The Supersymmetric t-J Model with Quantum Group Invariance

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Abstract

An integrable quantum group deformation of the supersymmetric t-J model is introduced. Open boundary conditions lead to an $spl_q(2,1)$ invariant hamiltonian. A general procedure to obtain such invariant models is proposed. To solve the model a generalized nested algebraic Bethe ansatz is constructed and the Bethe ansatz equations are obtained. The quantum supergroup structure of the model is investigated.

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1. INTRODUCTION

Investigations of one dimensional integrable quantum chains have a long history, starting with Bethe's [1] famous analysis of the XXX Heisenberg model. Solutions of the Schrödinger equation are obtained by the Bethe ansatz. In addition the $SU(2)$ symmetry of the model turns out to be crucial to find a complete set of eigenstates of the hamiltonian. Later, the deformed anisotropic XXZ Heisenberg model (only $U(1)$ invariant) was also analyzed by means of the Bethe ansatz [2]. However, the problem of finding a complete set of eigenstates has not been solved. The integrability of these models is based on an underlying Yang-Baxter algebra. For the anisotropic case this algebra is of the "trigonometric" type. Investigating these Yang-Baxter algebras Jimbo [3] and Drinfeld [4] were lead to the new mathematical concept of quantum groups. It is natural to use these new concepts for the analysis of anisotropic quantum chains. The Heisenberg hamiltonian H_{XXZ} with periodic boundary conditions transforms nontrivially with respect to the underlying $Sl_q(2)$ quantum group [5,6]. However, one easily obtains an invariant hamiltonian by $H = \sum_{j=1}^{N-1} P_j$, where P_j is a projector on a representation π of a symmetry group or quantum group associated to the lattice site j and $j+1$. Here π is to be considered as $\pi \subset \pi_0 \otimes \pi_0$ and the representation π_0 describes the states associated to each lattice site ($j = 1, \ldots, N$) (see also ref. [7]). For the quantum group case these hamiltonians have typically formally nonhermitean terms. However, the energy eigenvalues turn out to be real [5,8]. The models are obviously integrable if their hamiltonians can be derived from a transfer matrix τ of a two dimensional vertex model which is constructed from a Yang-Baxter algebra. This origin implies for all "spectral parameters" λ, λ' that $[\tau(\lambda), \tau(\lambda')] = 0$ which produces a set of conservation laws.

Cherednik [9] and Sklyanin [10] introduced techniques to handle transfer matrices derived from Yang-Baxter algebras with more general boundary conditions. Using these techniques Sklyanin [10] and Destri et al [11] solved the algebraic Bethe ansatz for an XXZ Heisenberg model with open boundary conditions such that the hamiltonian is quantum group invariant. Its spectrum was earlier determined numerically by Alcaraz et al [8] using the coordinate Bethe ansatz. In the construction of refs. [10,11], in addition to the matrix S defining the vertex weights, two new matrices $K^{\pm}$ that take into account the boundary conditions are introduced. The explicit form of these matrices is determined by the requirement of integrability. It is pointed out that the choice of boundary conditions is essential to get "quantum group" invariance. Other integrable models, as the Toda-chain [10,12], non-linear Schrödinger equation [10], the XY model [13], the XYZ model [10] and A_{n-1} vertex models [14] were also considered in connection with open boundary conditions. However, the Bethe ansatz equations were derived only for the XXZ chain [10,11]. Therefore, the problem of finding the spectrum of open chains has not been solved yet in its full extent and is presently a subject of increasing activity [7].

In this work we present a quantum supergroup version of the supersymmetric t-J hamiltonian

$$
\begin{aligned}
\mathcal{H} = &- \mathcal{P}\{\sum_{j=1}^{L-1} \sum_{\sigma} (c_{j,\sigma}^{\dagger} c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger} c_{j,\sigma})\}\mathcal{P} \\
&- 2\sum_{j=1}^{L-1} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \cos\gamma \ \ (S_j^z S_{j+1}^z - \frac{n_j n_{j+1}}{4}) \right) - \cos\gamma \sum_{j=1}^{L} n_j \ , \\
&+ i\sin\gamma (n_1 - n_L) - i\sin\gamma \sum_{j=1}^{L-1} (n_j S_{j+1}^z - S_j^z n_{j+1})
\end{aligned}
\tag{1.1}
$$

which is $spl_q(2,1)$ invariant (see also ref. [7]). In the isotropic limit $\gamma \to 0$ it reduces to the supersymmetric t-J model proposed by Anderson [15]. It has been solved exactly by means of the Bethe ansatz metod [16,17]. We show that the hamiltonian (1.1) can be derived from the transfer matrix of a "graded" 15-vertex model with anisotropy adapted to open boundary conditions. Through a generalization of the Cherednik and Sklyanin algebraic approach to the case of a graded three states vertex model we solve the eigenvalue problem of the transfer matrix in a recurrent way (nested Bethe ansatz method with two levels). Moreover, we propose a new approach where the matrix K^+ defining the boundaries is given by the Markov trace associated with the supergroup $spl_q(2,1)$. This is natural since in the theory of links Markov traces are employed to build up invariant objects (see e.g. ref. [18]). In fact, we can verify that this choice leads to an integrable and quantum supergroup invariant model. We stress that the use of Markov traces allows for a straightforward generalization for more complex cases.

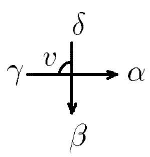
Obviously, the deformed t-J hamiltonian (1.1) is not hermitean, however it has real eigenvalues, as in the case of the quantum XXZ chain [5,8]. We remark that the choice of *periodic* boundary conditions for the "graded" 15-vertex model with anisotropy results in a hamiltonian with complex eigenvalues, as indicated by a direct diagonalization even for 3 sites [19]. This is in contrast to the 6-vertex model, where the eigenvalues are also real for periodic boundary conditions. From the Yang-Baxter algebra of the model we construct in the limit $v \to \pm i\infty$ a deformation of the "graded" Lie algebra $spl(2,1)$. More precisely, the generators of the quantum supergroup $spl_q(2,1)$ emerge from these special limits of the spectral parameter in the monodromy matrix. We show explicitly that they commute with the transfer matrix.

This paper is organized as follows. In sect. 2 we introduce the $spl_q(2,1)$ vertex model and define some basic quantities: the vertex weights S and the matrices $K^{\pm}$ defining the boundary conditions, monodromy and transfer matrices. The relation between the transfer matrix of the vertex model and the hamiltonian of the supersymmetric t-J model is also discussed for the deformed case. The spectrum of the model is presented in sect. 3 by means of the algebraic nested Bethe ansatz method adapted to open boundary conditions. In sect. 4 we show that the integrable vertex model under consideration provides an explicit realization of the quantum supergroup $spl_q(2,1)$. A summary of our main results is presented in sect. 5.

