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YANG-BAXTER ALGEBRA - BETHE ANSATZ - CONFORMAL QUANTUM FIELD THEORIES - QUANTUM GROUPS*

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ABSTRACT. This report is based on investigations [1,2] of six-vertex models by means of Bethe ansatz methods. A review on applications of Yang-Baxter algebras and the algebraic Bethe ansatz is presented. A Θ-vacuum like angle appearing in the RSOS-models is discussed. The continuum limit in the critical case of these statistical models is performed to obtain the "minimal models" of conformal quantum field theory. The six-vertex model possesses $SU(2)_q$ "quantum symmetry". Representations of the quantum group obtained by the Bethe ansatz are investigated.

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1. Introduction: Applications of Yang-Baxter algebras.

There exists a large number of fields in theoretical and mathematical physics where **Yang-Baxter Algebras** play a central role for solving problems. Yang-Baxter algebras are closely connected with the theory of integrable systems. Since some years this algebraic structure has entered also pure mathematical spheres as knot theory, von Neumann algebras etc.. Some applications are:

(i) Quantum chains in 1-dimension.
E.g. Heisenberg models, as investigated by Bethe in 1931 [3]*. These models are solved by the "Bethe ansatz" which, in its algebraic formulation invented by Faddeev [4], is based on the Yang-Baxter algebra.

(ii) Quantum field theories in 1+1-dimensions.
E.g. the nonlinear Schrödinger field theory, as investigated by Yang in 1967 [5]*. The relations used in this context for the coefficients of the coordinate Bethe ansatz are now called Yang-Baxter relations.

(iii) Classical statistical lattice models in 2-dimensions.
E.g. the eight-vertex model, as investigated by Baxter in 1972 [6]. These models are solved by means of the "star-triangle equations" which are equivalent to Yang-Baxter equations.

(iv) Soliton S-matrices in 1+1-dimensions.
The "factorization equations" [7]* which have been used to determine soliton S-matrices are Yang-Baxter equations.

Let me explain the algebraic structure of Yang-Baxter algebras in terms of soliton S-matrices. Soliton scattering does not change the momenta of the particles but only quantum numbers are exchanged. The N-particle S-matrix is a product of two particle ones. In particular the three-particle scattering is given by two possible orders of the factors which have to give the same result

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} \tag{1.1}$$

where S_{ij} depends on the rapidity difference $\vartheta_{ij} = \vartheta_i - \vartheta_j$. Let V_i denotes the vector space corresponding to quantum numbers $\alpha_i = 1, \ldots, n_i$ of the particle

*and later works

with rapidity ϑ_i. Then both sides of eq. (1.1) act in the tensor product space $V_1 \otimes V_2 \otimes V_3$ and S_{ij} in the embedding of $V_i \otimes V_j$.

The factorization or Yang-Baxter equation (1.1) reads more explicitly

$$S_{\alpha' \beta'}^{\alpha'' \beta''}(\vartheta_{12})\; S_{\alpha \gamma'}^{\alpha' \gamma''}(\vartheta_{13})\; S_{\beta \gamma}^{\beta' \gamma'}(\vartheta_{23}) = S_{\beta' \gamma'}^{\beta'' \gamma''}(\vartheta_{23})\; S_{\alpha' \gamma}^{\alpha'' \gamma'}(\vartheta_{13})\; S_{\alpha \beta}^{\alpha' \beta'}(\vartheta_{12}). \quad (1.2)$$

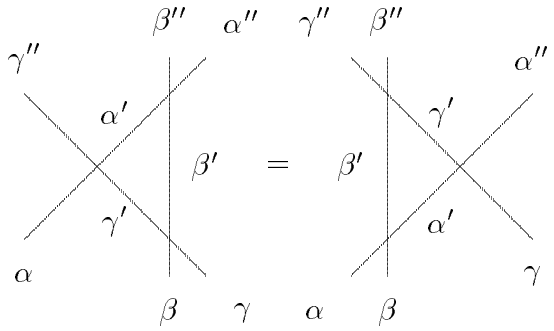


Fig. 1. *The factorization or Yang-Baxter equation for soliton S-matrices: Two different orderings of a three-particle scattering process obtained by parallel shifting a line .*

Generalizing the concept of Lie-algebras we say the T's fulfilling

$$T_{12} T_{13} S_{23} = S_{23} T_{13} T_{12} \quad (1.3)$$

generated a **"Yang-Baxter algebra"** where the "structure constants" S fulfil equation (1.1). The product in eq. (1.3) has to be understood analogously to eq. (1.2). However, the index α now correspond to an abstract space V_1 and T_{1i} is an operator in the space $V_1 \otimes V_i$.

(v) "Quantum groups". [8]
Sometimes (i.e. for the trigonometric case) the generators of a Yang-Baxter algebra $T(M^a, \vartheta, \gamma)$ may be written in terms of generators M^a and some complex parameters ϑ and γ (cf. section 2 and 3). In the limit

$$\vartheta \to i\infty \quad \text{and} \quad \gamma \to 0 \quad (1.4)$$

typically one finds that the Yang-Baxter relations (1.3) imply Lie-group commutation rules

$$[M^a, M^b] = f^{abc} M^c. \quad (1.5)$$

For nonvanishing γ one finds nonlinear commutation relations

$$[M^a, M^b] = f^{ab}(\gamma, M^c). \tag{1.6}$$

Interpreting eq. (1.5) as a Poisson algebra and γ as $\hbar$ one says eq. (1.6) describes the quantum version of the algebra defined by eq. (1.5). Or writing $q = e^{i\gamma}$ one says eq. (1.6) describes a "q-deformation" of the Lie-group. In section 6 we discuss the quantum group $SU(2)_q$.

There exists another type of deformations of Lie-groups: the "Yangians", corresponding to the following diagram:

$$\text{Yang-Baxter algebra} \rightarrow \left\{ \begin{array}{ccc} \overset{\vartheta \to i\infty}{\longrightarrow} & \text{"quantum group"} & \overset{\gamma \to 0}{\longrightarrow} \\ \underset{\gamma \to 0}{\longrightarrow} & \text{"Yangians"} & \underset{\vartheta \to i\infty}{\longrightarrow} \end{array} \right\} \rightarrow \text{Lie-group.}$$

(vi) Braids. [9]
Consider N points on a line in R^3 and the corresponding set of points on a parallel shifted line. A braid is given by set of nonintersecting monotonic curves connecting the points on one line with those on the other, see fig. 2. Two set of curves which can be continuously deformed into each other define the same braid.

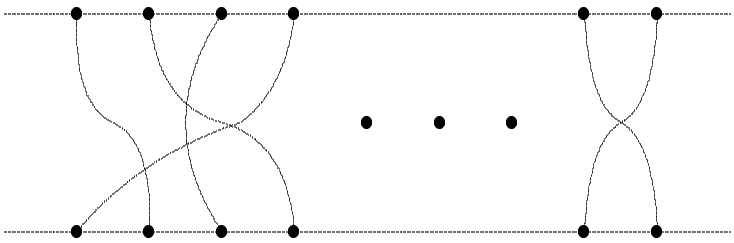
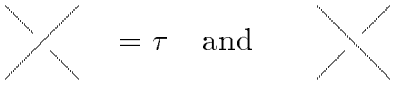


Fig. 3. *A braid given by nonintersecting curves in* R^3

Defining a product by putting two braids on top of each other they form a group. Artin has shown that a braid can by written as a product of generators of the form



These generators can be represented by Yang-Baxter matrices

$$\tau = S(\vartheta = i\infty) \quad \text{and} \quad \tau^{-1} = S(\vartheta = -i\infty). \tag{1.7}$$

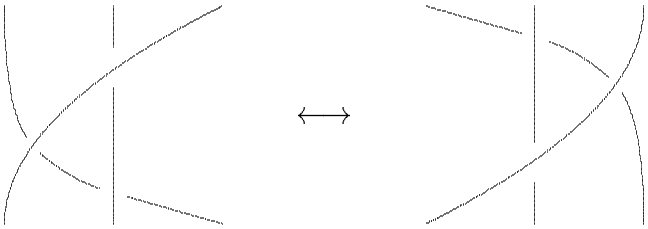


Fig. 3. *Reidemester move*

This follows from the fact that the deformation property, the "Reidemeister move" of fig. 3, agrees obviously with the Yang-Baxter equation of fig. 1.

One has to take the "spectral parameter" ϑ at infinity, because otherwise the sum rules of the ϑ's in eq. (1.1) cannot be fulfilled. Apparently, both, quantum groups and braids, correspond to Yang-Baxter algebras for discrete (infinite large) values of the spectral parameter.

(vii) Statistics matrix in 2-dimensional quantum field theories. [10]
Consider spacelike commutation rules of fields

$$\phi_A(x)\phi_B(x') = \phi_D(x')\phi_C(x)\ S_{AB}^{CD}(\epsilon(x_1 - x_1')). \tag{1.8}$$

In contrast to four dimensions the statistics is not necessarily bosonic or fermionic, but the statistics matrix S might be more general. Commuting three field $\phi_A\phi_B\phi_C$ in two different ways one finds that S fulfills braid- or Yang-Baxter commutation rules (1.2) for discrete ϑ's.

(viii) Conformal quantum field theories in 2-dimensions. [11]
A conformal field in two dimensions can be written as a sum of products of "light cone fields"

$$\phi(x) = \sum_i A_i(x+t)B_i(x-t) \tag{1.9}$$

Analogously to eq. (1.8) the light cone fields fulfil the commutation rule

$$A_i(u)A_j(u') = A_l(u')A_k(u)\ R_{ij}^{kl}(\epsilon(u-u')). \tag{1.10}$$

There exists a number of models (e.g. the "minimal" models considered below) with highly nontrivial braid representations R, whereas for all known examples the statistics matrices of eq. (1.8) are usually much simpler.

