) = \sum_{conf.} \exp(-E^{6-vert.} + i\ \Theta\, 2\, \hat{\Sigma}). \tag{5.31}$$

Note that this partition function has the same form as that for gauge theories with Θ-vacuum. The "charge flow lines" around the cylinder play the role of the instantons.

We will later see that the decomposition of the RSOS-model by eq. (5.29) may be interpreted as a decomposition with respect to "soliton sectors" of the RSOS-model. $\Theta = \gamma$ correspond to the vacuum sector and $\Theta = m\ \gamma,\ m = 2, \ldots, \nu - 1$

to the $(m-1)$-soliton sectors. The fields $\phi_m(x)$ introduced in eq. (5.11) belong to these sectors.

6. Conformal quantum field theories in two dimensions.

I interrupt the investigations of integrable systems in order to give a short introduction to conformal quantum field theories in two dimensions. In contrast to higher dimensions, in two dimensions conformal invariance is a rather strong restriction since the transformation group is infinite dimensional. It is related to the Virasoro algebra [18]

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \tfrac{1}{12}c(n^3-n)\delta_{n,-m} \\ c &= \text{"central charge"} \end{aligned} \tag{6.1}$$

for which a well developed theory exists [19]. Belavin, Polyakov and Zamolodchikov [16] classified the "minimal models" of conformal quantum field theories in terms of the central charge

$$c = 1 - \frac{6}{\nu(\nu-1)} \tag{6.2}$$

where ν is a rational number. This follows from the assumption that there exists only a finite number of "primary fields", i.e. fields transforming under conformal transformations

$$\begin{aligned} z &= x + iy \to w(z) \\ \bar{z} &= x - iy \to \bar{w}(\bar{z}) \end{aligned} \tag{6.3}$$

like

$$A(z,\bar{z}) \to \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} A(w,\bar{w}). \tag{6.4}$$

The transformations (6.3) are generated by $Vir \otimes \overline{Vir}$ a product of two Virasoro algebras. The conformal dimensions can assume values given by the Kac-formula [19]

$$\begin{aligned} &\Delta, \bar{\Delta} = h_{mn} = \frac{\left((\nu-1)m - \nu n\right)^2 - 1}{4\nu(\nu-1)}, \\ &0 < m < \nu, \quad 0 < n < \nu - 1, \qquad m,n = \text{integer}. \end{aligned} \tag{6.5}$$

Friedan, Qiu and Shenker [20] have shown that unitarity restricts the number ν to

$$\nu = \text{integer} \geq 3 \quad \text{or} \quad c \geq 1. \tag{6.6}$$

It would be interesting to find integrable lattice models which yield in the scaling limit conformal quantum field theories corresponding to these unitary representations of the Virasoro algebra. Therefore it is useful to calculate the central charge and the conformal dimensions for integrable critical statistical models. This can be done by analyzing finite-size effects.

For a statistical system on a periodic $N \times M$ square lattice one introduces a transfer matrix τ. The partition function can be written as

$$Z = \mathrm{tr}\, \tau^M = \sum_i \lambda_i^M \tag{6.7}$$

where the sum extends over all eigenvalues of the transfer matrix. Cardy [21] has shown that conformal invariance implies for the maximal eigenvalue

$$\lambda_{max} \approx \exp(-Nf + \frac{1}{N}\frac{\pi}{6}c) \ , \qquad \text{for} \quad N \to \infty \tag{6.8}$$

where f is the free energy per site and c is the central charge of the Virasoro algebra (cf. eq. (6.1)). Hence, the central charge can be calculated from the finite-size behavior of the maximal eigenvalue corresponding to the ground state of a transfer matrix.

In addition the conformal dimensions of operators of the model can be determined if one looks for excited states with energy $E = -\Re \log \lambda$ and momentum $P = -\Im \log \lambda$. As is argued in ref. [21] one has for $N \to \infty$

$$\begin{aligned} E_i - E_0 &\approx \frac{2\pi}{N} d_i \\ P_i - P_0 &\approx \frac{2\pi}{N} s_i \end{aligned} \tag{6.9}$$

where $d_i = \Delta + \bar{\Delta}$ ($s_i = \Delta - \bar{\Delta}$) is the "scaling dimension" ("spin") of the operator A associated to the excitation ψ_i, i.e. $\langle 0|A|\psi_i\rangle \neq 0$.

A direct calculation of the central charge and the conformal dimensions for specific models is usually not simple. Therefore it seems worthwhile to determine them for integrable models from the Bethe ansatz solution. This will be done in the following for the six-vertex model and for some modified six-vertex models related to Potts- and RSOS-models.

7. Finite size corrections.

The methods explained here have been developed by many authors [22] (see also [1] and [2]).

7.1 The models.

The q-state Potts models are a generalizations of the Ising model where the spin at each site can accept q values instead of two. For $q = 2, 3, 4$ the models possess a critical point which we consider in the following. The partition function on a cylinder [23] can be written in terms of a six-vertex model with a "seam" from the bottom to the top of the cylinder. This seam carries extra Boltzmann weights which produce an extra phase factor in the Bethe ansatz equations (4.11) and the equations (4.8) for the eigenvalue. These modifications have the same consequence as the Θ-angle for the RSOS-model. For the vacuum and the order parameter sectors one has $\Theta = \gamma,\ \pi/2$, respectively. The number q ($q = 2, 3, 4$) is related to the six-vertex coupling $\gamma = \pi/\nu$ ($\nu = 4, 6, \infty$) by

$$q = 4\cos^2\gamma. \tag{7.1}$$

We consider the modified six-vertex model defined in sect. 5

$$Z^{6-vert.}(\Theta) = \sum_{conf.} \exp(-E^{6-vert.} + i\ \Theta\, 2\,\hat{\Sigma}) \tag{7.2}$$

for the discrete set of parameter $\gamma = \pi/\nu,\ \nu = 3, 4, 5, \ldots$. The parameter Θ assumes for the various models the following values

(i) six-vertex model: $\Theta = 0$

(ii) RSOS-model: $\Theta = m\ \pi/\nu,\ m = 1, 2, \ldots, \nu - 1$.