2. THE VERTEX MODEL, YANG-BAXTER ALGEBRA AND OPEN BOUNDARY CONDITIONS

We consider an integrable generalization of the graded 15-vertex model [17] with an anisotropy parameter γ and open boundary conditions. It is a three states vertex model characterized by $\alpha = 1, 2, 3$, which can be bosonic (B) or fermionic (F). In what follows we will adopt the convention 1 =B, 2 =B, 3 =F. We begin by introducing the matrix of vertex weights S, which in terms of a generic spectral parameter v reads [20,21]

$$S^{\gamma\delta}_{\alpha\beta}(v) \quad = $$



$$= \left(\begin{array}{ccc|ccc|ccc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & c_- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & c_- & 0 & 0 \\
\hline
0 & c_+ & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & c_- & 0 \\
\hline
0 & 0 & c_+ & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_+ & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w
\end{array}\right) \qquad (2.1)$$

where the indices α , β , γ and δ run from 1 to 3 and

$$a=\sin(v+\gamma),\quad b=\sin v,\quad c_{+}=e^{iv}\sin\gamma,\quad c_{-}=e^{-iv}\sin\gamma,\quad w=\sin(-v+\gamma). \tag{2.2}$$

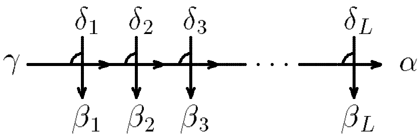
It provides a trigonometric solution of the Yang-Baxter equation

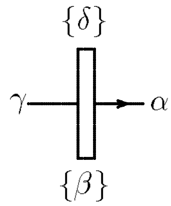
$$S_{\alpha'\beta'}^{\alpha''\beta''}(v-v')S_{\alpha\ \gamma'}^{\alpha'\gamma''}(v)S_{\beta\ \gamma}^{\beta'\gamma'}(v')=S_{\beta'\gamma'}^{\beta''\gamma''}(v')S_{\alpha'\ \gamma}^{\alpha''\gamma'}(v)S_{\alpha\ \beta}^{\alpha'\beta'}(v-v'). \tag{2.3}$$

In the limit $\gamma\to 0$ it reduces to the rational S matrix [17]. This S matrix acts in the tensor product of two 3-dimensional auxiliary spaces $\mathbf{C}^3\times\mathbf{C}^3$. The notation in the figure of eq. (2.1) indicates that the spectral parameter is $v=v_1-v_2$ where v_1 is associated to the horizontal line and v_2 to the vertical one.

The standard row-to-row monodromy matrix for a $L\times L$ square lattice is defined as the matrix product over the S's

$$T_{\alpha\{\beta\}}^{\gamma\{\delta\}}(v)=\ S_{\alpha_2\beta_1}^{\gamma\ \delta_1}(v)S_{\alpha_3\beta_2}^{\alpha_2\delta_2}(v)S_{\alpha_4\beta_3}^{\alpha_3\delta_3}(v)\dots S_{\alpha\ \beta_L}^{\alpha_L\delta_L}(v)$$

$=$ 

$=$ 

(2.4)

This matrix T acts in the tensor product of an auxiliary space and a "quantum space" $\mathbf{C}^3\times\mathbf{C}^{3L}$ and can be arranged as a 3×3 matrix of matrices acting in the "quantum space"

$$T_\alpha^\gamma(v)=\begin{pmatrix} A & B_2 & B_3\\ C_2 & D_2^2 & D_3^2\\ C_3 & D_2^3 & D_3^3\end{pmatrix}. \tag{2.5}$$

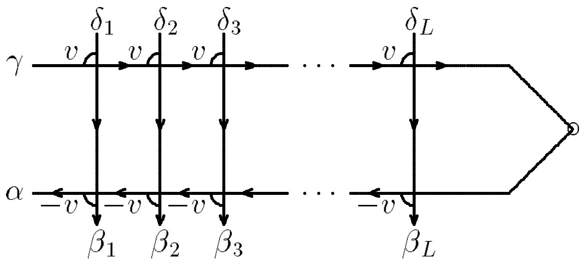
Furthermore, it also fulfills the Yang-Baxter relation

$$S_{\alpha'\beta'}^{\alpha''\beta''}(v-v')T_{\alpha\ \{\gamma'\}}^{\alpha'\{\gamma''\}}(v)T_{\beta\ \{\gamma\}}^{\beta'\{\gamma'\}}(v')=T_{\beta'\ \{\gamma'\}}^{\beta''\{\gamma''\}}(v')T_{\alpha'\ \{\gamma\}}^{\alpha''\{\gamma'\}}(v)S_{\alpha\ \beta}^{\alpha'\beta'}(v-v'), \tag{2.6}$$

as follows from eq.(2.3).

In order to construct the model with open boundary conditions let us introduce the "doubled" monodromy matrix $\mathcal{U}$ (see refs. [10,11] for the $Sl_q(2)$ case)

$$\mathcal{U}_{\alpha\{\beta\}}^{\gamma\{\delta\}}(v)\ =\ T_{\alpha'\{\beta'\}}^{\gamma\{\delta\}}(v)T^{-1}{}_{\alpha\{\beta\}}^{\alpha'\{\beta'\}}(-v)$$

$=$ 

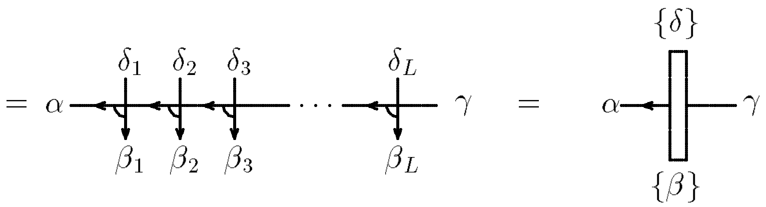
, (2.7)

where the symbol $\circ$ indicates that at this point the spectral parameter v associated to the line changes to $-v$ (in refs. [10,11] a matrix K^- is employed to indicate this change). T^{-1} is the inverse of T in the auxiliary and "quantum" spaces, i.e.,

$$T^{\gamma\ \{\delta\}}_{\alpha'\{\beta'\}}(v)T^{-1\,\alpha'\{\beta'\}}_{\ \ \alpha\ \{\beta\}}(v) = T^{-1\,\gamma\ \{\delta\}}_{\ \ \alpha'\{\beta'\}}(v)T^{\alpha'\{\beta'\}}_{\alpha\ \{\beta\}}(v) = \delta^{\gamma}_{\alpha}\delta^{\{\delta\}}_{\{\beta\}}, \tag{2.8}$$