Braids appear also in higher dimensions, if the fields are not localized at points, but in regions which extend to infinity.

The paper is organized as follows. Section 2 contains a short introduction to vertex and the (R)SOS-models. In section 3 it is demonstrated how the generalized Bethe ansatz for the eight-vertex model introduces a Θ-angle [4] which will play an important role for the RSOS-model in the following sections.

Conformal invariance of two dimensional quantum field theories has become a subject of main interest because of its connection to string theory. Statistical mechanical systems at second order phase transition points (i.e. critical systems) should possess conformal invariance. We shall use this fact to obtain information on conformal quantum field theories. Especially, we look for **integrable** lattice models which yield these conformal quantum field theories in the scaling limit. Since a direct calculation of the central charge of the Virasosro algebra and the conformal dimensions for specific models is usually not simple, it seems worthwhile to determine them for integrable models. This will be done in section 4, using finite size corrections of Bethe ansatz results, for the six-vertex and (R)SOS-models. In section 5 the continuum limit of the RSOS-model is discussed. In section 6 the "quantum group" $Sl(2)_q$ for q = root of unity is obtained from the underlying Yang-Baxter algebra of the RSOS-model. Properties of the representations obtained by the Bethe ansatz are investigated.

2. Eight-vertex model and (R)SOS-model.

The "Solid On Solid"- models are very interesting in the context of conformal quantum field theoretic models (see below). Baxter [6] introduced these models in order to solve the eight-vertex model a noncritical generalization of the six-vertex model, which has been solved before [12] by the Bethe ansatz method. Since we will at the end construct conformal quantum field theories as the scaling limit of lattice models, we are interested in the critical case. However, the transformation between the SOS-model and the eight-vertex model is singular for the critical case the six-vertex model.[1] Therefore let me introduce both models away from the critical point.

[1] In fact the situation is more complicated: There exits of course the limit which gives the ordinary six-vertex model. But we are looking for the limit which yields the critical RSOS-model.

2.1 Eight-vertex model.

The partition function of the eight-vertex model on an $N \times M$ periodic square lattice L is

$$Z^{8-vertex} = \sum_{conf.} \prod_{x \in L} S(x) \tag{2.1}$$

where the sum extends over all allowed "bond configurations". Each bond can accept one of two states (denoted by arrows or $\pm$). The configuration is allowed if at each vertex x there are an even number of incoming and out going arrows. The vertex weights $S(x)$ are determined by the eight bond configurations at the lattice site x. They assume the following values

$$\begin{aligned}
S^{++}_{++} &= \leftarrow^{\uparrow}_{\uparrow}\leftarrow = S^{--}_{--} = \rightarrow^{\downarrow}_{\downarrow}\rightarrow \;= a = \;\mathrm{sn}(\gamma - \vartheta) \\
S^{+-}_{+-} &= \leftarrow^{\downarrow}_{\downarrow}\leftarrow = S^{-+}_{-+} = \rightarrow^{\uparrow}_{\uparrow}\rightarrow \;= b = \;\mathrm{sn}\vartheta \\
S^{+-}_{-+} &= \leftarrow^{\downarrow}_{\uparrow}\rightarrow = S^{-+}_{+-} = \rightarrow^{\uparrow}_{\downarrow}\leftarrow \;= c = \;\mathrm{sn}\gamma \\
S^{++}_{++} &= \leftarrow^{\uparrow}_{\downarrow}\rightarrow = S^{--}_{--} = \rightarrow^{\downarrow}_{\uparrow}\leftarrow \;= d = k\;\mathrm{sn}\gamma\;\mathrm{sn}\vartheta\;\mathrm{sn}(\gamma - \vartheta)
\end{aligned} \tag{2.2}$$

where sn is the Jacobian elliptic functions of modulus k and periods $4K$ and $2iK'$.

By this notation the weights have been arranged as a $4 \times 4-$ matrix

$$S^{\gamma\delta}_{\alpha\beta}(\vartheta,\gamma) = \;\delta\; \begin{matrix} \gamma \\ -|- \\ \alpha \end{matrix} \beta \; = \begin{pmatrix} \leftarrow^{\uparrow}_{\uparrow}\leftarrow & & & \leftarrow^{\uparrow}_{\downarrow}\rightarrow \\ & \leftarrow^{\downarrow}_{\downarrow}\leftarrow & \leftarrow^{\downarrow}_{\uparrow}\rightarrow & \\ & \rightarrow^{\uparrow}_{\downarrow}\leftarrow & \rightarrow^{\uparrow}_{\uparrow}\rightarrow & \\ \rightarrow^{\downarrow}_{\uparrow}\leftarrow & & & \rightarrow^{\downarrow}_{\downarrow}\rightarrow \end{pmatrix} = \begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix}. \tag{2.3}$$

The parameterization in terms of the "spectral parameter" ϑ, the "coupling" γ and the modulus k has been introduced for later convenience (cf. eq. (3.1)). A trivial overall factor is set equal to one. In the critical limit ($k \to 0, K \to \frac{1}{2}\pi, K' \to \infty$ and $\mathrm{sn} \to \sin$) the weight d vanishes and we obtain the six-vertex model.

2.2 SOS-model.

The partition function of the SOS-model on an $N \times M$ periodic square lattice L is

$$Z^{SOS} = \sum_{conf.} \prod_{x \in L} R(x). \tag{2.4}$$

A "height" l equal to an integer number is assigned to each plaquette of the lattice, such that heights of neighbor plaquettes differ by ± 1. The Boltzmann weights

$$\begin{matrix} i & | & k \\ - & R & - \\ j & | & l \end{matrix} \;=\; R(i,j;k,l) \tag{2.5}$$

corresponding to the six possible configurations at a vertex x

$$\begin{array}{c} l \;\; |l-1 \\ \hline l+1| \;\; l \end{array} \quad \begin{array}{c} l \;\; |l+1 \\ \hline l-1| \;\; l \end{array} \quad \begin{array}{c} l-1| \;\; l \\ \hline l \;\; |l+1 \end{array} \quad \begin{array}{c} l+1| \;\; l \\ \hline l \;\; |l-1 \end{array} \quad \begin{array}{c} l-1| \;\; l \\ \hline l \;\; |l-1 \end{array} \quad \begin{array}{c} l+1| \;\; l \\ \hline l \;\; |l+1 \end{array}$$

can be found in Baxter's papers [6].

Baxter proved that the eight-vertex and the SOS-model can be related by means of two matrices X and Y:

$$\sum_{\alpha\beta} S^{\gamma\delta}_{\alpha\beta}\; X^{\alpha}(j,l)\; Y^{\beta}(k,l) = \sum_{i} X^{\gamma}(i,k)\; Y^{\delta}(i,j)\; R(i,j;k,l). \tag{2.6}$$

We do not need the explicit expressions for X and Y, they can be found in refs. [6] and [4].

From this relation Baxter concluded the equivalence of both models[2]

$$Z^{8-vertex} \simeq Z^{SOS}. \tag{2.7}$$

For generic parameters γ the SOS-model is an "unrestricted" one, that means the heights l assume all values. Andrews, Baxter and Forrester [13] found that for a discrete set of the parameter

$$\gamma = 2K/\nu, \quad \nu = 3, 4, 5, \ldots \tag{2.8}$$

the SOS-model splits into an "unrestricted" and a "restricted" one, the RSOS-model

$$\begin{cases} SOS & -\infty < l < \infty \\ RSOS & 0 < l < \nu. \end{cases} \tag{2.9}$$

In ref. [13] was shown that there are four different regimes of the parameters where the RSOS-model has different "ground state" structures (cf. fig. 4)

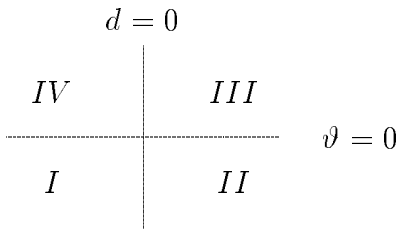


Fig. 6. *The four different regimes of the RSOS-model.*

[2] On a finite lattice with periodic boundary conditions (torus) this equivalence is true up to one plaquette.

We are interested in regime III near to the critical line $d = 0$. There are $\nu - 2$ "ground states" labeled by $\varphi = 1, \ldots, \nu - 2$. An order parameter corresponding to these states can be expressed in terms of the heights at two neighbor points

$$\varphi(x) = \tfrac{1}{2}(l(x) + l(x+1) - 1). \tag{2.10}$$

Andrews, Baxter and Forrester calculated the critical exponent α and the probability

$$P_l^{(\varphi)} = \langle \delta_{l,l(x)} \rangle^{(\varphi)} \tag{2.11}$$

that at a point x the height l is assumed for boundary conditions corresponding to φ. Using this result Huse [14] showed that the one-point functions of the fields

$$\phi_m(x) = \sin\left(\pi \frac{\varphi(x)}{\nu - 1}\right) \sin\left(\pi m \frac{\varphi(x)}{\nu - 1}\right), \qquad m = 1, \ldots, \nu - 2 \tag{2.12}$$

behave near to the critical point ($d = 0$) in regime III like

$$\langle \phi_m \rangle_{III} \approx (t - t_{crit.})^{\beta_m} \tag{2.13}$$

where the exponents β_m correspond to the scaling dimensions of the fields $\phi_m(x)$ (using a scaling relation)

$$d_m = \frac{2\beta_m}{2 - \alpha} = \frac{m^2 - 1}{2\nu(\nu - 1)}. \tag{2.14}$$

These are just the scaling dimensions of some fields appearing in the "minimal models" of conformal quantum field theories investigated by Belavin, Polyakov and Zamolodchikov [15] (see section 4). This result suggests that in the scaling limit (where the lattice constant goes to zero) the RSOS-models in regime III at a critical point should give a realization of the "minimal" conformal quantum field theories.