(iii) q-state Potts model: $\Theta = \pi/\nu,\ \pi/2$

7.2 Bethe ansatz equations.

The eigenvalue problem of the transfer matrix is equivalent to the Bethe ansatz equations

$$e^{iNp(q_j)} \prod_{i=1}^{n} e^{\Phi(q_j - q_i)}\ e^{i2\Theta} = -1 \qquad j = 1, \ldots, n = N/2. \tag{7.3}$$

As in section 4 we introduce the phase function, the density of roots and of holes

$$z(q) = Np(q) + \sum_{i=1}^{n} \Phi(q - q_i) + 2\Theta \tag{7.4}$$

$$\rho(q) = 2\pi \sum_{i=1}^{n} \delta(q - q_i) \ , \qquad \varphi(q) = \frac{dz(q)}{dq} - \rho(q). \tag{7.5}$$

Then the logarithm of the Bethe ansatz equations yields

$$2\pi I_j = z(q_j), \quad I_j \in \mathbf{Z} + \tfrac{1}{2}, \quad j = 1, \ldots, n. \tag{7.6}$$

From eq. (7.4) we derive

$$z' = \rho + \varphi = N p' + \Phi' * \rho \tag{7.7}$$

and

$$\rho = \rho_0 - R * \varphi \tag{7.8}$$

where $R = (1 - \Phi')^{-1}$ and $\rho_0 = N\, R * p'$ is the ground state density of roots. These equations are still exact for all N. The function $\varphi(q)$ contains in addition to the holes all finite-size corrections

$$\varphi(q) = 2\pi \sum_{holes} \delta(q - q_h) \ + \ \text{"finite-size corrections"}. \tag{7.9}$$

Let us consider first real roots only.

7.2 Real roots.

The eigenvalues (5.23) of the transfer matrix are given by

$$\log \lambda_A(\vartheta, \gamma) = -Nf + \int \frac{dq}{2\pi} \varphi(q) \epsilon(q) - i\Theta \tag{7.10}$$

where the free energy per lattice site defined by eq. (2.10) can be calculated from eq. (4.26) and $\epsilon(q)$ follows from eqs. (7.8), (4.13, 22 and 24)

$$\epsilon(q) = R * e\ (q) = i\ \sin^{-1}\big(\tanh(\tfrac{1}{2}\nu(q + 2i\vartheta))\big). \tag{7.11}$$

We are interested in the ground state and low-energy excitations corresponding to the primary fields. From eq. (4.28) we know that that these states possess a finite number of holes "near to $q = \pm\infty$". Hence, we consider first for simplicity the case where all roots are inside the interval $[q_1, q_n]$ and the holes outside. For large N the bounds q_1 and q_n will approach $\mp\infty$, respectively.

In order to discuss finite-size behavior we have to investigate the second terms in eqs. (7.8) and (7.10) containing the function

$$\varphi(q) = z'(q) - 2\pi \sum_{i=1}^{n} \delta(q - q_i). \tag{7.12}$$

Let us first consider eq. (7.10). We have to analyze

$$\int_{-\infty}^{+\infty} \frac{dq}{2\pi}\epsilon(q)\varphi(q) = \int_{q_1}^{q_n} \frac{dq}{2\pi}\epsilon(q)\varphi(q) + \left(\int_{q_n}^{+\infty} + \int_{-\infty}^{q_1}\right) \frac{dq}{2\pi}\epsilon(q)z'(q) \qquad (7.13)$$

The first term on the r.h.s. contains a difference of an integral and a sum which can be approximated for large N by an Euler-Maclaurin formula (cf. for example ref. [1])

$$-\frac{1}{2}(\epsilon(q_n)+\epsilon(q_1)) - \frac{\pi}{6}\left(\frac{\epsilon'(q_n)}{z'(q_n)} - \frac{\epsilon'(q_1)}{z'(q_1)}\right) \qquad (7.14)$$

The second term on the r.h.s. of eq. (7.13) can be calculated using the asymptotic expansion of $\epsilon(q)$

$$\epsilon(q) \approx \tfrac{1}{2}i\pi - 2ie^{-\frac{1}{2}\nu(q+i2\vartheta)}, \qquad \text{for} \quad q \to \infty \qquad (7.15)$$

we obtain

$$\tfrac{1}{2}i\pi\tilde{\tau}_+(0) - 2ie^{-i\nu\vartheta}\ e^{-\frac{1}{2}\nu q_+}\ \tilde{\tau}_+(\tfrac{1}{2}i\nu) \qquad (7.16)$$

where the following notation has been used

$$\tilde{\tau}_+(x) = \int_0^{\infty} \frac{dq}{2\pi}e^{iqx}\tau_+(q), \qquad \tau_+(q) = z'(q+q_n). \qquad (7.17)$$

A similar expression to eq. (7.16) where q_n is replaced by q_1 is obtained for the third term in eq. (7.13).

In order to calculate $\tilde{\tau}_\pm(\frac{1}{2}i\nu)$ we apply analogous techniques as above to eq. (7.8) which determines $z'(q)$ as

$$z' = \rho + \varphi = \rho_0 - J * \varphi \qquad (7.18)$$

where $J = R - 1 = (1-\Phi')^{-1} * \Phi'$. We first concentrate on this function at $q > q_n$ to determine τ_+. The decomposition of eq. (7.18) analogously to eq. (7.13) leads after the shift $q \to q + q_n$ to an integral equation of Wiener-Hopf type for the function $\tau_+(q)$

$$\tau_+(q) = f(q) - \int_0^{\infty} \frac{dq'}{2\pi}J(q-q')\ \tau_+(q') \qquad (7.19)$$

where for large N the function $f(q)$ is obtained in the same way as eqs. (7.14)

$$f(q) \approx \rho_0(q+q_n) + \tfrac{1}{2}J(q) - \frac{\pi}{6}\frac{1}{z'(q_n)}J'(q) \qquad (7.20)$$

This equation can be solved (see e.g. ref. [1]).