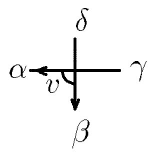
and can be constructed as the matrix product over the S^{-1} in the following way

$$T^{-1\,\gamma\{\delta\}}_{\ \ \alpha\{\beta\}}(v) = S^{-1\,\alpha_2\delta_1}_{\ \ \alpha\ \beta_1}(v)S^{-1\,\alpha_3\delta_2}_{\ \ \alpha_2\beta_2}(v)S^{-1\,\alpha_4\delta_3}_{\ \ \alpha_3\beta_3}(v)\dots S^{-1\,\gamma\ \delta_L}_{\ \ \alpha_L\beta_L}(v).$$



(2.9)

Here S^{-1} is the inverse of S

$S^{-1\,\gamma\delta}_{\ \ \alpha\beta}(v) =$


, (2.10)

$$S^{\gamma\ \delta}_{\alpha'\beta'}(v)S^{-1\,\alpha'\beta'}_{\ \ \alpha\ \beta}(v) = S^{-1\,\gamma\ \delta}_{\ \ \alpha'\beta'}(v)S^{\alpha'\beta'}_{\alpha\ \beta}(v) = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}$$

and it can be obtained from the matrix S as

$$S^{-1\,\gamma\ \delta}_{\ \ \alpha'\beta'}(v,\gamma) = -\frac{S^{\delta\ \gamma}_{\beta'\alpha'}(-v,\gamma)}{\sin(v+\gamma)\sin(v-\gamma)}. \tag{2.11}$$

The notation for the elements of S^{-1} and T^{-1} is the same as in (2.1) and (2.5) with a signal "$\tilde{\ }$" to distinguish them from the elements of S and T. Analogously, the "doubled" monodromy matrix $\mathcal{U}$ can also be written as a 3×3 matrix

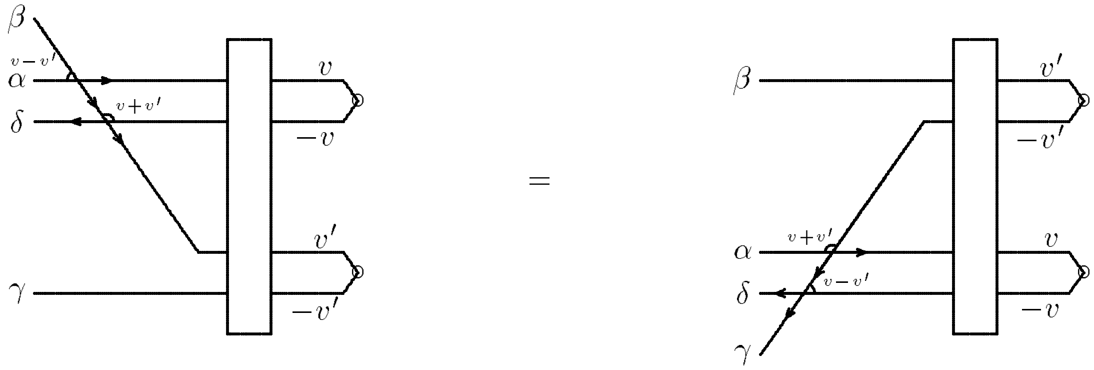
$$\mathcal{U}^{\gamma}_{\alpha}(v) = \begin{pmatrix} \mathcal{A} & \mathcal{B}_2 & \mathcal{B}_3 \\ \mathcal{C}_2 & \mathcal{D}^2_2 & \mathcal{D}^2_3 \\ \mathcal{C}_3 & \mathcal{D}^3_2 & \mathcal{D}^3_3 \end{pmatrix}. \tag{2.12}$$

It can easily be shown that the matrix S satisfies Cherednik's reflection property [9]

$$S^{\alpha\ \beta}_{\alpha'\beta'}(v-v')S^{\beta'\alpha'}_{\gamma\ \delta}(v+v') = S^{\alpha\ \beta}_{\alpha'\beta'}(v+v')S^{\beta'\alpha'}_{\gamma\ \delta}(v-v'). \tag{2.13}$$

Using eqs.(2.6), (2.7) and (2.13) we can prove that the "doubled" monodromy matrix $\mathcal{U}$ fulfills a modified Yang-Baxter algebra of the following form (for simplicity, we omit the indices of the "quantum" space)

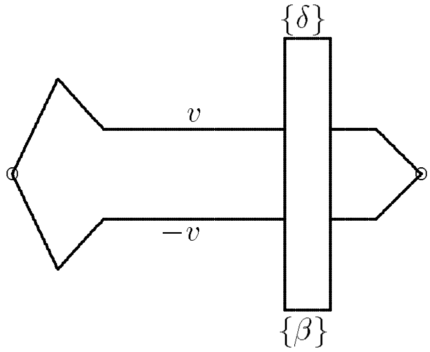
$$S^{\alpha\ \beta}_{\alpha'\beta'}(v-v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\beta'\delta'}_{\gamma'\delta}(v+v')\mathcal{U}^{\gamma'}_{\gamma}(v') = \mathcal{U}^{\beta}_{\beta'}(v')S^{\alpha\ \beta'}_{\alpha'\gamma'}(v+v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\gamma'\delta'}_{\gamma\ \delta}(v-v'). \quad (2.14)$$



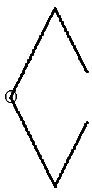
It is pointed out that we have a new structure of the Yang-Baxter algebra due to the presence of two vertex weights S at each side of the equation above. In fact, vertex models with open boundary conditions are more complex to handle in comparison with vertex models with periodic boundary conditions. Besides the fact that the monodromy matrix is constructed from two horizontal lines, in contrast to the usual one with only one line, the Yang-Baxter algebra has a more complicated structure.

The transfer matrix is defined as the Markov trace associated with the superalgebra $spl_q(2,1)$ of the "doubled" monodromy matrix in the auxiliary space

$$\tau^{\{\delta\}}_{\{\beta\}}(v) = \sum_{\alpha} K_{\alpha\alpha}\mathcal{U}^{\alpha\{\delta\}}_{\alpha\{\beta\}}(v), \quad (2.15)$$



where

$K =$  $$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & -q^2 \end{pmatrix} \tag{2.16}$$