3. Algebraic Bethe ansatz and "Θ-angle".

3.1 Yang-Baxter algebra.

The eight-vertex model is integrable. This follows from the fact that the vertex weights S given by eq. (2.2) for fixed γ and k fulfil the Yang-Baxter equation

$$S(\vartheta)S(\vartheta')S(\vartheta'-\vartheta) = S(\vartheta'-\vartheta)S(\vartheta')S(\vartheta). \tag{3.1}$$

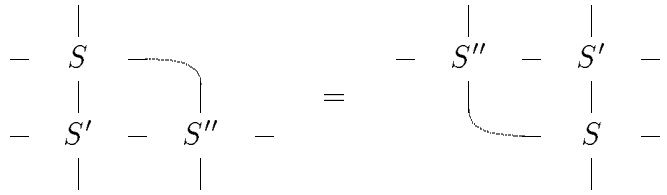


Fig. 5. *The Yang-Baxter equation $SS'S'' = S''S'S$ for the matrices of the statistical weights of the eight-vertex model.*

where the products are to be taken as in eq. (1.2), see fig. 5.

In order to express the partition function eq. (2.1) in terms of the matrix S we define the **monodromy matrix**

$$T^{\{\gamma\}\delta}_{\{\alpha\}\beta}(\vartheta,\gamma) = S^{\gamma 1\ \delta}_{\alpha_1\delta_2}(\vartheta,\gamma)S^{\gamma_2\delta_2}_{\alpha_2\beta_2}(\vartheta,\gamma)\dots S^{\gamma_N\delta_N}_{\alpha_N\beta}(\vartheta,\gamma) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\delta\ \{\gamma\}\ \alpha\ \{\alpha\}\ \beta\ = \ \dots = \begin{pmatrix} \leftarrow\leftarrow & \leftarrow\rightarrow \\ \rightarrow\leftarrow & \rightarrow\rightarrow \end{pmatrix}. \tag{3.2}$$

We have a matrix product with respect to the "horizontal space" and a tensor product with respect to N "vertical spaces".

We define the **transfer matrix** as a trace in the horizontal space

$$\tau(\vartheta,\gamma) = \mathrm{tr}_h\ T(\vartheta,\gamma) = A + D. \tag{3.3}$$

Then we can write the partition function (2.1) as a trace in the tensor product of the vertical spaces

$$Z = \mathrm{tr}_v\, \tau^M. \tag{3.4}$$

If we are able to solve the eigenvalue problem of the transfer matrix

$$\tau\psi = \lambda\psi \tag{3.5}$$

we can calculate the partition function in terms of the eigenvalues

$$Z = \sum_i \lambda_i^M. \tag{3.6}$$

In the thermodynamic limit only the maximal eigenvalue contribute to the free energy per lattice site

$$f = -kT\frac{1}{NM}\log Z \approx -kT\frac{1}{NM}\log \lambda_{max}^{M} \qquad \text{for } N, M \to \infty. \tag{3.7}$$

From eq. (3.1) follows by iterations the Yang-Baxter relation for the monodromy matrix which therefore generates a Yang-Baxter algebra:

$$T(\vartheta)T(\vartheta')S(\vartheta' - \vartheta) = S(\vartheta' - \vartheta)T(\vartheta')T(\vartheta). \tag{3.8}$$

This will be the basic formula in the following. It reflects the integrability of the model. Note that it implies the commutativity of the transfer matrix for different spectral parameters

$$[\tau(\vartheta), \tau(\vartheta')] = 0 \tag{3.9}$$

which represents (an infinite set of) conservation laws.

$$[c_n, c_m] = 0 \quad , \quad \text{where} \quad \log\tau = -\sum c_n \vartheta^n. \tag{3.10}$$

The lowest operators are related to the momentum and the Heisenberg model Hamiltonian

$$c_0 = iP \quad , \qquad c_1 = H_{xxz}. \tag{3.11}$$

From the Yang-Baxter relations eq. (3.8) follow commutation rules for the matrix elements A, B, C and D of monodromy matrix T defined by eq. (3.2), e.g.

$$A(\vartheta)B(\vartheta') = \frac{a(\vartheta' - \vartheta)}{b(\vartheta' - \vartheta)}B(\vartheta')A(\vartheta) - \frac{c(\vartheta' - \vartheta)}{b(\vartheta' - \vartheta)}B(\vartheta)A(\vartheta'). \tag{3.12}$$

These relations will be used to solve the eigenvalue equation of the transfer matrix

$$\tau(\vartheta)\psi = (A(\vartheta) + D(\vartheta))\psi = \lambda\psi \tag{3.13}$$

by means of the Bethe ansatz. I explain here very shortly the algebraic formulation developed in ref. [4]. This method is quite analogous to the algebraic procedure used in quantum mechanics to solve eigenvalue problems (as e.g. for the angular momentum) or more general to find irreducible representations of Lie-algebras. The Yang-Baxter relations (3.8) correspond to the Lie-algebra structure relations. The diagonal elements A and D of T play the rôle of the "weights". The off diagonal elements B and C are creation and annihilation operators of "pseudo-particles", respectively.

3.2 The six-vertex model. [3]

The six-vertex case is simple since the weight d vanishes which means "charge conservation". Therefore a "pseudo-vacuum" state

$$\phi^{\{\alpha\}} = \prod_{i=1}^{N} \delta_{\alpha_i +} = \uparrow \ldots \uparrow . \tag{3.14}$$

exists which is annihilated by the operator C

$$C(\vartheta)\ \phi = \rightarrow \overset{|}{\uparrow} - \ldots - \overset{|}{\uparrow} \leftarrow = 0 \qquad \forall\ \vartheta\ . \tag{3.15}$$

It is an eigenstate of A and D with eigenvalues a^N and b^N, respectively. The operator B creates "pseudo-particles". The **algebraic Bethe ansatz** represents an eigenstate of τ as an n-pseudo-particle state

$$\psi = B(\vartheta_1) \ldots B(\vartheta_n)\ \phi = \begin{array}{ll} \leftarrow \overset{|}{|} - \ldots - \overset{|}{|} \rightarrow & B(\vartheta_1) \\ \vdots \qquad\qquad \vdots & \\ \leftarrow \underset{1}{\overset{|}{\uparrow}} - \ldots - \underset{N}{\overset{|}{\uparrow}} \rightarrow & B(\vartheta_n) \end{array} \tag{3.16}$$

with the "Bethe-spin"

$$J_3 = N/2 - n, \quad n = 0, \ldots, N/2. \tag{3.17}$$

The "rapidities" ϑ_i are not free but have to be determined from the eigenvalue equation (3.13).

We follow the general strategy [4] of the algebraic Bethe ansatz method to solve eq. (3.13). We apply $A(\vartheta)$ and $D(\vartheta)$ to ψ and push them through all the B's using the commutation rules (3.12). From the first terms on the r.h.s. of eq. (3.12) we get the "wanted" contributions proportional to ψ

$$(A(\vartheta) + D(\vartheta))\ \psi = (\lambda_A(\vartheta) + \lambda_D(\vartheta))\ \psi\ +\ \text{"unwanted terms"} \tag{3.18}$$

where from eq. (3.12) we find

$$\lambda_A(\vartheta) = a^N(\vartheta) \prod_{i=1}^{n} \frac{a(\vartheta_i - \vartheta)}{b(\vartheta_i - \vartheta)} \qquad \lambda_D(\vartheta) = b^N(\vartheta) \prod_{i=1}^{n} \frac{a(\vartheta - \vartheta_i)}{b(\vartheta - \vartheta_i)}. \tag{3.19}$$

[3] The six-vertex model was first solved by means of the Bethe ansatz by Lieb [12].

The "unwanted terms" arise from the second terms in eqs. (3.12). They contain a $B(\vartheta)$ and so they can never give a vector proportional to ψ. The unwanted terms vanish if the **Bethe ansatz equations** hold

$$\left(\frac{b(\vartheta_j)}{a(\vartheta_j)}\right)^N \prod_{i=1}^{n} \frac{a(\vartheta_j-\vartheta_i)}{b(\vartheta_j-\vartheta_i)}\frac{b(\vartheta_i-\vartheta_j)}{a(\vartheta_i-\vartheta_j)} = -1, \quad j=1,\ldots,n, \quad n=0,\ldots,N/2. \tag{3.20}$$

We have transformed the eigenvalue problem of the transfer matrix (3.13) to sets of transcendental equations (3.20) for the parameters $\vartheta_1,\ldots,\vartheta_n$, where $n=0,\ldots,\frac{1}{2}N$ (Neven).

Solutions of Bethe ansatz equations are available for small or very large N and n. We are interested in the latter case. Each Bethe ansatz state is characterized by a set of "roots" u_i where $\vartheta_i = \frac{1}{2}(iu_i+\gamma)$. The ground state (for the disordered case γ = real) correspond to a completely filled "Dirac sea" of real roots.

For "excited states" there may be, in addition to holes in the "Dirac sea", complex roots u_i: the k-strings and "anti-pseudo-particles". A typical set of roots of Bethe ansatz equations corresponding to an excited state is shown in fig. 6.