In addition to the integral equation (7.19), there are further restrictions on $\tau_+(q)$. The value of $z'(q_n)$ appearing in the inhomogeneous term $f(q)$ can be calculated from $\tilde{\tau}_+(x)$ by

$$z'(q_n) = \tau_+(0+) = -2\pi i \lim_{x\to\infty} x\tilde{\tau}_+(x). \tag{7.21}$$

From the Bethe ansatz equations (7.6) we have the condition

$$\int_{q_j}^{\infty} \frac{dq}{2\pi} z_k'(q) = \frac{z(\infty)}{2\pi} - I_j \tag{7.22}$$

where for the upper bound of the numbers I_j we find from eqs. (7.4) and (4.11)

$$\frac{z(\infty)}{2\pi} = N - (N-2n)\frac{\gamma}{\pi} + \frac{\Theta}{\pi}. \tag{7.23}$$

Introducing the number of holes $h_+ = N - \frac{1}{2} - I_n$ above q_n and the Bethe spin $\Sigma = N/2 - n$ we obtain

$$\tilde{\tau}_+(0) = \tfrac{1}{2} + h_+ - \Sigma/\nu + \Theta/\pi. \tag{7.24}$$

Analogous formula hold for the function $\tau_-(q) = z'(q_1 - q)$.

Note that there might be some restriction on the number $h_\pm$ if Σ and Θ are large and ν is small. This restriction comes from the condition that the "last" roots should not be shifted to $\pm\infty$. They will not relevant in here since we are interested in low energy excitation.

After a long calculation [1] we obtain a formula for the eigenvalue of the transfer matrix including $1/N$ corrections

$$\begin{aligned}\log \lambda_A(\vartheta,\gamma) = {} & -Nf + \tfrac{1}{2} i\pi(h_+ - h_-) \\ & + \frac{\pi}{N}\left\{ \frac{i}{2} e^{-i\nu\vartheta}\left(\frac{1}{6} - \frac{\nu}{\nu-1}(h_+ - \Sigma/\nu + \Theta/\pi)^2\right)\right. \\ & \left. - \frac{i}{2} e^{i\nu\vartheta}\left(\frac{1}{6} - \frac{\nu}{\nu-1}(h_- - \Sigma/\nu - \Theta/\pi)^2\right)\right\}\end{aligned} \tag{7.25}$$

If there are also holes inside the bunch of real roots, above q_1 and below q_n, then the only change in eq. (7.25) is, that one has to replace $\frac{1}{6} \to \frac{1}{6} - K_\pm$, respectively, where $K_\pm$ are some integers > 0.

7.3 Complex roots.

For solutions of the Bethe ansatz equations including complex roots the calculation is a little more complicated. But the result is quite similar to eq. (7.25). If there are $n_\pm^{(k)}$ k-strings "near to $q_\pm$",[5] then one has to replace the numbers of holes $h_\pm$ by

$$\hat{h}_\pm = h_\pm - 2\sum_k n_\pm^{(k)}(k-1). \tag{7.26}$$

7.4 Applications to six-vertex, RSOS- and Potts-models.

In sect. 7.2 the eigenvalue of the transfer matrix has been calculated, including finite-size corrections for a large N. Using the result (7.25) for the ground state where no holes and complex roots are present we find

$$\log \lambda_{A,max} \approx -Nf + \frac{1}{N}\frac{\pi}{6}\left(1 - \frac{6}{\nu(\nu-1)}\left(\frac{\Theta}{\pi}\right)^2\right)\sin\nu\vartheta. \tag{7.27}$$

We expect conformal invariance in the scaling limit for the isotropic models. Therefore we take $\nu\vartheta = \pi/2$ where vertex weights fulfil $a = b$. Comparing eq. (7.27) and Cardy's formula (6.8) we conclude that the central charges of the Virasoro algebra of the six-vertex, the RSOS- and the q-state Potts models are

$$c^{6-vertex} = 1 \tag{7.28a}$$

$$c^{RSOS} = 1 - \frac{6}{\nu(\nu-1)} \tag{7.28b}$$

$$c^{Potts} = 1 - \frac{6}{\nu(\nu-1)}. \tag{7.28c}$$

We have obtained these results in ref. [1]. For a) and c) they have been expected from renormalization group arguments (see e.g. [24]) and for b) from Huse's observation (5.13).

Excitations possess "holes" and complex roots. We obtain for these states by means of the general formula (7.24) taking Cardy's formulas (6.9) into account[6]

$$\log \lambda_A - \log \lambda_{A,max} \approx -\frac{2\pi}{N}\{(\Delta + \bar{\Delta})\sin\nu\vartheta + i(\Delta - \bar{\Delta})\cos\nu\vartheta\}. \tag{7.29}$$

[5] The k-strings which do not approach $\mp\infty$ like the last real roots $q_{1,n}$ as $N \to \infty$ will not affect the $1/N$ corrections.

[6] Note that the imaginary term correspond to the momentum since $\tau(\vartheta = 0) = \exp(-iP)$.

The primary fields correspond to low-energy excitations with a small number of holes $h_\pm$ "near to the last roots" $q_{1,n}$ which approach $\pm\infty$ for $N \to \infty$. In addition there may be a number $n_\pm^{(k)}$ of k-strings "near to $q_{1,n}$". For these configurations of roots we obtain from eq. (7.25) for the six-vertex model the conformal dimensions

$$\Delta = \frac{(\nu \hat{h}_+ - \Sigma)^2}{4\nu(\nu-1)} + K_+ \ , \qquad \bar{\Delta} = \frac{(\nu \hat{h}_- - \Sigma)^2}{4\nu(\nu-1)} + K_- . \tag{7.30}$$

$\hat{h}_\pm$ is given by eq. (7.26), $\Sigma = \frac{1}{2}h - \sum_k n_k(k-1)$ is the "spin"[7] of the Bethe state and h (n_k) is the total number of holes (k-strings). These results agree with Kadanoff's works on the Gaussian model (see e.g. ref. [25]).