Here q is the quantum group deformation parameter related to the anisotropy parameter γ by $q = e^{i\gamma}$. The Markov trace (2.16) is obtained from the projector on the trivial representation in $\mathbf{C}^3 \times \mathbf{C}^{3^*}$ [22]. In fact, Markov traces play a crucial role in the theory of links, where they are used to construct invariants from braids (see e.g. refs.[18,23]). We prove in sect. 4 that this construction assures $spl_q(2,1)$ invariance of the transfer matrix of the model. We can show [17] that the transfer matrix (2.15) commutes for different spectral parameters

$$[\tau(v), \tau(v')] = 0, \tag{2.17}$$

i.e., the model is integrable. Notice that in the construction of refs. [10,11] the matrix K depends on the spectral parameter v whereas in the Markov trace (2.16) v does not appear explicitly. This is due to the fact that a symmetric form of the matrix S defining the vertex weights was employed in contrast to our case (see eq.(2.1)). We stress that the use of Markov traces is more useful, since it allows for a systematic treatment of more general cases. The transfer matrix (2.15) is related to a "q-deformed" version of the one-dimensional supersymmetric t-J model

$$
\begin{aligned}
\mathcal{H} = & -\mathcal{P}\{\sum_{j=1}^{L-1}\sum_{\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\}\mathcal{P} \\
& -2\sum_{j=1}^{L-1}\left(S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \cos\gamma\ (S^z_j S^z_{j+1} - \frac{n_j n_{j+1}}{4})\right) - \cos\gamma\sum_{j=1}^{L} n_j\,, \\
& + i\sin\gamma(n_1 - n_L) - i\sin\gamma\sum_{j=1}^{L-1}(n_j S^z_{j+1} - S^z_j n_{j+1})
\end{aligned}
\tag{2.18}
$$

in the following way

$$
\frac{\partial}{\partial v}\tau(v)|_{v=0} = \frac{-\sin\gamma}{4}\mathcal{H} + 2\cot\gamma. \tag{2.19}
$$

In eq. (2.18) L is the number of sites of the quantum chain. The $c^{(\dagger)}_{j\pm}$ are spin up or down annihilation (creation) operators, the $\vec{S}_j$ spin matrices and the n_j occupation numbers of electrons at lattice site j. The projector $\mathcal{P} = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy at one lattice point. Therefore, at each lattice site we have three possibilities $(1,2,3) \equiv (\uparrow,\downarrow,0)$, i.e. an electron with spin up or down or no electron (hole). Notice that the $spl_q(2,1)$ invariant hamiltonian (2.18) has special imaginary terms. In fact, this feature has also appeared in the $SU_q(2)$ invariant XXZ hamiltonian. However, in contrast to that case, the hamiltonian (2.18) has not only an imaginary boundary term $(n_1 - n_L)$ but also an imaginary term (last operator in (2.18)) that contributes, in principle, for all sites of the chain. In particular it gives a local contribution of $-i\sin\gamma(i\sin\gamma)$ when two first neighboring sites of the chain are occupied by $\downarrow\uparrow$ $(\uparrow\downarrow)$ and consequently it is relevant for configurations $\downarrow\uparrow$ $(\uparrow\downarrow)$ which are separated by holes, e.g., chains of the type $\uparrow\downarrow\ 0 \uparrow\downarrow\ 0 \ldots \uparrow\downarrow\ 0$. It is interesting to observe that this term breaks the spin parity of the model, which means that the hamiltonian (2.18) is formally not invariant by inverting all spins anymore. At half filling (one electron per site, i.e. $n_j = 1$ for $j = 1,\ldots,L$), this term reduces to the boundary term of the H_{XXZ} hamiltonian (see ref. [11]) and the boundary $(n_1 - n_L)$ cancels out, as expected. Although the deformed t-J hamiltonian is not hermitean, it possesses real eigenvalues. Indeed, under an auxiliar scalar product it is hermitean. One easily finds that the hamiltonian (2.18) fulfills

$$
(P_{1,\ldots,N}\mathcal{H})^{\dagger} = P_{1,\ldots,N}\mathcal{H} \tag{2.20}
$$

where $P_{1,\dots,N}$ is the permutation operator in the space $\mathbf{C}_1^3 \otimes \cdots \otimes \mathbf{C}_N^3$: $P_{1,\dots,N} a_1 \otimes \cdots \otimes a_N = a_N \otimes \cdots \otimes a_1$ and † is the canonical hermitean adjoint. Since $P_{1,\dots,N}$ is hermitean with respect to the canonical scalar product $\langle\psi|\phi\rangle$, for the auxiliar one $\langle\psi|P_{1,\dots,N}|\phi\rangle$, the usual argument implies for an eigenstate ϕ of $\mathcal{H}$

$$\begin{aligned}\langle\phi|P_{1,\dots,N}\mathcal{H}|\phi\rangle &= E\langle\phi|P_{1,\dots,N}|\phi\rangle \\ &= \langle\phi|P_{1,\dots,N}\mathcal{H}|\phi\rangle^* = E^*\langle\phi|P_{1,\dots,N}|\phi\rangle.\end{aligned} \tag{2.21}$$

Therefore E is real if $\langle\phi|P_{1,\dots,N}|\phi\rangle \neq 0$. We will show below that the hamiltonian $\mathcal{H}$ is q-invariant such that the eigenstates belong to invariant subspaces. It can be shown [24] that for such states the "norm" $\langle\phi|P_{1,\dots,N}|\phi\rangle \neq 0$ never vanishes.

3. CONSTRUCTION OF BETHE EIGENVECTORS FOR OPEN BOUNDARY CONDITIONS

In this section we will solve the eigenvalue problem of the transfer matrix (2.15)

$$\tau\Psi = \lambda\Psi, \tag{3.1}$$

following the algebraic nested Bethe ansatz method with two levels adapted to open boundary conditions. This construction is based on the Yang-Baxter algebra of the "doubled" monodromy matrices

$$S^{\alpha}_{\alpha'}{}^{\beta}_{\beta'}(v-v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\beta'}_{\gamma'}{}^{\delta'}_{\delta}(v+v')\mathcal{U}^{\gamma'}_{\gamma}(v') = \mathcal{U}^{\beta}_{\beta'}(v')S^{\alpha}_{\alpha'}{}^{\beta'}_{\gamma'}(v+v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\gamma'}_{\gamma}{}^{\delta'}_{\delta}(v-v'). \tag{3.2}$$

As already pointed out in the last section the "doubled" monodromy matrix $\mathcal{U}$ can be written as a 3×3 matrix

$$\left(\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_2 & \mathcal{B}_3 \\ \hline \mathcal{C}_2 & \mathcal{D}_2^2 & \mathcal{D}_3^2 \\ \mathcal{C}_3 & \mathcal{D}_2^3 & \mathcal{D}_3^3 \end{array}\right). \tag{3.3}$$

and the transfer matrix is given by the Markov trace (2.16) of the "doubled" monodromy matrix $\mathcal{U}$ (see eq.(2.15)). According to the first level Bethe ansatz the eigenvector of the transfer matrix can be written as (in what follows we will omit the quantum space indices and write them only whenever necessary)

$$\Psi = \mathcal{B}_{\alpha_1}(v_1)\mathcal{B}_{\alpha_2}(v_2)\dots\mathcal{B}_{\alpha_N}(v_N)\Phi\Psi^{\{\alpha\}}_{(1)}, \tag{3.4}$$

where the summations over the α_i ($i = 1, \ldots, N$) are restricted to $\alpha_i = 2,\ 3$. The coefficients $\Psi_{(1)}^{\{\alpha\}}$ are determined by the second level Bethe ansatz and Φ is the first level "pseudovacuum" defined by the equation

$$\mathcal{C}_\gamma \Phi = 0 \qquad \text{for} \quad \gamma = 2, 3, \tag{3.5}$$

whose solution is

$$\Phi^{\{\beta\}} = \prod_{i=1}^{L} \delta_{\beta_i 1} = \begin{array}{cccc} \beta_1 & \beta_2 & & \beta_L \\ | & | & \cdots & | \\ 1 & 1 & & 1 \end{array} \quad . \tag{3.6}$$