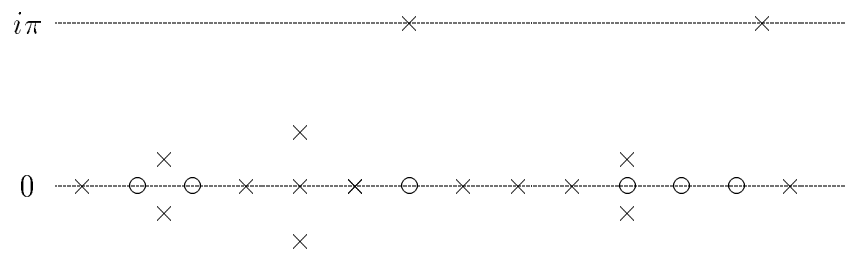


Fig. 6. *A typical set of roots of Bethe ansatz equations in the complex q-plane, including real roots, holes ∘, k-strings and "anti-pseudo-particles"*

3.3 The eight-vertex and the SOS-model.

The Bethe ansatz in the simple version of section 3.2 does not work for the eight-vertex model. The "pseudo-vacuum" ϕ defined by $C\phi = 0$ does not exist because of the d-vertex. Baxter invented the transformation to the SOS-model (2.6) in order to construct a generalized Bethe ansatz. I will follow here the formulation of Takhtadzhan and Faddeev [4].

We transform the eight-vertex monodromy matrix by means of the transformation matrix Y of eq. (2.6) w.r.t. the horizontal space

$$T^{\{\gamma\}}_{\{\alpha\}}(i,j;k,l) = Y^{-1}_{\delta}(i,j)\; T^{\{\gamma\}\delta}_{\{\alpha\}\beta}\; Y^{\beta}(k,l) \tag{3.21}$$

or in matrix notation

$$T_{j,l} = \begin{pmatrix} A_{j,l} & B_{j,l} \\ C_{j,l} & D_{j,l} \end{pmatrix} = \begin{pmatrix} T(j-1,j;l-1,l) & T(j-1,j;l+1,l) \\ T(j+1,j;l-1,l) & T(j+1,j;l+1,l) \end{pmatrix} \tag{3.22}$$

(and obtain a hybrid of eight-vertex and SOS-monodromy matrix). Generalized Bethe ansatz states for the eight-vertex model (analogously to eq. (3.16)) are

$$\psi_l(\vartheta_1,\ldots,\vartheta_n) = B_{l+1,l-1}(\vartheta_1)\ldots B_{l+n,l-n}(\vartheta_n)\ \phi_{l+n,l-n}\ . \tag{3.23}$$

The generalized "pseudo-vacuum" is

$$\begin{aligned} \phi^{\{\alpha\}}_{l_0,l_N} &= X^{\alpha_1}(l_0,l_1)\ldots X^{\alpha_N}(l_{N-1},l_N), \quad l_{i+1} = l_i - 1 \\ &\text{with} \qquad C_{j,l}\ \Phi_{j,l} = 0. \end{aligned} \tag{3.24}$$

In contrast to the six-vertex Bethe ansatz the number of B-operators is not arbitrary. In eq. (3.23) we have $l_0 = l+n$ and $l_N = l-n$. Therefore $l_{i+1} = l_i - 1$ in eq. (3.24) implies

$$n = N/2 \quad \text{or Bethe-spin} \ \ J_3 = N/2 - n = 0. \tag{3.25}$$

Because of eq. (3.21), the transfer matrix of the eight-vertex model can be expressed in terms of the transformed monodromy matrices independently of l

$$A_{l,l} + D_{l,l} = \tau^{8-vert.}. \tag{3.26}$$

Analogously to the six-vertex case (cf. eq. (3.18)) one can show

$$(A_{l,l} + D_{l,l})\ \psi_l = \lambda_A\ \psi_{l-1} + \lambda_D\ \psi_{l+1} + \ \text{"unwanted terms"} \tag{3.27}$$

where the λ's are given by eq. (3.19). Hence, the eigenstates of the eight-vertex model have the following form

$$\psi_\Theta(\vartheta_1,\ldots,\vartheta_n) = \sum_{l=-\infty}^{\infty} e^{-i\Theta l}\ \psi_l(\vartheta_1,\ldots,\vartheta_n) \tag{3.28}$$

$$\tau^{8-vert.}\ \psi_\Theta = \lambda\ \psi_\Theta \tag{3.29}$$

and the eigenvalues are

$$\lambda = e^{-i\Theta}\ \lambda_A + e^{i\Theta}\ \lambda_D. \tag{3.30}$$

The Bethe ansatz equations, which are equivalent to the vanishing of the "unwanted terms" read (analogously to eq. (3.8))

$$\left(\frac{b(\vartheta_j)}{a(\vartheta_j)}\right)^N \prod_{i=1}^{n} \frac{a(\vartheta_j-\vartheta_i)}{b(\vartheta_j-\vartheta_i)} \frac{b(\vartheta_i-\vartheta_j)}{a(\vartheta_i-\vartheta_j)}\, e^{i2\Theta} = -1, \quad j=1,\ldots,n, \quad n=N/2. \tag{3.31}$$

In the critical limit $k \to 0$ we recover the Bethe ansatz equations (3.20) for the six-vertex model. However, we lost the freedom having $n \neq N/2$, but, instead, obtained the new parameter Θ. For generic values of Θ the sum in eq. (3.28) will vanish. So Θ has to be determined together with the solution of the Bethe ansatz equations from the condition $\psi_\Theta \neq 0$.

For the special values of the parameter $\gamma = 2K/\nu$, leading to the RSOS-models (cf. eq. (2.8)), the problem simplifies. Everything is periodic with respect to the heights l with period ν. So the admissible values are (see ref. [4])

$$\Theta = m\ \pi/\nu, \qquad m = 0,1,\ldots,\nu-1. \tag{3.32}$$

The sum in eq. (3.28) can be restricted to one period $l = 0,1,\ldots,\nu-1$ and the condition (3.25) has to be replaced by $n = N/2 \bmod \nu$.

3.4 Critical RSOS-model and Θ-angle.

Now let us turn to the SOS-model. All eigenvalues of the eight-vertex transfer matrix are also eigenvalues of the SOS-transfer matrix. We specialize the model now in two steps:

(a) $\gamma = 2K/\nu, \quad \nu = 3,4,5,\ldots$

As mentioned above, for these special values of the parameter γ the SOS-model splits into an "unrestricted" and a "restricted" one, the RSOS-model where the values of $\Theta = m\gamma$ are limited to $m \neq 0$

$$\begin{cases} SOS & -\infty < l < \infty \quad \Theta = m\ \pi/\nu \quad m = 0,1,\ldots,\nu-1 \\ RSOS & 0 < l < \nu \quad \Theta = m\ \pi/\nu \quad m = 1,\ldots,\nu-1. \end{cases} \tag{3.33}$$

In the following we will concentrate on the RSOS-model.

(b) the critical case $d \to 0,\ k \to 0,\ K \to \pi/2$

In this limit the vertex weights a and b approach the six-vertex expressions. Therefore the Bethe ansatz equations of both models eqs. (3.20) and (3.31) coincide, with two exceptions:

(i) for the RSOS case there is the extra constant phase shift 2Θ and

(ii) the "Bethe-spin"

$$J_3 = N/2 - n \tag{3.34}$$

can assume only the values $J_3 = 0 \mod \nu$, whereas for the six-vertex model it can assume all values $J_3 = 0, 1, \ldots, N/2$.

So we may write formally for the eigenstate of the models taking eqs. (3.20), (3.31) and (3.33) into account

$$\{\text{RSOS}\} = \sum_{m=1}^{\nu-1} \{\text{6-vertex}(\Theta = m\gamma, J_3 = 0 \mod \nu)\}. \tag{3.35}$$

Here we have introduced a modified six-vertex model depending on Θ and J_3. This model is defined on a cylinder of height M and circumference N. We will be interested in the limit $M \gg N \to \infty$. The boundary conditions are periodic around and fixed at the bottom and the top of the cylinder. The numbers of up- minus down-arrows at the bottom and the top are $2J_3$. This means there is a total "charge flow" along the cylinder of $2J_3$ (note that for the six-vertex model we have "charge" conservation).

We introduce a "seam" from the bottom to the top of the cylinder and write

$$2\hat{J}_3 = \text{number of left- minus right-arrows on the seam .} \tag{3.36}$$

Then for each six-vertex configuration on the cylinder, $2\hat{J}_3$ is a well defined topological number equal to the total "charge flow" around the cylinder. The modified six-vertex model in eq. (5.24) is defined by the partition function

$$Z^{6-vert.}(\Theta) = \sum_{conf.} \exp(-E^{6-vert.} - i\ \Theta\, 2\, \hat{J}_3)\ \delta(J_3). \tag{3.37}$$

Note that this partition function has the same form as that for gauge theories with Θ-vacuum. The "charge flow lines" around the cylinder play the role of the instantons.

The modified monodromy and transfer matrices are

$$T(\Theta) = \begin{pmatrix} Ae^{-i\Theta} & B \\ C & De^{i\Theta} \end{pmatrix}, \quad \tau(\Theta) = Ae^{-i\Theta} + De^{i\Theta} \tag{3.38}$$

where A, B, C and D are defined by eq. (3.2).

We will later see that the decomposition of the RSOS-model by eq. (3.35) may be interpreted as a decomposition with respect to "soliton sectors" of the RSOS-model. $\Theta = \gamma$ correspond to the vacuum sector and $\Theta = m\ \gamma,\ m = 2, \ldots, \nu - 1$ to the $(m-1)$-soliton sectors. The fields $\phi_m(x)$ introduced in eq. (2.12) belong to these sectors.