The conformal dimensions for the RSOS-model in the "$(m-1)$-soliton sector" are

$$\Delta = \frac{(\nu \hat{h}_+ - \Sigma + m)^2 - 1}{4\nu(\nu-1)} + K_+ \ , \qquad \bar{\Delta} = \frac{(\nu \hat{h}_- - \Sigma - m)^2 - 1}{4\nu(\nu-1)} + K_- \tag{7.31}$$

One can show [1] that one gets minimal values for Δ and $\bar{\Delta}$ (i.e. the integers K_+ and K_- on the r.h.s. of eqs. (7.31) vanish) if some conditions are fulfilled: There are no holes inside $[q_1, q_n]$, $n_+^{(k)} = 0$ for all k, $n_-^{(k)} = 1$ for only one k and the number of holes have to fulfil $h_+ = h_- = k-1$.

The extra phase $\Theta = m\gamma$ in the Bethe ansatz equations imply that for the minimal Δ and $\bar{\Delta}$ only the k-strings with

$$k = 2, 3, \ldots, \nu - m - 1 \tag{7.32}$$

contribute. This has been shown in ref. [1]. Hence, for the RSOS-model in the $(m-1)$-soliton sector the k-strings for $k \geq \nu - m - 1$ do not contribute to the primary fields. A similar argument holds for the "anti-pseudo-particle". We consider fixed boundary conditions at the top and the bottom of the cylinder with Bethe spin $\Sigma = 0 \bmod \nu$ (cf. eq. (5.29)). Comparing eqs. (7.31) for $\Sigma = 0$ with the Kac-formula (6.5) we find for the RSOS-model the conformal dimensions of the primary fields

$$\begin{aligned} &\Delta = \bar{\Delta} = h_{mn} \, , \qquad n = m + k - 1, \\ &k = 1, 2, \ldots, \nu - m - 1, \qquad m = 1, 2, \ldots, \nu - 2. \end{aligned} \tag{7.33}$$

This results "proves" the suggestion of Huse [15] that the RSOS-models correspond to the "minimal models" [16] of conformal quantum field theory.

[7] Σ must not be confused with the "spin" s of the field operator (cf. eq. (6.9)).

For the q-state Potts model the conformal dimensions are given by eq. (7.33) where $m = 1$ for the vacuum sector (corresponding to the "energy operators") and $\nu/2$ for the order parameter sector (see [1] and [26]).

The Bethe ansatz states which belong to the conformal dimensions given by eq. (7.33) are characterized by the following set of roots (see fig. 7): outside the bunch of real roots between q_1 and q_n there are $k-1$ holes on both sides and in addition one k-string on the left hand side.

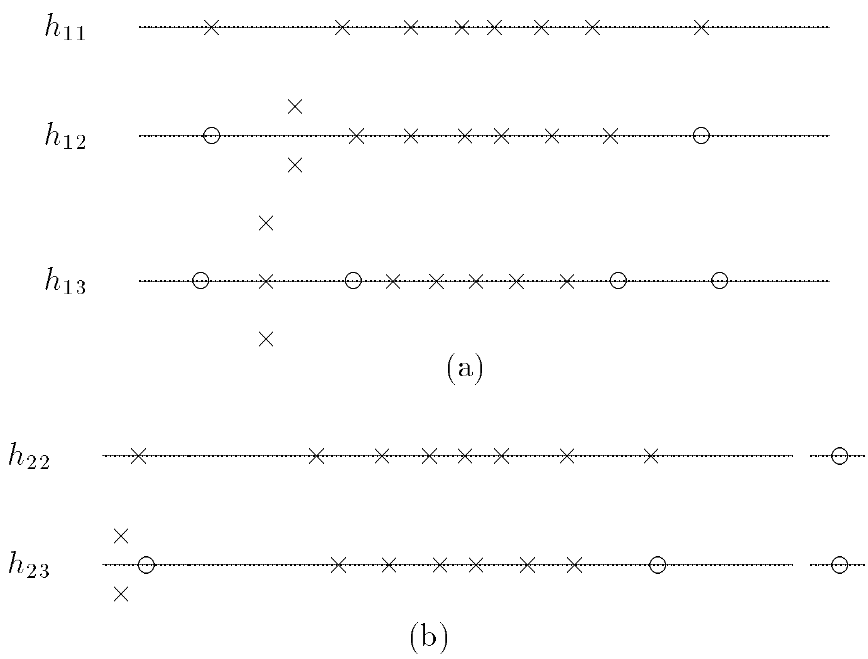


Fig. 7. *Typical sets of roots $\times$ and holes $\circ$ of Bethe ansatz equations in the complex q-plane for the RSOS-model with $\gamma = \pi/6$ on an N=16 chain. Some root configurations corresponding to primary fields ϕ_{mn} with dimensions $\Delta = \bar{\Delta} = h_{mn}$ are depicted: (a) $m = 1$, i.e. $\Theta = \gamma$, (b) $m = 2$, i.e. $\Theta = 2\gamma$.*

8. Scaling limit.

Before performing the limit, lattice constant to zero, we transform the critical RSOS-model into an appropriated form. We start with the model in the form of the modified six-vertex models given by eq. (5.29). The transformation is done in several steps:

(i) Vertex model $\longrightarrow$ Heisenberg model.

