State sum invariants of compact 3-manifolds with boundary and 6j-symbols *

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In a recent article Turaev and Viro [TV] have constructed nontrivial "quantum" invariants of closed compact 3-manifolds M^3 in the form of state sums (called partition functions in statistical physics and vacuum functionals in quantum field theory) associated with the quantized universal enveloping algebra $U_q(sl(2,\mathbf{C}))$ when q is a complex root of unity of a certain degree $2r > 4$. The state sum is first defined for a given triangulation X of M and then shown to be independent of the triangulation X of M thus giving rise to a well defined invariant of M, which thus depends on q. This result may be viewed as a rigorous mathematical construction of what is called a topological quantum field theory. In fact, in the language of physicists, a triangulation corresponds to the introduction of a high-energy cut-off. Now topological quantum field theories have trivial dynamics, in other words they only deal with the vacuum sector, are scale invariant and more generally independent of any metrics. Invariance under subdivision is just the statement that the associated renormalization group transformation is trivial. This result suggests that the familiar techniques from algebraic topology should become useful to construct and discuss other topological quantum field theories. Moreover, similar combinatorial techniques could become helpful in discussing particle structures of low dimensional quantum field theories, whose vacuum sector is given in terms of a topological quantum field theory.

Another interesting novel feature of the Turaev Viro approach is that the states in the state sum are labelled by elements of the dual of the (quantum) group, i.e. by irreducible representations. This contrast with the familar ansatz in lattice gauge theories, where the states are labelled by elements of the group.

In a recent article [KMS] the authors have extended the Turaev Viro construction to the case where M has nonempty boundary. This leads in particular to a simplified proof of the main result of Turaev and Viro, which is the invariance of the state sum under triangulations. Furthermore by this extension one has introduced observables into the Turaev-Viro approach in the form of certain closed (piecewise) smooth 2-submanifolds consist of several copies of 2-tori, these 2-tori may be viewed as blown up links, i.e. the boundaries of tubular neighborhoods of such links. Invariance is then just the statement that these 2-tori around the links may be chosen arbitrary "small".

To construct the state sum for manifolds with boundary, we first briefly recall the abstract set-up of Turaev and Viro. Let K be a commutative ring with unit. By K^* we denote the set of invertible elements in K. Let I be a finite set, $\omega \epsilon K^*$ a distinguished element and $i \mapsto \omega_i$ a map from I into K^*. We set

$$\tilde{\omega}^2 = \sum_{i \epsilon I} \omega_i^4. \tag{1}$$

We assume there is given a nonempty set of unordered triples $(i, j, k) \epsilon I$ called admissible. We set $\delta(i, j, k) = 1$ if (i, j, k) is admissible and zero otherwise. An ordered 6-tuple (i, j, k, ℓ, m, n) is called admissible if the 4 unordered 3-triples $(i, j, k), (k, \ell, m), (i, m, n)$ and (j, ℓ, n) are admissible. To each such admissible 6-tuple we assume there is associated an element of K, the abstract 6j-symbol, denoted by

$$\begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix}$$

satisfying the following symmetry relations

$$\begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & \ell & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ \ell & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ \ell & j & k \end{vmatrix}. \tag{2}$$

In addition we impose three conditions. First, the following "orthogonality" relations are supposed to hold

$$\sum_{k \epsilon I} \omega_k^2 \omega_n^2 \begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} \begin{vmatrix} i & j & k \\ \ell & m & n' \end{vmatrix} = \delta_{n,n'}. \tag{3}$$

The summation is such that all symbols are supposed to be defined, i.e. both 6-tuples (i, j, k, ℓ, m, n) and (i, j, k, ℓ, m, n') are admissible. Secondly, we assume that

$$\omega^2 = \omega_k^{-2} \sum_{ij} \omega_i^2 \omega_j^2 \delta(i, j, k) \tag{4}$$

holds for all $k \epsilon I$. Assume now I to be irreducible, i.e. to any $(i, j) \epsilon I$ there exists a sequence $(\ell_1, \ldots \ell_n) \epsilon I$ with $\ell_1 = i, \ell_n = j$ and $\delta(\ell_\nu, \ell_{\nu+1}, \ell_{\nu+2}) = 1$ for all $| \leq \nu \leq n - 2$. Then one may prove [KMS] that

$$\sum_i \omega_i^2 \delta(i, j, k) = \frac{\omega^2}{\tilde{\omega}^2} \omega_j^2 \omega_k^2. \tag{5}$$

For the special case of quantum 6j-symbols associated to $U_q(sl(2, C))$ with $q = \exp(i\pi s/r)$ (r and $s \epsilon \mathbf{Z}$ relatively prime) one has

$$\omega_i = (\sqrt{-1})^{2i} \left(\frac{q^{2i+1} - q^{-2i-1}}{q - q^{-1}} \right)^{\frac{1}{2}} \qquad (i = 0, \frac{1}{2}, 1, \ldots \frac{r}{2} - 1) \tag{6}$$