4. Finite size corrections and conformal quantum field theories in two dimensions.

Belavin, Polyakov and Zamolodchikov [15] classified the "minimal models" of conformal quantum field theories in terms of the central charge of the Virasoro algebra

$$c = 1 - \frac{6}{\nu(\nu-1)}. \tag{4.1}$$

The conformal dimensions can assume values given by the Kac-formula [16]

$$\begin{aligned} &\Delta, \bar{\Delta} = h_{mn} = \frac{\big((\nu-1)m - \nu n\big)^2 - 1}{4\nu(\nu-1)}, \\ &0 < m < \nu, \quad 0 < n < \nu - 1, \qquad m, n = \text{integer}. \end{aligned} \tag{4.2}$$

Friedan, Qiu and Shenker [17] have shown that unitarity restricts the number ν to

$$\nu = \text{integer} \geq 3 \quad \text{or} \quad c \geq 1. \tag{4.3}$$

It is interesting to find integrable lattice models which yield in the continuum limit conformal quantum field theories corresponding to these unitary representations of the Virasoro algebra. Therefore it is useful to calculate the central charge and the conformal dimensions for integrable critical statistical models. This can be done by analyzing finite-size effects.

For a statistical system on a periodic $N \times M$ square lattice one introduces a transfer matrix τ. The partition function can be written as

$$Z = \operatorname{tr} \tau^M = \sum_i \lambda_i^M$$

where the sum extends over all eigenvalues of the transfer matrix. Cardy [18] has shown that conformal invariance implies for the maximal eigenvalue

$$\lambda_{max} \approx \exp(-Nf + \frac{1}{N}\frac{\pi}{6}c)\ , \qquad \text{for} \quad N \to \infty \tag{4.4}$$

where f is the free energy per site and c is the central charge of the Virasoro algebra. Hence, the central charge can be calculated from the finite-size behavior of the maximal eigenvalue corresponding to the ground state of a transfer matrix.

In addition the conformal dimensions of operators of the model can be determined if one looks for excited states with energy $E = -\Re \log \lambda$ and momentum

$P = -\Im \log \lambda$. As is argued in ref. [18] one has for $N \to \infty$

$$\begin{aligned} E_i - E_0 &\approx \frac{2\pi}{N} d_i \\ P_i - P_0 &\approx \frac{2\pi}{N} s_i \end{aligned} \tag{4.5}$$

where $d_i = \Delta + \bar{\Delta}$ ($s_i = \Delta - \bar{\Delta}$) is the "scaling dimension" ("spin") of the field operator A associated to the excitation ψ_i, i.e. $\langle 0|A|\psi_i\rangle \neq 0$.

A direct calculation of the central charge and the conformal dimensions for specific models is usually not simple. Therefore it seems worthwhile to determine them for integrable models from the Bethe ansatz solution using finite size correction methods [1, 2 and 19]. This has be done in ref. [1] for the six-vertex model and for some modified six-vertex models related to Potts- and RSOS-models. The results are as follows.

For the modified six-vertex model (3.38) with $\Theta = m\gamma$ we obtained after a long calculation [1] a formula for the eigenvalue of the transfer matrix, including $1/N$ corrections

$$\begin{aligned} \log \lambda_A(\vartheta, \gamma, \Theta = m\gamma) = -Nf + \tfrac{1}{2} i\pi (h_+ - h_-) \\ + \frac{\pi}{N} \left\{ \frac{1}{6} \sin \nu\vartheta - 2ie^{-i\nu\vartheta} \left(\frac{(\nu \hat{h}_+ - J_3 + m)^2}{4\nu(\nu-1)} + K \right) \right. \\ \left. + 2ie^{i\nu\vartheta} \left(\frac{(\nu \hat{h}_- - J_3 - m)^2}{4\nu(\nu-1)} + \bar{K} \right) \right\} \end{aligned} \tag{4.6}$$

where

$$\hat{h}_\pm = h_\pm + [(1 \mp m)/\nu] - 2 \sum_k n_\pm^{(k)} (k-1). \tag{4.7}$$

The integers K and $\bar{K}$ vanish for low energy excitations corresponding to primary fields. The integers $h_\pm$ ($n_\pm^{(k)}$) are the numbers of holes (k-strings) near to the last root u_n or the first root u_1, respectively.

We apply these Bethe ansatz results to the six-vertex, the SOS- and the RSOS-model. The various models are characterized as follows:

(a) six-vertex model: $\Theta = 0$, $J_3 = 0, 1, 2, \ldots$,
ground state: $h_\pm = n_\pm^{(k)} = J_3 = 0$;

(b) SOS-model: $\Theta = m\ \pi/\nu$, $m = 0, 1, 2, \ldots, \nu - 1$, $J_3 = 0$,
ground state: $h_\pm = n_\pm^{(k)} = m = 0$;

(c) RSOS-model: $\Theta = m\ \pi/\nu$, $m = 1, 2, \ldots, \nu - 1$, $J_3 = 0$,

ground state: $h_\pm = n_\pm^{(k)} = 0$, $m = 1$.

We expect conformal invariance in the continuum limit for the isotropic models. Therefore we take $\nu\vartheta = \pi/2$ where vertex weights fulfil $a = b$. Comparing eq. (4.6) for the ground state and Cardy's formula (4.4) we conclude that the central charges of the Virasoro algebra of the six-vertex, the SOS- and the RSOS-models are

$$c^{6-vertex} = 1 \tag{4.8a}$$

$$c^{SOS} = 1 \tag{4.8b}$$

$$c^{RSOS} = 1 - \frac{6}{\nu(\nu-1)} \ . \tag{4.8c}$$

We have obtained these results in ref. [1]. they have been expected from renormalization group arguments (see e.g. [20]) and for c) from Huse's observation (3.14).

Excitations possess "holes" and complex roots. We obtain for these states by means of the general formula (4.6) taking Cardy's formulas (4.5) into account

$$\Delta^{6-vertex} = \frac{(\nu\hat{h}_+ - J_3)^2}{4\nu(\nu-1)} + K_+ \ , \qquad \bar{\Delta}^{6-vertex} = \frac{(\nu\hat{h}_- - J_3)^2}{4\nu(\nu-1)} + K_- . \tag{4.9a}$$

$$\Delta^{SOS} = \frac{(\nu\hat{h}_+ + m)^2}{4\nu(\nu-1)} + K_+ \ , \qquad \bar{\Delta}^{SOS} = \frac{(\nu\hat{h}_- - m)^2}{4\nu(\nu-1)} + K_- \tag{4.9b}$$

$$\Delta^{RSOS} = \frac{(\nu\hat{h}_+ + m)^2 - 1}{4\nu(\nu-1)} + K_+ \ , \qquad \bar{\Delta}^{RSOS} = \frac{(\nu\hat{h}_- - m)^2 - 1}{4\nu(\nu-1)} + K_- \tag{4.9c}$$

One can show [1] that one gets minimal values for Δ and $\bar{\Delta}$ (i.e. the integers K_+ and K_- on the r.h.s. of eqs. (4.9c) vanish) if some conditions are fulfilled [1], e.g. if $h_+ + [(1-m)/\nu] = h_- = m - n$, $n_-^{(m-n)} = 1$, else $n_\pm^{(k)} = 0$, where $n \leq m$. Comparing eqs. (4.6) with the Kac-formula (4.2) we find for the RSOS-model the conformal dimensions of the primary fields

$$\Delta = \bar{\Delta} = h_{mn}. \tag{4.10}$$

These results confirm the suggestion of Huse [14] that the RSOS-models correspond to the "minimal models" [15] of conformal quantum field theory.

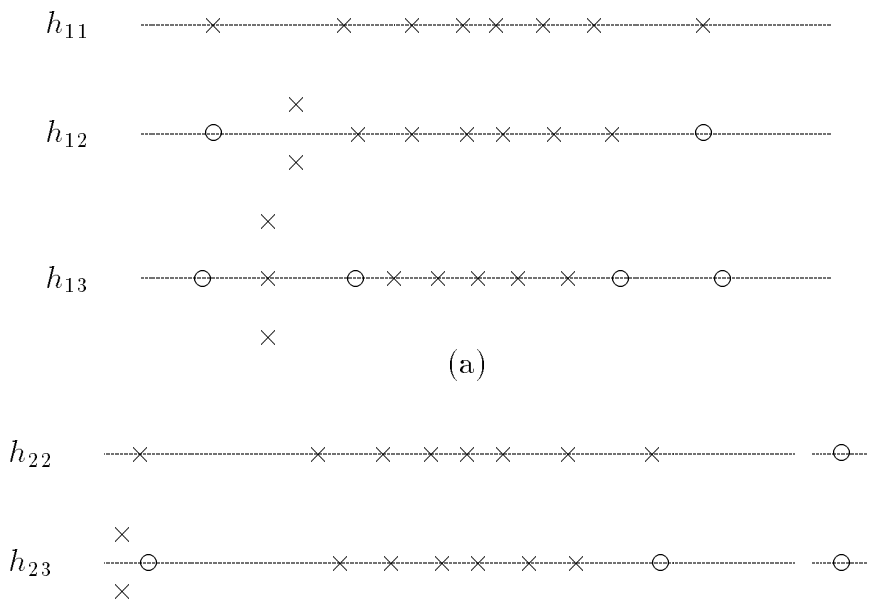


Fig. 7. *Typical sets of roots* $\times$ *and holes* $\circ$ *of Bethe ansatz equations in the complex u-plane for the RSOS-model with* $\gamma = \pi/6$ *on an N=16 chain. Some root configurations corresponding to primary fields* ϕ_{mn} *with dimensions* $\Delta = \bar{\Delta} = h_{mn}$ *are depicted: (a)* $m = 1$, *i.e.* $\Theta = \gamma$ *and (b)* $m = 2$, *i.e.* $\Theta = 2\gamma$.