We expand the logarithm of the modified six-vertex transfer matrix

$$\tau^{6-vert.}(\Theta,\vartheta,\gamma) = \mathrm{Tr}\left(T^{6-vert.}(\vartheta,\gamma)\; e^{i\Theta\sigma^3}\right), \quad \gamma = \pi/\nu \tag{8.1}$$

in powers of the spectral parameter ϑ

$$-\log\tau^{6-vert.}(\Theta,\vartheta,\gamma) = c_0 + c_1\vartheta + c_2\vartheta^2 + \dots. \tag{8.2}$$

It follows for c_1 the Heisenberg Hamiltonian (see e.g. ref. [23]) in terms of the Pauli matrices σ^k

$$H_{xxz} = -\sum_{i=1}^{N} J_k\sigma_i^k\sigma_{i+1}^k\,, \quad J_1 = J_2, \quad J_3/J_1 = -\cos\gamma \tag{8.3}$$

where the extra phase factor in eq. (8.1) produces the boundary conditions

$$\sigma_{N+1}^{\pm} = \sigma_1^{\pm}e^{\mp i2\Theta}\;, \quad \sigma^{\pm} = \tfrac{1}{2}(\sigma^1 \pm i\sigma^2). \tag{8.4}$$

(ii) Heisenberg model $\longrightarrow$ lattice fermion model. [27]

We apply the Jordan-Wigner transformation

$$\sigma_j^3 = e^{i\pi n_j}, \quad \sigma_j^- = i^j a_j \exp\big(i\pi\sum_{i=1}^{j-1} n_i\big), \quad n_j = a_j^+a_j. \tag{8.5}$$

The operators a_j fulfil fermion commutation rules

$$\{a_i, a_j^+\} = \delta_{ij}. \tag{8.6}$$

Up to a term proportional to the particle number and a constant the Hamiltonian writes in terms of lattice fermions

$$H = -4J_1\sum_{i=1}^{N}\big[\tfrac{i}{2}(a_i^+a_{i+1} - a_{i+1}^+a_i) - \cos\gamma\; n_in_{i+1}\big]. \tag{8.7}$$

The boundary conditions follow from eq. (8.4)

$$a_{N+1} = e^{i\pi\Sigma + i2\Theta}a_1 \tag{8.8}$$

where the spin operator Σ is defined by eq. (5.28) (note that $\Sigma = 0$ mod ν).

(iii) Continuum limit: Lattice fermion model $\longrightarrow$ Thirring model.

Introducing the Dirac spinor

$$\psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} a_i^+ + a_{i+1}^+ \\ a_i - a_{i+1} \end{pmatrix}, \quad x = ai, \quad i \text{ even} \tag{8.9}$$

one finds after wave function renormalization in the limit, lattice constant $a \to 0$, the Thirring model defined on a cylinder of height $L' = Ma$ and circumference $L = Na$. The Lagrangian is

$$\mathcal{L} = \bar{\psi} i\gamma\partial\psi - \tfrac{1}{2} g(\bar{\psi}\gamma_\mu\psi)^2 \tag{8.10}$$

with the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.11}$$

The boundary conditions follow from eq. (8.8)

$$\psi(L) = e^{-i2\Theta\gamma_5}\psi(0). \tag{8.12}$$

The Thirring coupling g is related to the six-vertex or RSOS-coupling by [27]

$$\gamma = \frac{\pi}{\nu} = \frac{\pi + 2g}{2\pi + 2g} \quad \text{or} \quad \frac{\nu - 1}{\nu} = \frac{\beta^2}{8\pi} \tag{8.13}$$

where the sine-Gordon coupling has been introduced by Coleman's relation $4\pi/\beta^2 = 1 + g/\pi$. Note that the RSOS-models are distinguished among the SOS-models by exactly those couplings which correspond also to distinguished sine-Gordon couplings. It has been observed [28] that, at these couplings, in the Coulomb gas picture there are thresholds of multipole condensation. In the Bethe ansatz approach of the sine-Gordon model the ground state shows phase transitions at these couplings [29].

(iv) Bosonization.

Following for example the work of Swieca [30] we write the Dirac fermion operator as a product of an order times a disorder parameter

$$\psi = \sigma\mu, \quad \sigma(x) = \exp\left(i\tfrac{1}{2}\beta\gamma_5\phi(x)\right), \quad \mu(x) = \exp\left(i\frac{2\pi}{\beta}\int_x^\infty dx'\dot{\phi}(x')\right). \tag{8.14}$$

Then the model can be formulated in terms of a free boson Lagrangian with special boundary condition on the cylinder

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2, \quad \phi(L) = \phi(0) - \frac{4\Theta}{\beta}. \tag{8.15}$$

The discontinuity at "infinity" depends on Θ. This motivates us to call the sectors with different Θ, soliton sectors of the RSOS-model.

There exits a very simple calculation of the central charge c of the model in this formulation. By means of the substitution $\phi \to \phi + \frac{4\Theta}{\beta}\frac{x}{L}$ the field ϕ become continuous and we obtain for the partition function

$$\begin{aligned} Z_\Theta &= \int \mathcal{D}\phi \exp\left[-\int d^2x \left(\tfrac{1}{2}(\partial_\mu\phi)^2 + \frac{8\Theta^2}{L^2\beta^2}\right)\right] \\ &= Z_0 \exp\left(-\frac{L'}{L}\frac{8\Theta^2}{\beta^2}\right) \end{aligned} \tag{8.16}$$

where Z_0 is the free boson partition function with periodic boundary conditions and $c = 1$. From Cardy's formula (6.8), eq. (8.13) and $\Theta = \pi/\nu$ we find

$$c = 1 - \frac{6}{\pi}\frac{8\Theta^2}{\beta^2} = 1 - \frac{6}{\nu(\nu-1)} \tag{8.17}$$

which agrees with our Bethe ansatz result eq. (7.27b).

Note that eqs. (8.15) only describes one soliton sector of the RSOS-model. The complete model has the soliton sectors corresponding to $\theta = m\pi/\nu, \quad m = 1, \ldots, \nu - 1$ (cf. eq. (5.29)).

9. Coulomb gas picture.

This section remains on a more formal level compared to the previous ones. Some suggestions are natural but are not proved rigorously.