This pseudovacuum is an eigenvector of $\mathcal{A}$ (see eq. (2.7))

$$\mathcal{A}(v)\Phi = a^L(v)\tilde{a}^L(-v)\Phi \tag{3.7}$$

and also of $\mathcal{D}_2^2$ and $\mathcal{D}_3^3$

$$\mathcal{D}_\alpha^\alpha(v)\Phi = -\Big(\frac{c_+(2v)}{a(2v)} - 1\Big) b^L(v)\tilde{b}^L(-v)\Phi + \frac{c_+(2v)}{a(2v)}\mathcal{A}(v)\Phi. \tag{3.8}$$

An easy way to calculate $\mathcal{D}_\alpha^\alpha \Phi$ for $\alpha = 2, 3$ is to write $\mathcal{D}_\alpha^\alpha$ in terms of the elements of the monodromy T and its inverse T^{-1} (see eq.(2.7))

$$\mathcal{D}_\alpha^\alpha(v) = C_\alpha(v)\tilde{B}_\alpha(-v) + \sum_{\beta=2}^{3} D_\beta^\alpha(v)\tilde{D}_\alpha^\beta(-v) \tag{3.9}$$

and then commute C_α with $\tilde{B}_\alpha$ through the Yang-Baxter equation for the monodromy matrices T and T^{-1}, which follows straightforwardly from eq.(2.6). Finally, using the fact that $C_\alpha \Phi = \tilde{D}_2^3 \Phi = \tilde{D}_3^2 \Phi = 0$ we find eq.(3.8). We remark that in eqs. (3.8) and (3.9) no summation over α is assumed and the summations over the internal lines in eq.(3.7) can assume only the value 1, due to the special form of the matrices S and S^{-1}.

Following the general strategy of the algebraic Bethe ansatz to solve eq.(3.1) we apply the transfer matrix τ (2.15) to the eigenvector Ψ (3.4)

$$\tau(v)\Psi = \Big(\mathcal{A}(v) + \tau_D(v)\Big)\Psi, \tag{3.10}$$

where

$$\tau_D(v) = q^2\Big(\mathcal{D}_2^2 - \mathcal{D}_3^3\Big). \tag{3.11}$$

Using the Yang-Baxter relation (3.2) and the expressions for the matrices S (2.1) and $\mathcal{U}$ (3.3) we obtain the commutation rules between $\mathcal{A}(v)$, $\mathcal{D}^{\alpha}_{\alpha'}(v)$ and $\mathcal{B}_{\alpha}(v')$ (α, $\alpha' = 2,3$). We do not write them here since we will not need them later. These relations simplify considerably if instead of using $\mathcal{D}^{\alpha}_{\alpha'}(v)$ we work with $\hat{\mathcal{D}}^{\alpha}_{\alpha'}(v)$ such that

$$\mathcal{D}^{\alpha}_{\alpha'}(v) = \chi(v)\frac{S^{\alpha\ \ \beta}_{\beta'\alpha'}(2v+\gamma)}{b(2v+\gamma)}\hat{\mathcal{D}}^{\beta'}_{\beta}(v) + \frac{c_{+}(2v)}{a(2v)}\delta^{\alpha}_{\alpha'}\mathcal{A}(v), \tag{3.12}$$

where $\chi(v)$ is given by

$$\chi(v) = q\frac{b(2v)a(2v+\gamma)}{a(2v)b(2v+\gamma)}. \tag{3.13}$$

More explicitly we have

$$\begin{aligned}\mathcal{A}(v)\mathcal{B}_{\alpha}(v') =& \frac{a(v'-v)b(v'+v)}{b(v'-v)a(v'+v)}\mathcal{B}_{\alpha}(v')\mathcal{A}(v) - \frac{c_{+}(v'-v)b(2v')}{b(v'-v)a(2v')}\mathcal{B}_{\alpha}(v)\mathcal{A}(v') \\ &- \frac{c_{-}(v'+v)}{a(v'+v)}\chi(v')\mathcal{B}_{\alpha'}(v)\hat{\mathcal{D}}^{\beta}_{\beta'}(v')\frac{S^{\alpha'\beta'}_{\beta\ \ \alpha}(2v'+\gamma)}{b(2v'+\gamma)}\end{aligned} \tag{3.14}$$

and correspondingly for $\hat{\mathcal{D}}^{\beta}_{\beta'}$

$$\begin{aligned}\hat{\mathcal{D}}^{\beta}_{\beta'}(v)\mathcal{B}_{\alpha}(v') =& \frac{S^{\alpha'\ \ \beta''}_{\alpha''\beta'}(v-v')}{b(v-v')}\frac{S^{-1\,\alpha''\beta}_{\ \ \ \alpha\beta'''}(-v-v'-\gamma)}{\tilde{b}(-v-v'-\gamma)}\mathcal{B}_{\alpha'}(v')\hat{\mathcal{D}}^{\beta'''}_{\beta''}(v) \\ &+ \frac{1}{\chi(v)}\frac{c_{+}(v+v')b(2v')}{a\ \ (v+v')a(2v')}\mathcal{B}_{\beta'}(v)\mathcal{A}(v')\delta^{\beta}_{\alpha} \\ &- \frac{\chi(v')}{\chi(v)}\frac{c_{+}(v-v')}{b(v-v')}\frac{S^{\beta\ \ \alpha''}_{\beta''\alpha}(2v'+\gamma)}{b(2v'+\gamma)}\mathcal{B}_{\beta'}(v)\hat{\mathcal{D}}^{\beta''}_{\alpha''}(v').\end{aligned} \tag{3.15}$$