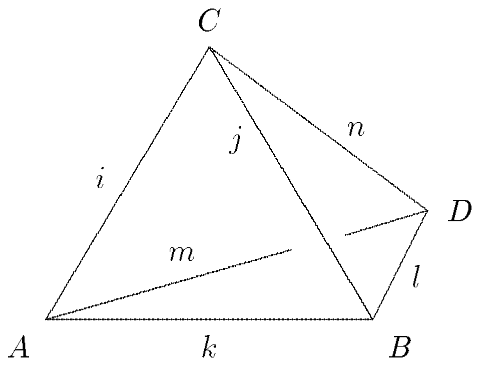
and

$$\omega^2 = \tilde{\omega}^2 = \frac{-2r}{(q-q^{-1})^2}. \tag{7}$$

Finally we assume the following polynomial relation to hold in K (the abstract Biedenharn-Elliot identity for 6j-symbols).

$$\sum_n \omega_n^2 \begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \begin{vmatrix} j & \ell & n \\ D & C & B \end{vmatrix} = \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & \ell & m \\ D & A & B \end{vmatrix} \tag{8_2}$$

again with restrictions similar to those in (3). This relation has a graphical representation in the form



Using (2), (3) and (4), relation (8_2) gives four additional relations

$$\begin{aligned} \begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} = \omega^{-2} \sum_{A,B,C,D} \omega_A^2 \omega_B^2 \omega_C^2 \omega_D^2 & \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \\ & \cdot \begin{vmatrix} k & \ell & m \\ D & A & B \end{vmatrix} \begin{vmatrix} j & n & \ell \\ D & B & C \end{vmatrix} \begin{vmatrix} i & m & n \\ D & C & A \end{vmatrix}, \end{aligned} \tag{8_0}$$

$$\begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} = \sum_B \omega_B^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & \ell & m \\ D & A & B \end{vmatrix} \begin{vmatrix} j & n & \ell \\ D & B & C \end{vmatrix}, \tag{8_1}$$

$$\begin{aligned} \sum_{\substack{\ell,m,n \\ D}} \omega_\ell^2 \omega_m^2 \omega_n^2 \omega_D^2 & \begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \\ & \cdot \begin{vmatrix} j & \ell & n \\ D & C & B \end{vmatrix} \begin{vmatrix} k & m & \ell \\ D & B & A \end{vmatrix} = \omega^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix}, \end{aligned} \tag{8_3}$$

$$\begin{aligned} \sum_{\substack{i,j,k,\ell,m,n \\ A,B,C,D}} \omega_i^2 \omega_j^2 \omega_k^2 \omega_\ell^2 \omega_m^2 \omega_n^2 \omega_A^2 \omega_B^2 \omega_C^2 \omega_D^2 & \begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \\ & \cdot \begin{vmatrix} j & \ell & n \\ D & C & B \end{vmatrix} \begin{vmatrix} k & m & \ell \\ D & B & A \end{vmatrix} \begin{vmatrix} i & k & j \\ B & C & A \end{vmatrix} = (\omega^2)^3 \tilde{\omega}^2. \end{aligned} \tag{8_4}$$

To construct the state sum for a compact 3-manifold M with boundary we now proceed as follows. Let X be a triangulation of M which induces a triangulation ∂X of the boundary ∂M of M.

Definition: An edge colouring of X is a map $\underline{j} : \sigma^1 \to j(\sigma^1)$ from the set of nonoriented 1-simplexes σ^1 of X into I. A vertex colouring of ∂X is a map $\underline{J} : \sigma^0 \to J(\sigma^0)$ from the set of vertices of ∂X into I.

To a given edge colouring of X and to every nonoriented 3-simplex $\sigma^3 \epsilon X$ we associate the 6j-symbol

$$(6\underline{j})(\sigma^3) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ j(\sigma_4^1) & j(\sigma_5^1) & j(\sigma_6^1) \end{vmatrix} \tag{9}$$

provided the 6 tuple $(j(\sigma_1^1), ... j(\sigma_6^1))$ is admissible. Here σ_i^1 and $\sigma_{i+2}^1 (i = 1, 2, 3)$ are opposite edges in $\partial\sigma^3$ and $[\sigma_1^1, \sigma_2^1, \sigma_3^1]$ is a two simplex on $\partial\sigma^3$.