Dotsenko and Fatteev [31] used techniques of the two dimensional Coulomb gas in order to derive correlation functions for the "minimal models" of conformal quantum field theory. Starting from an approach of Feigin and Fuchs on representation theory of the Virasoro algebra, the authors of [31] used the Coulomb gas as a calculation framework together with a number of consistency conditions to derive their results. They had to introduce, ad hoc, a "charge at infinity" and "screening charges". This has the consequence that the field theoretic aspects of the models are not very obvious.

Here we start with the results of sect. 8 and "derive" the Coulomb gas of refs. [31] and [32] in a more field theoretic language. We introduce the "seam" of eqs. (5.30) and (5.31) which runs along a curve $\mathcal{C}$ from the bottom to the top of the cylinder in a Gaussian path integral. Then the partition function of the model given by eq. (8.15) writes as follows

$$Z_\Theta = \int \mathcal{D}\phi \exp\left[-\int d^2x \tfrac{1}{2}(\partial_\mu\phi)^2 + i\frac{4\Theta}{\beta}\int_{\mathcal{C}} dx_\mu \epsilon_{\mu\nu}\partial_\nu\phi\right]. \tag{9.1}$$

We introduce the order or "electric charge" operator of charge α

$$V_\alpha(x) = e^{i\alpha\phi(x)} \tag{9.2}$$

In order to simplify the notation we have changed the normalization of the free bose field $\phi \to \phi/\sqrt{8\pi}$. In terms of the complex "light cone" coordinates

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned} \tag{9.3}$$

the field equation of ϕ reads

$$\partial_z\partial_{\bar{z}}\phi = 0 \tag{9.4}$$

which implies that $\phi(x)$ is a sum of two functions depending only on z or on $\bar{z}$, respectively,

$$\phi(x) = \phi(z) + \phi(\bar{z}). \tag{9.5}$$

Hence, the electric charge operator decomposes into a product of two "light cone" vertex operators

$$V_\alpha(x) = e^{i\alpha\phi(x)} = V_\alpha(z)V_\alpha(\bar{z}) \tag{9.6}$$

The disorder or "magnetic charge" operator of charge α is defined by

$$\tilde{V}_\alpha(x) = e^{i\alpha\tilde{\phi}(x)} = V_\alpha(z)V_{-\alpha}(\bar{z}) \tag{9.7}$$

where $\tilde{\phi}(x) = \phi(z) - \phi(\bar{z})$.

Obviously, the partition function eq. (9.1), including the "seam term", is proportional to the two point function of magnetic charges sitting at the ends of the seam at the bottom "$-\infty$" and the top "∞" of the cylinder

$$Z_\Theta = Z_0\ \langle\tilde{V}_{-\alpha_0}(-\infty)\tilde{V}_{\alpha_0}(\infty)\ \rangle_0, \quad \alpha_0 = \frac{1}{2}\frac{1}{\sqrt{\nu(\nu-1)}} \tag{9.8}$$

where Z_0 and $\langle\ldots\rangle_0$ correspond to the periodic case. The expectation value of a product of vertex operators with respect to the periodic gaussian model is [30]

$$\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle_0 = \delta\left(\sum\alpha_i\right)\ \prod_{i,j}(z_i - z_j)^{-2\alpha_i\alpha_j}. \tag{9.9}$$

Next let us consider the primary fields $\phi_{mm}(x)$ with conformal dimensions $\Delta = \bar{\Delta} = h_{mm}$ (cf. eq. (5.13)). They correspond to Huse's operators of the RSOS-model given by eqs. (5.11). Using the procedure of sect. 8 one finds that the order parameter $\delta(x)$ defined by eq. (5.9) gives in the continuum limit

$$\delta(x) \to \frac{\beta}{2\pi} \int^x dy \dot{\phi} = \tfrac{\nu-1}{\pi}\, \alpha_0\, \tilde{\phi}(x). \tag{9.10}$$

Hence, we obtain from eq. (5.11)

$$\phi_{mm}(x) = \sin \alpha_0 \tilde{\phi}(x) \ \sin m\alpha_0 \tilde{\phi}(x). \tag{9.11}$$

If we use eq. (9.9) and calculate the two point function

$$\langle \phi_{mm}(x)\phi_{mm}(0)\rangle = \left(x^2\right)^{-d} \tag{9.12}$$

neglecting the presence of the "seam" and charge neutrality we find for the four possible terms

$$d = \begin{cases} -2(1 \mp m)^2 \alpha_0^2 \\ -2(1-m^2)\alpha_0^2 = h_{mm} \left(c = 1 - \frac{6}{\nu(\nu-1)}\right) \end{cases} \tag{9.13}$$

corresponding to total charges 0 and $\pm 2\alpha_0$, respectively. The upper case obviously belongs to the SOS-model with $c = 1$, whereas the lower case belongs to the RSOS-model with $c < 1$. In order to preserve charge neutrality Dotsenko and Fateev [31] introduced a charge $-2\alpha_0$ at infinity.

Apparently, eq. (9.13) shows that our Coulomb gas picture of the continuum limit of the RSOS-model is still not complete. We are interested in the model in regime III of fig. 6. However, we are sitting at $\vartheta = 0$ and $d = 0$ where all regimes meet. This is because in eq. (8.1) we have expanded the transfer matrix for $d = 0$ around $\vartheta = 0$ to obtain the Heisenberg Hamiltonian. In order to generate a perturbation in direction of regime III we have to replace the Lagrangian (8.15) by

$$\begin{gathered} \mathcal{L} + g_+ \sin \alpha_+ \tilde{\phi} + g_- \cos \alpha_- \tilde{\phi} \\ \text{with} \quad \alpha_+ = \sqrt{\tfrac{\nu}{\nu-1}}\,, \quad \alpha_- = -\sqrt{\tfrac{\nu-1}{\nu}} \end{gathered} \tag{9.14}$$

where $g_\pm$ are infinitesimal couplings. The first additional term is contained in c_2 of eq. (8.2) and causes a shift in direction of regimes III and IV. The second term causes a shift away from criticality in direction of regime III. The limits $g_\pm \to 0$

have to be taken in a suitable way to obtain the conformal invariant "minimal models". The additional contributions $q_+ = \sin\alpha_+\tilde{\phi}$ and $q_- = \cos\alpha_-\tilde{\phi}$ in the Lagrangian (9.14) are the screening charges of ref. [31].