All indices of the auxiliary space in eqs.(3.12), (3.14) and (3.15) assume only the values 2 and 3. We observe in these relations the presence of an additional term $\mathcal{B}_{\alpha}(v)\hat{\mathcal{D}}^{\beta}_{\beta'}(v')$ in (3.14) and $\mathcal{B}_{\beta'}(v)\mathcal{A}(v')$ in (3.15) in comparison with the corresponding relations using the Yang-Baxter algebra for the monodromy T (2.6). Nevertheless, the scheme of the algebraic Bethe ansatz method can be applied, as for the H_{XXZ} chain [10]. Notice that the action of the new operators $\hat{\mathcal{D}}^{\alpha'}_{\alpha}$ on the first level "pseudovacuum" Φ

$$\hat{\mathcal{D}}^{\alpha'}_{\alpha}(v)\Phi = \sigma^{\alpha'}_{\alpha}\frac{1}{q^2}\frac{b^2(2v+\gamma)}{a(2v+\gamma)w(2v+\gamma)}\Big(\frac{c_{+}(2v)}{a(2v)} - 1\Big)b^L(v)\tilde{b}^L(-v)\Phi, \tag{3.16}$$

yields a simpler expression than the corresponding one for $\mathcal{D}^{\alpha'}_{\alpha}(v)$. In fact, an examination of the action of $\mathcal{D}^{\alpha'}_{\alpha}(v)$ over the pseudovacuum Φ suggests how the operator $\hat{\mathcal{D}}^{\alpha'}_{\alpha}$

shall be defined. The matrix σ in eq.(3.16) is just a diagonal matrix given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.17}$$

By successive applications of eq.(3.14) we commute $\mathcal{A}$ through all $\mathcal{B}_\alpha$ towards Φ. From the first term on the right hand side of (3.14) we get the "wanted" contributions

$$\mathcal{A}(v)\Psi = \lambda_A(v)\Psi + u.t.(\mathcal{A}), \tag{3.18}$$

where the coefficient λ_A is given by

$$\lambda_A = a^L(v)\tilde{a}^L(-v)\prod_{i=1}^{N}\frac{a(v_i-v)b(v_i+v)}{b(v_i-v)a(v_i+v)}. \tag{3.19}$$

The terms generated from the second and third terms in eqs. (3.14) are called "unwanted" since they can not originate a vector proportional to Ψ. Correspondingly, we get from the commutation relation (3.15) wanted and unwanted contributions

$$\begin{aligned}\tau_D(v)\Psi =& q\chi(v)\hat{\mathcal{D}}^\beta_\beta\Phi = b^L(v)\tilde{b}^L(-v)\Big(\frac{c_+(2v)}{a(2v)}-1\Big)\frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)} \\ &\times\prod_{i=1}^{N}\Big(\frac{1}{b(v-v_i)}\frac{1}{\tilde{b}(-v-v_i-\gamma)}\Big)\mathcal{B}_{\alpha'_1}(v_1)\mathcal{B}_{\alpha'_2}(v_2)\dots\mathcal{B}_{\alpha'_N}(v_N)\Phi \\ &\times\tau_{(1)}{}^{\{\alpha'\}}_{\{\alpha\}}(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Psi^{\{\alpha\}}_{(1)} + u.t.(\mathcal{D}),\end{aligned} \tag{3.20}$$

where $\tau_{(1)}$ is the second level transfer matrix

$$\tau_{(1)}(v,\{v_i\}) = \sum_{\beta=2}^{3}\sigma^\beta_\beta\ \mathcal{U}_{(1)}{}^\beta_\beta(v,\{v_i\}), \tag{3.21}$$

which is defined as the Markov trace associated to the superalgebra $Sl_q(1,1)$ of the second level "doubled" monodromy matrix $\mathcal{U}_{(1)} = S(v-v_N)S^{-1}(v-v_N)\dots S(v-v_1)S^{-1}(v-v_1)$. It is pointed out that all indices range from 2 to 3, as in the internal block of the matrix $\mathcal{U}$ (3.3). This suggests the identification $\mathcal{A}_{(1)} \equiv \mathcal{U}_{(1)}{}^2_2$, $\mathcal{B}_{(1)} \equiv \mathcal{U}_{(1)}{}^2_3$, $\mathcal{C}_{(1)} \equiv \mathcal{U}_{(1)}{}^3_2$ and $\mathcal{D}_{(1)} \equiv \mathcal{U}_{(1)}{}^3_3$. Our task is to solve the eigenvalue problem

$$\tau_{(1)}(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Psi_{(1)} = \lambda_{(1)}\Psi_{(1)}, \tag{3.22}$$

in order to get the eigenvalue $\lambda_{\mathcal{D}}$ in (3.20). The eigenvector $\Psi_{(1)}$ is defined by the second level Bethe ansatz

$$\Psi_{(1)} = \mathcal{B}_{(1)}(\nu_1+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\mathcal{B}_{(1)}(\nu_2+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\dots\mathcal{B}_{(1)}(\nu_M+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Phi_{(1)}, \quad (3.23)$$

where $\Phi_{(1)}$ is the second level "pseudovacuum", which takes the form $\Phi_{(1)}^{\{\alpha\}} = \prod_{i=1}^{N}\delta_{\alpha_i,2}$ and is annihilated by $\mathcal{C}_{(1)}$

$$\mathcal{C}_{(1)}(v,\{v_i\})\Phi_{(1)} = 0. \quad (3.24)$$

Indeed, as in the previous step, it is more convenient to work with the operator $\hat{\mathcal{D}}_{(1)}$ defined as

$$\hat{\mathcal{D}}_{(1)}(v,\{v_i\}) = \mathcal{D}_{(1)}(v,\{v_i\}) - \frac{c_-(2v)}{a(2v)}\mathcal{A}_{(1)}(v,\{v_i\}). \quad (3.25)$$

Then the action of $\mathcal{A}_{(1)}$ and $\mathcal{D}_{(1)}$ on the pseudovaccum $\Phi_{(1)}$ is given by

$$\mathcal{A}_{(1)}(v,\{v_i\})\Phi_{(1)} = \prod_{i=1}^{N} a(v-v_i)\tilde{a}(-v-v_i)\Phi_{(1)}, \quad (3.26)$$

$$\hat{\mathcal{D}}_{(1)}(v,\{v_i\})\Phi_{(1)} = \Big(1-\frac{c_-(2v)}{a(2v)}\Big)\prod_{i=1}^{N} b(v-v_i)\tilde{b}(-v-v_i)\Phi_{(1)}. \quad (3.27)$$

The commutation relations for $\mathcal{A}_{(1)}$, $\mathcal{D}_{(1)}$and $\mathcal{B}_{(1)}$ follow from the Yang-Baxter algebra (3.2), which is also valid for $\mathcal{U}_{(1)}(v,\{v_i\})$

$$\begin{aligned}\mathcal{A}_{(1)}(v)\mathcal{B}_{(1)}(v') =& \frac{a(v'-v)b(v'+v)}{b(v'-v)a(v'+v)}\mathcal{B}_{(1)}(v')\mathcal{A}_{(1)}(v) - \frac{c_-(v'-v)b(2v')}{b(v'-v)a(2v')}\mathcal{B}_{(1)}(v)\mathcal{A}_{(1)}(v') \\ &- \frac{c_+(v'+v)}{a(v'+v)}\mathcal{B}_{(1)}(v)\hat{\mathcal{D}}_{(1)}(v'),\end{aligned} \quad (3.28)$$

and

$$\begin{aligned}\hat{\mathcal{D}}_{(1)}(v)\mathcal{B}_{(1)}(v') =& \frac{a(v'-v)b(v'+v)}{b(v'-v)a(v'+v)}\mathcal{B}_{(1)}(v')\hat{\mathcal{D}}_{(1)}(v) + \frac{c_-(v-v')b(2v)}{b(v-v')a(2v)}\mathcal{B}_{(1)}(v)\hat{\mathcal{D}}_{(1)}(v') \\ &- \frac{c_-(v'+v)b(2v)b(2v')}{a(v'+v)a(2v)a(2v')}\mathcal{B}_{(1)}(v)\mathcal{A}_{(1)}(v').\end{aligned} \quad (3.29)$$