Typical root configuration of Bethe ansatz equations corresponding to some "Kac dimensions" are depicted in fig. 7. The symmetry of the Kac-formula $(m, n) \leftrightarrow (\nu - m, \nu - 1 - n)$ correspond to $u \leftrightarrow -u$.

5. Continuum limit.

Before performing the limit, lattice constant to zero, we transform the critical RSOS-model into an appropriated form. We start with the model in terms of the modified six-vertex models given by eqs. (3.35), (3.37) and (3.38). The transformation is done in several steps:

(i) Vertex model $\longrightarrow$ Heisenberg model.

We expand the logarithm of the modified six-vertex transfer matrix in powers of the spectral parameter ϑ

$$-\log \tau^{6-vert.}(\Theta, \vartheta, \gamma) = c_0 + c_1\vartheta + c_2\vartheta^2 + \dots . \tag{5.1}$$

It follows for c_1 the Heisenberg Hamiltonian (see e.g. ref. [21]) in terms of the Pauli matrices σ^k

$$H_{xxz} = -\sum_{i=1}^{N} J_k \sigma_i^k \sigma_{i+1}^k\,, \quad J_1 = J_2, \quad J_3/J_1 = -\cos\gamma. \tag{5.2}$$

(ii) Heisenberg model $\longrightarrow$ lattice fermion model. [22]

We apply the Jordan-Wigner transformation and obtain the Hamiltonian in terms of lattice fermions

$$H = -4J_1 \sum_{i=1}^{N} \big[\tfrac{i}{2}(a_i^\dagger a_{i+1} - a_{i+1}^\dagger a_i) - \cos\gamma\; a_i^\dagger a_i\, a_{i+1}^\dagger a_{i+1}\big]. \tag{5.3}$$

(iii) Continuum limit: Lattice fermion model $\longrightarrow$ Thirring model.

Introducing the Dirac spinor one finds in the limit, lattice constant $a \to 0$, the Thirring model with the Lagrangian

$$\mathcal{L} = \bar\psi i\gamma\partial\psi - \tfrac{1}{2}g(\bar\psi\gamma_\mu\psi)^2. \tag{5.4}$$

The Thirring coupling g is related to the six-vertex or RSOS-coupling by [22]

$$\gamma = \frac{\pi}{\nu} = \pi\frac{\pi+2g}{2\pi+2g} \quad \text{or} \quad \frac{\nu-1}{\nu} = \frac{\beta^2}{8\pi} \tag{5.5}$$

where the sine-Gordon coupling has been introduced by Coleman's relation $4\pi/\beta^2 = 1 + g/\pi$. Note that the RSOS-models are distinguished among the SOS-models by exactly those couplings which correspond also to distinguished sine-Gordon couplings. It has been observed [23] that, at these couplings, in the Coulomb gas picture there are thresholds of multipole condensation. In the Bethe ansatz approach of the sine-Gordon model the ground state shows phase transitions at these couplings [24].

(iv) Bosonization.

The model can be formulated in terms of a free boson Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2. \tag{5.6}$$

defined on a cylinder of height $L' = Ma$ and circumference $L = Na$ with special boundary condition on the cylinder, which follow from the topological term in the partition function (3.37)

$$\phi(L) = \phi(0) - \frac{4\Theta}{\beta}. \tag{5.7}$$

The discontinuity at "infinity" depends on Θ. This motivates us to call the sectors with different Θ, soliton sectors of the RSOS-model.

Note that eqs. (5.6) and (5.7) only describes one soliton sector of the RSOS-model. The complete model has the soliton sectors corresponding to $\theta = m\pi/\nu$, $m = 1, \ldots, \nu - 1$ (cf. eq. (3.35)).

(v) Coulomb gas picture.

Dotsenko and Fatteev [25] used techniques of the two dimensional Coulomb gas in order to derive correlation functions for the "minimal models" of conformal quantum field theory. Starting from an approach of Feigin and Fuchs on representation theory of the Virasoro algebra, the authors of [25] used the Coulomb gas as a calculation framework together with a number of consistency conditions to derive their results. They had to introduce, ad hoc, a "charge at infinity" and "screening charges". This has the consequence that the field theoretic aspects of the models are not very obvious.

Here we start with the results of (iv) and "derive" the Coulomb gas of refs. [25] and [26] in a more field theoretic language. We introduce the "seam" of eqs. (3.25) and (3.26) which runs along a curve $\mathcal{C}$ from the bottom to the top of the cylinder in a Gaussian path integral. Then the partition function of the model given by eqs. (5.6) and (5.7) writes as follows

$$Z_\Theta = \int \mathcal{D}\phi \exp\left[- \int d^2x \tfrac{1}{2} (\partial_\mu \phi)^2 + i \frac{4\Theta}{\beta} \int_{\mathcal{C}} dx_\mu \epsilon_{\mu\nu} \partial_\nu \phi \right]. \tag{5.8}$$

We introduce the order or "electric charge" operator of charge α as a product of two "light cone" vertex operators

$$V_\alpha(x) = e^{i\alpha\phi(x)} = V_\alpha(z) V_\alpha(\bar{z}). \tag{5.9}$$

The disorder or "magnetic charge" operator of charge α is

$$\tilde{V}_\alpha(x) = e^{i\alpha\tilde{\phi}(x)} = V_\alpha(z) V_{-\alpha}(\bar{z}) \tag{5.10}$$

where $\phi(x) = \phi(z) + \phi(\bar{z})$ and $\tilde{\phi}(x) = \phi(z) - \phi(\bar{z})$.

Obviously, the partition function eq. (5.8), including the "seam term", is proportional to the two point function of magnetic charges sitting at the ends of the seam at the bottom "$-\infty$" and the top "∞" of the cylinder

$$Z_\Theta = Z_0 \, \langle \tilde{V}_{-\alpha_0}(-\infty) \tilde{V}_{\alpha_0}(\infty) \, \rangle_0, \quad \alpha_0 = \frac{1}{2} \frac{1}{\sqrt{\nu(\nu-1)}} \tag{5.11}$$

where Z_0 and $\langle \ldots \rangle_0$ correspond to the periodic case. The expectation value of a product of vertex operators with respect to the periodic gaussian model is

$$\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) \rangle_0 = \delta \left(\sum \alpha_i \right) \prod_{i<j} (z_i - z_j)^{-2\alpha_i \alpha_j}. \tag{5.12}$$

Next let us consider the primary fields $\phi_{mm}(x)$ with conformal dimensions $\Delta = \bar{\Delta} = h_{mm}$ (cf. eq. (4.2)). They correspond to Huse's operators of the RSOS-model given by eqs. (2.12). One finds

$$\phi_{mm}(x) = \sin \alpha_0 \tilde{\phi}(x) \ \sin m\alpha_0 \tilde{\phi}(x). \tag{5.13}$$

Our Coulomb gas picture of the continuum limit of the RSOS-model is still not complete. We are interested in the model in regime III of fig. 4. However, we are sitting at $\vartheta = 0$ and $d = 0$ where all regimes meet. This is because in eq. (5.1) we have expanded the transfer matrix for $d = 0$ around $\vartheta = 0$ to obtain the Heisenberg Hamiltonian. In order to generate a perturbation in direction of regime III we have to replace the Lagrangian (5.6) by

$$\begin{gathered} \mathcal{L} + g_+ \sin \alpha_+ \tilde{\phi} + g_- \cos \alpha_- \tilde{\phi} \\ \text{with} \quad \alpha_+ = \sqrt{\tfrac{\nu}{\nu-1}}\,, \quad \alpha_- = -\sqrt{\tfrac{\nu-1}{\nu}} \end{gathered} \tag{5.14}$$

where $g_\pm$ are infinitesimal couplings. The first additional term is contained in c_2 of eq. (5.1) and causes a shift in direction of regimes III and IV. The second term causes a shift away from criticality in direction of regime III. The limits $g_\pm \to 0$ have to be taken in a suitable way to obtain the conformal invariant "minimal models". The additional contributions $q_+ = \sin \alpha_+ \tilde{\phi}$ and $q_- = \cos \alpha_- \tilde{\phi}$ in the Lagrangian (5.14) are the screening charges of ref. [25].

Since $\alpha_+ + \alpha_- = 2\alpha_0$ the charges $q_\pm$ will screen the charges $\pm\alpha_0$ at the ends of the seam, such that the total charge at infinity is $\mp 2\alpha_0$. In addition, in a vacuum expectation value of products of vertex operators they will contribute such that the total charge is $\pm 2\alpha_0$. The final result is the following: The n-point function of magnetic operators (5.13) of charges $\alpha_1, \ldots, \alpha_n$ is

$$\begin{aligned} &\langle \tilde{V}_{\alpha_1}(x_1) \ldots \tilde{V}_{\alpha_n}(x_n) \rangle \\ &\qquad = \sum \int d^2 y_1 \ldots d^2 y_k \langle \tilde{V}_{\alpha_1}(x_1) \ldots \tilde{V}_{\alpha_n}(x_n) q_+(y_1) \ldots q_-(y_k) \rangle_0 \end{aligned} \tag{5.15}$$

where the sum is finite and extends over $\alpha_1 + \ldots + \alpha_n \pm \alpha_+ \pm \ldots \pm \alpha_- = \pm 2\alpha_0$ and $\langle \ldots \rangle_0$ is given by eq. (5.12). As noted in ref. [25] for non-critical theories

infinitely many terms like (5.15) sum up to produce a mass as for the sine-Gordon model.