To a given edge colouring $\underline{j}$ of X, vertex colouring $\underline{J}$ of ∂X and nonoriented 2-simplex $\sigma^2 \epsilon \partial X$ we associate the 6j-symbol

$$(6\underline{j}, \underline{J})(\sigma^2) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ j(\sigma_1^0) & j(\sigma_2^0) & j(\sigma_3^0) \end{vmatrix} \tag{10}$$

provided the 6-tuple $(j(\sigma_1^1), j(\sigma_2^1), j(\sigma_3^1), j(\sigma_1^0), j(\sigma_2^0), j(\sigma_3^0))$ is admissible. Here σ_i^0 are the vertices opposite to the edges $\sigma_i^1 (i = 1, 2, 3)$. Given $\underline{j}$ and $\underline{J}$ we define the "Boltzmann-Gibbs" weight factor

$$\begin{aligned} W(X)(\underline{j}, \underline{J}) = \prod_{\sigma^0 \epsilon X} \omega^{-2} \prod_{\sigma^0 \epsilon \partial X} \omega^2_{J(\sigma^0)} \prod_{\sigma^1 \epsilon X} \omega^2_{j(\sigma^1)} \\ \prod_{\sigma^3 \epsilon X} (6\underline{J})(\sigma^3) \prod_{\sigma^2 \epsilon \partial X} (6\underline{j}, \underline{J})(\sigma^2). \end{aligned} \tag{11}$$

For a given triangulation X of M we define the state sum to be given by

$$Z(X) = \sum_{\underline{j}, \underline{J}} W(X)(\underline{j}, \underline{J}) \tag{12}$$

with a summation over all $\underline{j}$ and $\underline{J}$ such that the weight factor (11) is defined. For $\partial M = \phi$ this state sum definition agrees with that of Turaev and Viro.

Theorem: $Z(X)$ is independent of the particular choice of the triangulation and hence defines an invariant $Z(M)$ of the manifold M.

The proof is based on the fact that the relations $(8_i)(0 \leq i \leq 4)$ have geometric interpretation in form of a local Stokes theorem.

We note that a local Stokes theorem is also responsible for the fact, that in the context of Regge calculus [R] the Lipschitz-Killing curvatures for piecewise linear manifolds are invariant under subdivisions [CMS]. In the present context, it is sufficient to consider invariance under local subdivisions like e.g. the Alexander moves. If this subdivision is localized in $Y \subset X$, then there are two cases

(i) If Y is near the boundary ∂X, we shift ∂X using invariance under isotopies of ∂X such that the state sum has no contribution from Y.

As an example consider the case where $\sigma^3 \epsilon X$ is such that $\partial\sigma^3$ has two 2-simplexes in common with ∂X. We now remove σ^3 from X. Then ∂X changes in the form that $\partial X \cap \partial\sigma^3$ is replaced by the two other 2-simplexes in $\partial\sigma^3$. Invariance of $Z(X)$ under this elementary isotopy then follows by using (8_2). If in general $\sigma^3 \epsilon X$ has $i (1 \leq i \leq 3)$ 2-simplexes in common with ∂X then one uses the relation (8_i).

(ii) If Y is not near the boundary ∂X, we first remove a $\sigma^3 \epsilon Y$ using (8_0), thus creating a hole in X with a new boundary component $\partial\sigma^3$. We then enlarge this new boundary using (i) to remove all of Y from X.

Invariance under a subdivision in Y is now obvious.

One may now use surgery techniques to show that e.g.

$$\begin{aligned} Z(D^3) &= \frac{\tilde{\omega}^2}{\omega^2}, \\ Z(S^3) &= \frac{\tilde{\omega}^2}{\omega^4}, \\ Z(S^2 \times S^1) &= 1. \end{aligned} \tag{13}$$

Note that the formula for $Z(D^3)$ is just a consequence of (8_4). In the context of $U_q(sl(2,C)$ one has

$$\omega^{-2} = \tilde{\omega}^{-2} = \frac{2}{r} \sin^2 \frac{\pi s}{r}. \tag{14}$$

This compares with the Chern-Simons topological quantum field theory at level k [Wi1]. Note that according to Reshetikhin and Turaev [RT] there is a rigorous definition $\tau_k(M)$ of the state sum for the Chern-Simons theory in case M is orientable such that

$$Z(M) = \tau_k(M)\tau_k(-M) = |\tau_k(M)|^2 \tag{15}$$

holds [T] with $k + 2 = r$.

Note added in proof:

Meanwhile the state sum has been calculated [KS] for the case $M = \Sigma_g \times S^1$ where Σ_g is a Riemann surface of genus g. One has

$$Z(\Sigma_g \times S^1) = \dim \mathcal{H}^{TV}_{\Sigma_g} \tag{16}$$

where $\mathcal{H}^{TV}_{\Sigma_g}$ is the Hilbert space for this topological quantum field theory. The calculation gives

$$\dim \mathcal{H}^{TV}_{\Sigma_g} = (\dim \mathcal{H}^{CS}_{\Sigma_g})^2 \tag{17}$$

where $\mathcal{H}^{CS}_{\Sigma_g}$ is the toplogical Hilbert space for the Chern Simons theory at level $k = r-2$. $\dim \mathcal{H}^{CS}_{\Sigma_g}$ is given by the Verlinde formula [V]

$$\dim \mathcal{H}^{CS}_{\Sigma_g} = \left(\frac{k+2}{2}\right)^{g-1} \sum_{1 \leq m \leq k+1} \sin^{2-2g}\left(\frac{m\pi}{k+2}\right) \tag{18}$$

(see e.g. [FG], [Wi2]). Note that relations (17) and (18) are compatible with (15).

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