In this level we also observe the presence of the terms $\mathcal{B}_{(1)}(v)$ $\hat{\mathcal{D}}_{(1)}(v')$ in (3.28) and $\mathcal{B}_{(1)}(v)$ $\mathcal{A}_{(1)}(v')$ in (3.29). Then, proceeding along the same lines as in the calculation of (3.10) we obtain wanted and unwanted contributions when $\mathcal{A}_{(1)}$ and $\mathcal{D}_{(1)}$ commutes with $\mathcal{B}_{(1)}$ toward $\Phi_{(1)}$. The "unwanted" terms are not proportional to $\Psi_{(1)}$ and hence they must cancel in order to get an eigenvector of the second level transfer matrix $\tau_{(1)}$ (3.22).

$$\begin{aligned}\tau_{(1)}\left(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\}\right)\Psi_{(1)} =& \Bigg(\left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)\mathcal{A}_{(1)}\left(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\}\right)\\ &-\hat{\mathcal{D}}_{(1)}\left(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\}\right)\Bigg)\Psi_{(1)} \\ &= \left(\lambda_{\mathcal{A}_{(1)}}+\lambda_{\mathcal{D}_{(1)}}\right)\Psi_{(1)}+u.t.(\mathcal{A}_{(1)})+u.t.(\mathcal{D}_{(1)})\end{aligned}\quad, \tag{3.30}$$

where $\lambda_{\mathcal{A}_{(1)}}$ and $\lambda_{\mathcal{D}_{(1)}}$ are given by

$$\lambda_{\mathcal{A}_{(1)}} = \left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)\prod_{i=1}^{N} a(v-v_i)\tilde{a}(-v-v_i-\gamma)\prod_{j=1}^{M}\frac{a(\nu_j-v)b(\nu_j+v+\gamma)}{b(\nu_j-v)a(\nu_j+v+\gamma)} \tag{3.31}$$

$$\lambda_{\mathcal{D}_{(1)}} = -\left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)\prod_{i=1}^{N} b(v-v_i)\tilde{b}(-v-v_i-\gamma)\prod_{j=1}^{M}\frac{a(\nu_j-v)b(\nu_j+v+\gamma)}{b(\nu_j-v)a(\nu_j+v+\gamma)} \tag{3.32}$$

Thus, substituting these results in (3.20) we find

$$\tau_{\mathcal{D}}(v)\Psi = \left(\lambda_{\mathcal{D}_I}(v)+\lambda_{\mathcal{D}_{II}}(v)\right)\Psi+u.t.(\mathcal{D}), \tag{3.33}$$

where the contributions to the eigenvalue $\lambda_{\mathcal{D}_I}$ and $\lambda_{\mathcal{D}_{II}}$ read

$$\begin{aligned}\lambda_{\mathcal{D}_I}(v) =& \frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)}\left(\frac{c_+(2v)}{a(2v)}-1\right)\left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)b^L(v)\tilde{b}^L(-v)\\ &\times\prod_{i=1}^{N}\frac{a(v-v_i)\tilde{a}(-v-v_i-\gamma)}{b(v-v_i)\tilde{b}(-v-v_i-\gamma)}\prod_{j=1}^{M}\frac{a(\nu_j-v)b(\nu_j+v+\gamma)}{b(\nu_j-v)a(\nu_j+v+\gamma)},\end{aligned} \tag{3.34}$$

$$\begin{aligned}\lambda_{\mathcal{D}_{II}}(v) =& -\frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)}\left(\frac{c_+(2v)}{a(2v)}-1\right)\left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)b^L(v)\tilde{b}^L(-v)\\ &\times\prod_{j=1}^{M}\frac{a(\nu_j-v)b(\nu_j+v+\gamma)}{b(\nu_j-v)a(\nu_j+v+\gamma)}.\end{aligned} \tag{3.35}$$

Then, combining eqs.(3.18) and (3.33) we get the eigenvalue $\lambda(v)$ of the transfer matrix $\tau(v)$ if the "unwanted terms" u.t.($\mathcal{A}$) and u.t.($\mathcal{D}$) cancel out

$$\tau(v)\Psi = \lambda(v)\Psi, \tag{3.36}$$

with

$$\lambda(v)=\lambda_{\mathcal{A}}(v)+\lambda_{\mathcal{D}_I}(v)+\lambda_{\mathcal{D}_{II}}(v). \tag{3.37}$$

All "unwanted" terms vanish if the Bethe ansatz equations (BAE) hold

$$\begin{aligned}\left(\frac{a(v_k)\tilde{a}(-v_k)}{b(v_k)\tilde{b}(-v_k)}\right)^L \prod_{i\neq k}^{N} &\frac{a(v_i-v_k)b(v_i+v_k)\tilde{b}(\ v_i+v_k+\gamma)}{a(v_k-v_i)a(v_i+v_k)\tilde{a}(-v_i-v_k-\gamma)} \\ &\times\prod_{j=1}^{M}\frac{a(\nu_j+v_k+\gamma)b(\nu_j-v_k)}{b(\nu_j+v_k+\gamma)a(\nu_j-v_k)}=1\ , \quad k=1,\dots,N,\end{aligned} \tag{3.38}$$

$$\prod_{i=1}^{N}\frac{a(\nu_l-v_i)\tilde{a}(-\nu_k-v_i-\gamma)}{b(\nu_l-v_i)\tilde{b}(-\nu_k-v_i-\gamma)}=1\ , \quad l=1,\dots,M. \tag{3.39}$$