At the end of this section some remarks on the other primary fields appearing in the Kac table (4.2). We start from the fields introduced above by eq. (2.12) and (5.13) corresponding to the diagonal $n = m$. In sect. 4 (cf. eq. (4.10) and fig. 7) we found that they belong to the Bethe ansatz states of lowest energy, the ground states, in the $(m-1)$-soliton sectors of the RSOS-model. In each sector the operator creating the lowest excitation from the ground state is

$$\sigma_+(x)\sigma_-(y) \to e^{i\frac{1}{2}\alpha_+\tilde{\phi}} \tag{5.16}$$

where the r.h.s. is the continuum limit of the l.h.s.. This suggests to replace for $n > m$ in eq. (5.13) $m\alpha_0$ by $m\alpha_0 + \frac{1}{2}(n-m)\alpha_+ = \frac{1}{2}(m\alpha_- + n\alpha_+)$. Then we obtain for the primary fields of the minimal models

$$\phi_{mn} = \sin\alpha_0\tilde{\phi}\ \ \sin\tfrac{1}{2}(m\alpha_- + n\alpha_+)\tilde{\phi}. \tag{5.17}$$

This formula agrees with the proposal in ref. [25] and reflects the symmetry of the Kac formula (4.2) with respect to $(m,n) \to (\nu - m, \nu - 1 - n)$, since $\nu\alpha_- + (\nu-1)\alpha_+) = 0$.

More complete derivations of the results of this section will be published elsewhere [27]. Another attempt to understand the Coulomb gas picture of the RSOS-models of Dotsenko and Fateev can be found in refs. [28].

6. Bethe ansatz, representation theory and quantum groups.

6.1 The quantum group $Sl(2)_q$.

Let us write the matrix S of statistical weights for the six-vertex model given by eqs. (2.2) and (2.3) in terms of

$$q = -e^{-i\gamma} \quad \text{and} \quad x = e^{i\vartheta}. \tag{6.1}$$

Note that for the RSOS q is a root unity since $\gamma = \pi/\nu$, $\nu = 3, 4, \ldots$.

The following will simplify if S only contains two terms with different x behavior. We get this form by mean of the "gauge transformation" in the horizontal space $S \to G^{-1}SG$ with $G = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)$. The transformed matrix is

$$S = \begin{pmatrix} xq^{s_3+\frac{1}{2}} - x^{-1}q^{-s_3-\frac{1}{2}} & x\,(q-q^{-1})s_- \\ x^{-1}(q-q^{-1})s_+ & xq^{-s_3+\frac{1}{2}} - x^{-1}q^{s_3-\frac{1}{2}} \end{pmatrix} \tag{6.2}$$

where $\vec{s} = \frac{1}{2}\vec{\sigma}$ ($s_\pm = s_1 \pm i s_2$) are the spin $\frac{1}{2}$-matrices in the vertical space.

It is not trivial but it can be shown that this transformed matrix fulfills the Yang-Baxter equations in the form (1.1), (1.2) or (3.1). Hence the monodromy matrix $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ defined by eq. (3.2) fulfills the Yang-Baxter relations (3.8) which imply commutation rules of the form (3.12) for the operators A, B, C and D.

We are interested in the limits $\vartheta \to \mp i\infty$ or $x \to \infty, 0$. The leading terms of the monodromy matrix are proportional to

$$T(x,q) \propto \begin{cases} \stackrel{x\to\infty}{\longrightarrow} & \begin{pmatrix} q^{J_3} & \alpha_+ J_- \\ 0 & q^{-J_3} \end{pmatrix} \\ \stackrel{x\to 0}{\longrightarrow} & \begin{pmatrix} q^{-J_3} & 0 \\ \alpha_- J_+ & q^{J_3} \end{pmatrix} \end{cases} \qquad \alpha_\pm = \pm q^{\mp\frac{1}{2}}(q - q^{-1}). \tag{6.3}$$

One easily finds

$$\begin{aligned} J_3 &= \{\sum_{i=1}^N s_3(i)\}_q = \sum_{i=1}^N s_3(i) \\ J_\pm &= \{\sum_{i=1}^N s_\pm(i)\}_q = \sum_{i=1}^N q^{s_3(1)+\ldots+s_3(i-1)} s_\pm(i) q^{-s_3(i+1)-\ldots-s_3(N)} \ . \end{aligned} \tag{6.4}$$

where we have introduced the "q-sum" of angular momenta $\{\sum \vec{J}\}_q$. We will not need these explicit expressions but only the commutation rules which can be derived from the Yang-Baxter algebra relations (3.8) in the form of eq. (3.12)

$$T_{12}T_{13}S_{23} = S_{23}T_{13}T_{12} \quad \Rightarrow \quad [J_3, J_\pm] = \pm J_\pm \ , \quad [J_+, J_-] = (2J_3)_q \tag{6.5}$$

where $(\ )_q$ is the "q-number" defined by

$$(X)_q = \frac{q^X - q^{-X}}{q - q^{-1}}. \tag{6.6}$$

These commutation rules describe the "quantum group" $Sl(2)_q$. It is a deformation of the Lie-group $Sl(2)$ which is obtained in the limit $q \to 1$.

Note that in general $\vec{J}$ is not selfadjoint[4]

$$J_+^\dagger \neq J_- \quad \text{for} \quad q \neq \text{ real.} \tag{6.7}$$

[4]In the literature on quantum groups often an indefinite metric is used such that $\vec{J}$ becomes selfadjoint.

6.2 Symmetry properties of the critical RSOS-model.

The $SU(2)$ symmetry of the XXX-Heisenberg model or the six-vertex model with $\gamma = 0$ or π is expressed by

$$[H_{XXX}, \vec{J}] = 0, \quad [\tau^{6-vertex}(\gamma = 0, \pi),\ \vec{J}] = 0 \tag{6.8}$$

where the components of $\vec{J}$ are the usual generators of $SU(2)$.

For the anisotropic Heisenberg model (H_{XXZ}) or the generic six-vertex model the symmetry group reduces to $U(1)$, i.e. eq. (6.8) remains true only for J_3. In the following it becomes obvious that the symmetry of the RSOS-model related to six-vertex models by (3.35) is much stronger. But it is not governed by a group as usual. One could ask: is it possible to generalize eq. (6.8) using the concept of "quantum groups" developed in the previous subsection

$$[\tau, \vec{J}_q\,] = 0 \quad ? \tag{6.9}$$

For the isotropic case it can be shown [29] that the invariance formula (6.8) follows directly from the Yang-Baxter algebra relations in the form of eq. (3.8). Generalizing this derivation for $q \neq \pm 1$ we start with the Yang-Baxter relations (c.f. fig. 8)

$$T(x)T(x')S(x'/x) = S(x'/x)T(x')T(x) \tag{6.10}$$

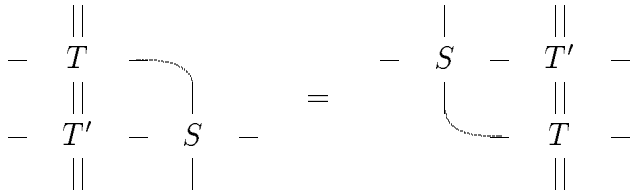


Fig. 8. *The Yang-Baxter algebra relation for the six- vertex monodromy matrix.*

and take the limit $x' \to \infty$. We obtain the set of commutation relations in matrix form

$$\begin{aligned} T(x) \begin{pmatrix} q^{J_3} & \alpha_+ J_- \\ 0 & q^{-J_3} \end{pmatrix} \begin{pmatrix} q^{s_3} & \alpha_+ s_- \\ 0 & q^{-s_3} \end{pmatrix} \\ = \begin{pmatrix} q^{s_3} & \alpha_+ s_- \\ 0 & q^{-s_3} \end{pmatrix} \begin{pmatrix} q^{J_3} & \alpha_+ J_- \\ 0 & q^{-J_3} \end{pmatrix} T(x) \end{aligned} \tag{6.11}$$

where $\vec{s}$ act in the "horizontal space" $V = \mathbf{C}^2$ and $\vec{J}$ in the "vertical space" the tensor product $V^{\otimes N}$ and $T(x)$ in the tensor product of both.

Using the "q-sum" of angular momenta defined eq. by (6.4)

$$\begin{aligned} \{J_3 + s_3\}_q &= J_3 + s_3 \\ \{J_\pm + s_\pm\}_q &= J_\pm q^{-s_3} + q^{J_3} s_\pm . \end{aligned} \tag{6.12}$$

we write eq. (6.11) as

$$\begin{aligned} T(x) \begin{pmatrix} q^{\{J_3+s_3\}_q} & \alpha_+ \{J_- + s_-\}_q \\ 0 & q^{-\{J_3+s_3\}_q} \end{pmatrix} \\ = \begin{pmatrix} q^{\{s_3+J_3\}_q} & \alpha_+ \{s_- + J_-\}_q \\ 0 & q^{-\{s_3+J_3\}_q} \end{pmatrix} T(x) \end{aligned} \tag{6.13}$$

The operators defined in eqs. (6.12) acting in the tensor product space of $\vec{J}$ and $\vec{s}$ are also called "coproducts"

$$\begin{aligned} \Delta(\vec{J}) = \{\vec{J} + \vec{s}\}_q = \quad & - \; \vec{J} \; - \; \vec{s} \; - \\ \Delta'(\vec{J}) = \{\vec{s} + \vec{J}\}_q = \quad & - \; \vec{s} \; - \; \vec{J} \; - \end{aligned} \tag{6.14}$$

and the underlying structure is called a Hopf-algebra. This structure emerges naturally from the Yang-Baxter algebra. (In eq. (6.14) the vertical lines correspond to the spaces in which $\vec{J}$ and $\vec{s}$ act, the horizontal lines to the matrix components of eqs. (6.3), (6.11) and (6,13) i.e. formally to the components of $\vec{J}$ and $\vec{s}$.) Note that for $q \neq \pm 1$ the "q-sum" is non-commutative i.e. $\Delta \neq \Delta'$

$$\{\vec{J} + \vec{s}\}_q \neq \{\vec{s} + \vec{J}\}_q \tag{6.15}$$

even if $\vec{J}$ belong to the same (spin-$\frac{1}{2}$) representation ($N = 1$) as $\vec{s}$.