Since $\alpha_+ + \alpha_- = 2\alpha_0$ the charges $q_\pm$ will screen the charges $\pm\alpha_0$ at the ends of the seam, such that the total charge at infinity is $\mp 2\alpha_0$. In addition, in a vacuum expectation value of products of vertex operators they will contribute such that the total charge is $\pm 2\alpha_0$. The final result is the following: The n-point function of magnetic operators (9.7) of charges $\alpha_1, \ldots, \alpha_n$ is

$$\begin{aligned} &\langle \tilde{V}_{\alpha_1}(x_1) \ldots \tilde{V}_{\alpha_n}(x_n)\rangle \\ &\quad = \sum \int d^2y_1 \ldots d^2y_k \langle \tilde{V}_{\alpha_1}(x_1) \ldots \tilde{V}_{\alpha_n}(x_n) q_+(y_1) \ldots q_-(y_k)\rangle_0 \end{aligned} \tag{9.15}$$

where the sum is finite and extends over $\alpha_1 + \ldots + \alpha_n \pm \alpha_+ \pm \ldots \pm \alpha_- = \pm 2\alpha_0$ and $\langle \ldots \rangle_0$ is given by eq. (9.9). As noted in ref. [31] for noncritical theories infinitely many terms like (9.15) sum up to produce a mass as for the sine-Gordon model.

At the end of this section some remarks on the other primary fields appearing in the Kac table (6.5). We start from the fields introduced above by eq. (5.11) and (9.11) corresponding to the diagonal $n = m$. In sect. 7 (cf. eq. (7.33) and fig. 7) we found that they belong to the Bethe ansatz states of lowest energy, the ground states, in the $(m-1)$-soliton sectors of the RSOS-model. In each sector the operator creating the lowest excitation from the ground state is

$$\sigma^+(x)\sigma^-(y) \to e^{i\frac{1}{2}\alpha_+\tilde{\phi}} \tag{9.16}$$

where the r.h.s. is the continuum limit of the l.h.s.. This suggests to replace for $n > m$ in eq. (9.11) $m\alpha_0$ by $m\alpha_0 + \frac{n-m}{2}\alpha_+ = \frac{1}{2}(m\alpha_- + n\alpha_+)$. Then we obtain for the primary fields of the minimal models

$$\phi_{mn} = \sin\alpha_0\tilde{\phi}\ \sin\tfrac{1}{2}(m\alpha_- + n\alpha_+)\tilde{\phi}. \tag{9.17}$$

This formula agrees with the proposal in ref. [31] and reflects the symmetry of the Kac formula (6.5) with respect to $(m, n) \to (\nu - m, \nu - 1 - n)$, since $\nu\alpha_- + (\nu - 1)\alpha_+) = 0$.

More complete derivations of the results of this section will be published elsewhere [33]. Another attempt to understand the Coulomb gas picture of the RSOS-models of Dotsenko and Fateev can be found in ref. [34]. In the present paper we concentrate on the AA-series of conformal field theories, in the language of ref. [35]. The other modular invariant models, where also fields with $\Delta \neq \bar{\Delta}$ appear, have to be constructed from the RSOS-models of ref. [36].

10. Conformal quantum field theories - braids - quantum groups.

This section contains a short review of some works of ref. [12], where braid properties of CQFT are investigated. We restrict here to the minimal models with central charge

$$c = 1 - \frac{6}{\nu(\nu-1)}\ , \quad \nu = 3, 4, 5, \ldots \tag{10.1}$$

and their relation to braids and quantum groups.

The state space decomposes into irreducible representation spaces $\mathcal{H}_a$ of $Vir \otimes \overline{Vir}$ with highest weight vectors

$$|a\rangle = \phi_a(0)|vac\rangle \tag{10.2}$$

generated by primary fields from the vacuum. The conformal dimensions are given by the Kac formula (6.5)

$$\Delta_a = h_{mn},\ \bar{\Delta}_a = h_{\bar{m}\bar{n}}; \quad m, \bar{m} = 1, \ldots, \nu-1,\ n, \bar{n} = 1, \ldots, \nu-2. \tag{10.3}$$

The vacuum and the 1-operator belong to h_{11}. The "conformal algebra" or the "fusion rule" is expressed by the expansion

$$\phi_a(x)\phi_b(0) = \sum_c \phi_c(0) c_{cab}(x) + \ldots \tag{10.4}$$

where the dots mean non-leading contribution from secondary fields. Introducing the projectors $\mathcal{P}_a$ on the spaces $\mathcal{H}_a$ we can write the fusion rules in terms of "three point functions" or projected fields

$$\mathcal{P}_c \phi_a \mathcal{P}_b \neq 0 \quad \text{if} \quad c_{cab} \neq 0. \tag{10.5}$$

The rules which give the nonvanishing c's can be found in ref. [16]. In the following we make two simplifications:

(i) The conformal fields are sums of products of light cone fields (cf. eq. (1.10)). We restrict here to "right movers".

(ii) We consider the elementary field $\varphi(u) = \phi_{21}(u)$ with dimension $\Delta = h_{21}$ and the spaces $\mathcal{H}_m$, $m = 1, \ldots, \nu-1$ belonging to h_{m1}.

The fusion rules for this simple case read

$$ {}^{m'}\varphi^{m} := \mathcal{P}_{m'}\varphi\mathcal{P}_{m} \neq 0 \quad \text{iff} \quad |m-m'|=1,\ 0<m,m'<\nu. \tag{10.6}$$

This looks like a $SU(2)$ sum rule with spin $= m/2$, "truncated" at $m=0$ and ν.

The object of interest is the n-point Wightman function $\langle\varphi(u_1)\ldots\varphi(u_n)\rangle$, especially, its behavior at $u_{i+1}\approx u_i$. This is determined by the monodromy properties of the "conformal blocks" ${}^{i}\varphi^{k}(1)\ {}^{k}\varphi^{j}(u)$ which are obtained by insertion of projectors into the n-point function. The monodromy is expressed by the commutation relations of the projected fields.

$$ {}^{i}\varphi^{k}(u_1)\ {}^{k}\varphi^{j}(u_2) = \sum_{k'} {}^{i}\varphi^{k'}(u_2)\ {}^{k'}\varphi^{j}(u_1)\ R_{12}(k',i;j,k). \tag{10.7}$$

This relation may be depicted by the graph of fig. 8. The lines denote the fields and the plaquettes between the lines the spaces $\mathcal{H}_m$, $m=1,\ldots,\nu-1$. The l.h.s. (r.h.s.) of eq. (10.7) correspond to the bottom (top) of the graph. The fusion rules (10.6) imply, that for neighbor plaquettes $|m-m'|=1$ holds.

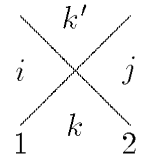
$R_{12}(k',i;j,k) =$ 

Fig. 8. *The braid matrix*

The matrices R depend only on the sign $\epsilon(u_1-u_2)$ and

$$R(-) = R^{-1}(+). \tag{10.8}$$

Commuting three field in two different ways one finds the Yang-Baxter equation with discrete spectral parameters (cf. fig. 9)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{10.9}$$

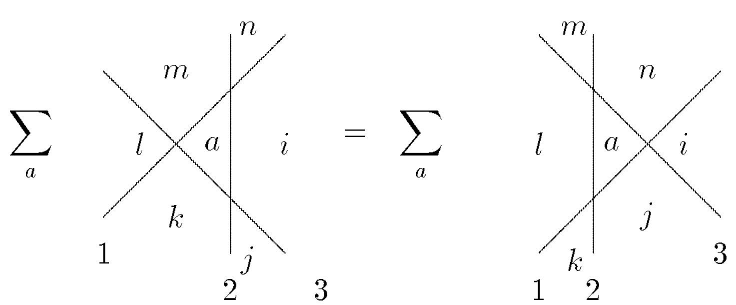


Fig. 9. *The hexagon equation*

Eqs. (10.8) and (10.9) are characteristic for braid generators. Indeed, eq. (10.7) defines a representation of the braid group. Comparing the properties of the braiding matrix R defined by eq. (10.7) with those of the RSOS-matrix R^{RSOS} of sect. 5, one suspects

$$R \propto R^{RSOS}(k = 0, \gamma = \pi/\nu, \vartheta = i\infty). \tag{10.10}$$

The choice of the parameters is motivated as follows: In order to have the discrete spectral parameters of eq. (10.9), one has to take $\vartheta = i\infty$. This limit is possible only for the trigonometric case $k = 0$. The truncation properties at $m = 0$ and ν correspond to $\gamma = \pi/\nu$.

The relation (10.10) was confirmed by Rehren [12], who solved eqs. (10.6-9) together with some consequences of conformal invariance. It must be stressed that the R-matrix here in the continuum field theory as a braid matrix play a completely different role compared to that in the lattice model of section. 5, although, they have the same couplings γ and k.

At the end let us note two points where the quantum group might play an important role in conformal quantum field theories. In sections 1 and 3 we have remarked that quantum groups typically are related to Yang-Baxter algebras at spectral parameter $\vartheta = i\infty$. We find this situation here, for the braid matrices given by eq. (10.10). Usually, in physics symmetries are described by groups. One could ask, what kind of symmetries correspond to quantum groups.

Equation (10.7) defines maps in the space of projected fields. These maps should lead to symmetry properties of Virasoro algebra representations. Indeed, Rehren [12] showed that by solving eqs. (10.6-9) one finds relations of the conformal dimensions which agree with the Kac formula, derived from representation theory of the Virasoro algebra.

Another type of maps of projected field follows from the fusion rules (10.4). The fields $\varphi_m := \phi_{m1}$ can be obtained iteratively from the elementary one $\varphi = \varphi_2$ by

$$\varphi_m \cdot \varphi_2 = \sum_{m'} \varphi_{m'} \; c_{m'm2} \;, \quad |m - m'| = 1. \tag{10.11}$$

For the projected fields the general "fusion matrix" N is defined by

$${}^{i}\varphi_a^k \cdot {}^{k}\varphi_b^j = {}^{i}\varphi_c^j \; N_{cab}^{ikj}. \tag{10.12}$$

which is graphically represented in fig. 10.

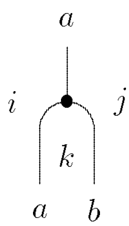
$N_{cab}^{ikj} =$ 

Fig. 10. *The fusion matrix*

Consistency of the braiding matrix R and the fusion matrix N is expressed by the "pentagon equation"(cf. fig. 11)

$$RN = NRR. \tag{10.13}$$

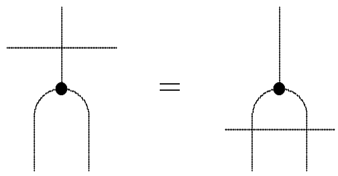


Fig. 11. *The pentagon equation*

There is a complete analogy of the braid matrices R of this section and the factorized S-matrices mentioned in the introduction. The fusion procedure for the braid matrices correspond to the S-matrix bound state problem developed in refs. [37]. Using these old results one finds for example relations of the fusion matrix N with the braiding matrix R.

There exists an extensive literature [9-12] and [38] on subjects concerning the content of this section. There are contacts with pure mathematical fields as knot theory, von Neumann algebras, e.t.c..

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