We write the set of equations (6.13) and the corresponding ones for $x \to 0$ as (c.f. fig. 9)

$$T(x)\{\vec{J} + \vec{s}\}_q = \{\vec{s} + \vec{J}\}_q T(x). \tag{6.16}$$

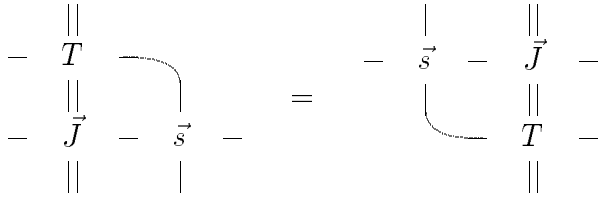


Fig. 9. *The $SU(2)_q$-invariance of the six-vertex monodromy matrix.*

This formula[5] expresses the "q-invariance" of the RSOS-model which becomes obvious if we apply it to the modified six-vertex transfer matrices for $\Theta = m\gamma$ defined by eq. (3.38)

$$\tau_m = tr_h(Tq^{2ms_3}). \tag{6.17}$$

The third component of $\vec{J}$ is an invariant

$$[\tau_m, J_3] = 0. \tag{6.18}$$

After some calculation one finds

$$J_\pm^k \tau_m = \tau_{m-k} J_\pm^k \mp \left\{ \begin{matrix} C \\ B \end{matrix} \right\} (k)_q J_\pm^{k-1} \left(q^{m-k\mp J_3} - q^{-m+k\pm J_3}\right). \tag{6.19}$$

This implies on a subspace of fixed value for J_3 the commutation rules

$$\begin{aligned} J_\pm^k \tau_m &= \tau_{m-k} J_\pm^k \quad \text{on states with} \quad J_3 = \pm(m-k) \\ \tau_m (J_\pm^k)^\dagger &= (J_\pm^k)^\dagger \tau_{m-k} \quad \text{on states with} \quad J_3 = \pm m. \end{aligned} \tag{6.20}$$

The second equation follows from the first one using crossing properties of S and T. The action of $J_\pm$ on the eigenstates of the modified six-vertex models are depicted in fig. 10. Especially, all states of the ordinary six-vertex model ($m = 0$) are mapped on states of the SOS-model ($J_3 = 0$) by the operators $(J_\pm^m)^\dagger$, which again shows that all eigenvalues of the six-vertex transfer matrix are also eigenvalues of the SOS-transfer matrix.

6.3 RSOS-Bethe ansatz states and highest weight states.

It is well known [29] that for the isotropic Heisenberg model and the six-vertex model for $q = \pm 1$ (i.e. the Lie-group case) all Bethe ansatz states

$$\psi = B_1 \dots B_n \phi \quad \text{with} \quad \tau\psi = \lambda\psi \tag{6.21}$$

are highest weight states

$$J_+ \psi = 0 \quad \text{for} \quad q = \pm 1. \tag{6.22}$$

This is not true for the quantum group case $q \neq \pm 1$, but eq. (6.22) can be generalized.

[5] In the literature on quantum groups eq. (6.16) is interpreted as the intertwining formula for the two comultiplications and T is called the R-matrix.

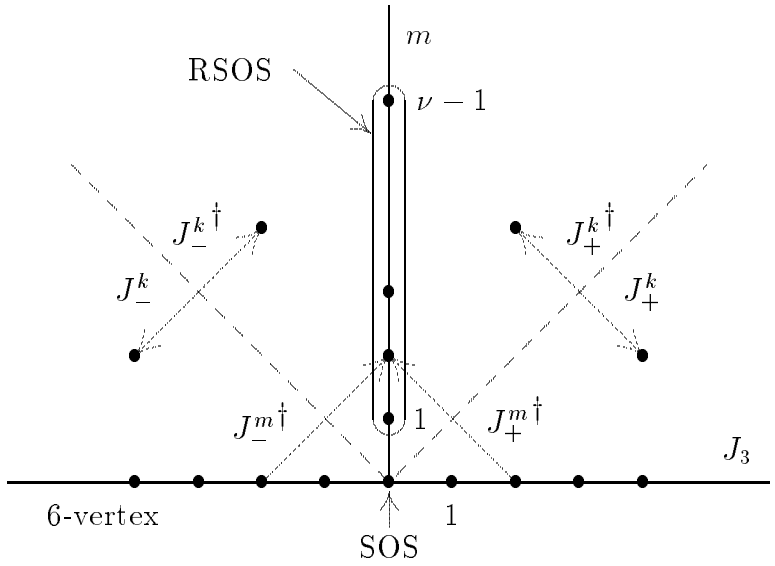


Fig. 10. *The action of $J_\pm$ on the eigenstates of the modified six-vertex transfer matrix (6.17). The various models correspond to the following points:*
(a) the ordinary six-vertex model: $m = 0$ and $J_3 = 0, \pm 1, \ldots$,
(b) the SOS- model: $m = 0, 1, \ldots, \nu - 1$ and $J_3 = 0$,
(c) the RSOS- model: $m = 1, \ldots, \nu - 1$ and $J_3 = 0$.

Consider the Bethe ansatz states of section 3 which are eigenstates of the modified six-vertex transfer matrix

$$\psi = B_1 \ldots B_n \phi, \quad \tau_m \psi = \lambda \psi \quad \text{with} \quad J_3 = N/2 - n. \tag{6.23}$$

Using the general invariance formula (6.16) one can show that

$$J_+^k \psi = 0 \quad \text{for} \quad k = m - J_3 \tag{6.24}$$

if the Bethe ansatz equations (3.31) hold, which are equivalent to the eigenvalue equation (6.23).

For the RSOS-model with $J_3 = 0$ we find

$$J_+^m \psi = 0, \quad m = 1, 2, \ldots, \nu - 1. \tag{6.25}$$

This means that the Bethe ansatz states in the $(m-1)$-soliton sector of the RSOS-model are "spin $m - 1$" states. Especially, the ground state is a "singlet state". On the other side, those Bethe ansatz states of the SOS-model which are not "restricted", i.e. those for $m = 0$, do not fulfil an equation like eq. (6.25). Hence they have "infinite spin", even the ground state.

6.4 The "second" quantum group.

We have seen that the RSOS-models with coupling

$$\gamma = \pi/\nu, \quad \nu = 3, 4, 5, \ldots \tag{6.26}$$

in the continuum limit should approach the "minimal models" of conformal quantum field theories introduced in ref. [15] with central charge

$$c = 1 - \frac{6}{\nu(\nu-1)}. \tag{6.27}$$

The conformal dimensions of these models $\Delta, \bar{\Delta} = h_{m,n}$ are given by the Kac-formula (4.2). The quantum group investigated above intertwines between the "soliton sectors" $m = 1, 2, \ldots, \nu - 1$.

There exists a duality in the minimal models interchanging

$$m \leftrightarrow n \quad \text{and} \quad \nu \leftrightarrow \nu - 1. \tag{6.28}$$

Hence there should exists a second quantum group

$$Sl(2)_{q'} \quad \text{with} \quad q' = -e^{-i\gamma'}, \quad \gamma' = \frac{\pi}{\nu - 1} \tag{6.29}$$

intertwining between the sectors $n = 1, 2, \ldots, \nu - 2$. One possibility to find in the RSOS-models such a quantum group is the following.

Consider the non-critical RSOS-model, i.e. the eight-vertex model. At the end we will take again the critical limit. The following is simpler in the continuum limit, however, the result should also hold on the lattice. In the continuum limit discussed in section 5 the eight-vertex model approaches the sine-Gordon model with coupling $\beta^2/8\pi = (\nu - 1)/\nu$ (c.f. eq. (5.5)). Because of charge conservation the soliton (antisoliton) S-matrix S^{SG} has the same form as the matrix of Boltzmann weights (2.3) of a six vertex-model (d=0). Using the fact that both matrices fulfil the Yang-Baxter equations following ref. [7] one finds that both matrices are proportional

$$S^{SG} \propto \begin{pmatrix} a' & & & \\ & b' & c' & \\ & c' & b' & \\ & & & a' \end{pmatrix} \tag{6.30}$$

where, however, we find for the amplitudes a', b' and c' instead of the parameter γ in eq. (2.2)

$$\gamma' = \frac{\pi}{\nu - 1}. \tag{6.31}$$

Therefore the underlying quantum group in the sense of section 6.1 is that of eq. (6.29), the "second" quantum group we are looking for. More detailed investigations on this point will be published elsewhere [27].

As mentioned in the introduction, in the limit where the spectral parameter ϑ approaches $\pm i\infty$ as discussed in section 6.1, Yang-Baxter algebras yield three objects:

i) quantum groups,
ii) braid representations and
iii) statistics matrices in two dimensional field theories.

Hence it is not surprising that in the limit $\vartheta \to \pm i\infty$ the Boltzmann weight matrices of the RSOS-model defined by eq. (2.6) are proportional to the light cone braid matrices in eq. (1.10) of the minimal models of conformal quantum field theories.

In addition there is a complete analogy of these braid matrices R and the factorized S-matrices mentioned in the introduction. For example the fusion procedure for the braid matrices is identical to the S-matrix bound state problem developed in refs. [30]. Using these old results one finds relations of the fusion matrix with the braiding matrix R.

There exists an extensive literature [8-11] and [31] on subjects concerning the content of this section. There are contacts with pure mathematical fields as knot theory, von Neumann algebras, e.t.c..

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