In this case Ψ (3.4) is an eigenstate of the transfer matrix τ (2.15). These equations can be obtained from the fact that the eigenvalue $\lambda(v)$ (3.37) has no poles at $v=v_i$ and $v=\nu_j$. As a matter of fact, one can use this property (a necessary condition for the eigenvalues) as a short-cut to derive the BAE. Using eq.(2.2) and making the change of variables $v\to\gamma(iv-1)/2, \quad \nu\to\gamma(i\nu-2)/2$ the BAE can be written as

$$\begin{aligned}\left(\frac{\sinh\frac{\gamma}{2}(v_k-i)}{\sinh\frac{\gamma}{2}(v_k+i)}\right)^{2L}\prod_{i\neq k}^{N}&\frac{\sinh\frac{\gamma}{2}(v_k-v_i+2i)\sinh\frac{\gamma}{2}(v_k+v_i+2i)}{\sinh\frac{\gamma}{2}(v_k-v_i-2i)\sinh\frac{\gamma}{2}(v_k+v_i-2i)} \\ &\times\prod_{j=1}^{M}\frac{\sinh\frac{\gamma}{2}(v_k-\nu_j-i)\sinh\frac{\gamma}{2}(v_k+\nu_j-i)}{\sinh\frac{\gamma}{2}(v_k-\nu_j+i)\sinh\frac{\gamma}{2}(v_k+\nu_j+i)}=1\ , \quad k=1,\dots,N,\end{aligned} \tag{3.40}$$

$$\prod_{i=1}^{N}\frac{\sinh\frac{\gamma}{2}(\nu_l-v_i-i)\sinh\frac{\gamma}{2}(\nu_l+v_i-i)}{\sinh\frac{\gamma}{2}(\nu_l-v_i+i)\sinh\frac{\gamma}{2}(\nu_l+v_i+i)}=1\ , \quad l=1,\dots,M. \tag{3.41}$$

Therefore, we have reduced the eigenvalue problem of the transfer matrix τ to the solution of a system of coupled transcendental equations for the parameters v and ν. These equations can be mapped into the BAE for periodic boundary conditions on a lattice with $2L$ sites, analogously to the H_{XXZ} chain [11].

4. QUANTUM GROUP STRUCTURE OF THE DEFORMED T-J MODEL

In this section we investigate the quantum group structure of the deformed t-J model (2.18). We show that for special limit values of the spectral parameter v the Yang-Baxter algebra leads to a deformation of the "graded" Lie algebra $spl(2,1)$. Furthermore, we prove that the transfer matrix τ (or, equivalently, the deformed t-J hamiltonian) is $spl_q(2,1)$ invariant.

In what follows we use $x = e^{iv}$ and $q = e^{i\gamma}$. First it is convenient to write the matrix S (2.1) as

$$S(x) = xS_+ - x^{-1}S_- \quad , \tag{4.1}$$

where S_+ and S_- are given by

$$S_+ = \begin{pmatrix} q^{w_1} & 0 & 0 \\ (q-q^{-1})q^{-\frac{w_3}{2}}f_1 & q^{w_2} & 0 \\ (q-q^{-1})[f_1,f_2] & (q-q^{-1})q^{-\frac{w_1}{2}-w_3}\sigma f_2 & \sigma q^{-w_3} \end{pmatrix}, \tag{4.2}$$

$$S_- = \begin{pmatrix} q^{-w_1} & -(q-q^{-1})e_1 q^{\frac{w_3}{2}} & -(q-q^{-1})[e_2,e_1] \\ 0 & q^{-w_2} & -(q-q^{-1})e_2 q^{\frac{w_1}{2}+w_3} \\ 0 & 0 & \sigma q^{w_3} \end{pmatrix}, \tag{4.3}$$

and w_i,f_i, e_i, $i = 1,2$ are the generators of the spl(2,1) superalgebra in the "graded" Cartan-Chevalley basis and the matrix σ was already defined in eq. (3.17). Further details may be found in the appendix A, where we recall the main properties of the superalgebra spl(2,1) and also the fundamental representation of the generators w_i,f_i, e_i. In the limit $x \to \infty(0)$, the leading terms T_+, (T_-) of the monodromy matrix T (2.5) are proportional to

$$T_+ \propto \begin{pmatrix} q^{-\frac{L}{2}}q^{W_1} & 0 & 0 \\ \alpha_- q^{-\frac{W_3}{2}}F_1 & q^{-\frac{L}{2}}q^{W_2} & 0 \\ q^{\frac{L}{2}}T_{+\,1}^{\;\,3} & \alpha_- q^{-\frac{W_1}{2}-W_3}\tilde{\sigma}F_2 & q^{-\frac{L}{2}}\tilde{\sigma}q^{-W_3} \end{pmatrix}, \tag{4.4}$$

$$T_- \propto \begin{pmatrix} q^{\frac{L}{2}}q^{-W_1} & -\alpha_+ E_1 q^{\frac{W_3}{2}} & q^{-\frac{L}{2}}T_{-\,3}^{\;\,1} \\ 0 & q^{\frac{L}{2}}q^{-W_2} & -\alpha_+ E_2 q^{\frac{W_1}{2}+W_3} \\ 0 & 0 & q^{\frac{L}{2}}\tilde{\sigma}q^{W_3} \end{pmatrix}, \tag{4.5}$$

with $\alpha_\pm = q^{\pm\frac{1}{2}}(q-q^{-1})$ and

$$\begin{aligned} q^{\pm W_i} &= q^{\pm w_i} \otimes q^{\pm w_i} \otimes \ldots \otimes q^{\pm w_i}, \quad i = 1,2,3 \\ \tilde{\sigma} &= \sigma \otimes \sigma \otimes \ldots \otimes \sigma \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
F_1 &= \sum_{j=1}^{L} q^{-\frac{h_1}{2}} \otimes \ldots \otimes q^{-\frac{h_1}{2}} \otimes \overset{j_{th}}{f_1} \otimes q^{\frac{h_1}{2}} \otimes \ldots \otimes q^{\frac{h_1}{2}} \\
F_2 &= \sum_{j=1}^{L} q^{-\frac{h_2}{2}} \otimes \ldots \otimes q^{-\frac{h_2}{2}} \otimes \overset{j_{th}}{f_2} \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \otimes \ldots \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \\
E_1 &= \sum_{j=1}^{L} q^{-\frac{h_1}{2}} \otimes \ldots \otimes q^{-\frac{h_1}{2}} \otimes \overset{j_{th}}{e_1} \otimes q^{\frac{h_1}{2}} \otimes \ldots \otimes q^{\frac{h_1}{2}} \\
E_2 &= \sum_{j=1}^{L} q^{-\frac{h_2}{2}} \otimes \ldots \otimes q^{-\frac{h_2}{2}} \otimes \overset{j_{th}}{e_2} \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \otimes \ldots \otimes \left(\sigma q^{\frac{h_2}{2}}\right)
\end{aligned} . \tag{4.7}
$$

Here $h_1 = w_1 - w_2$ and $h_2 = w_2 + w_3$ (see appendix A). In the last four expressions we have a kind of "q-sum" of one-site operators extended to all sites. For $L = 2$ these formulae define a co-product : $\Delta^{(2)}(f_1) = F_1$, etc. The matrix σ takes into account that F_2 and E_2 are odd operators. In the literature we also find the definition of the coproduct without σ, but then we shall use the graded-tensor product law instead of usual tensor product law [25]. Both methods are equivalent. By taking appropriate limits ($x \to \infty, 0$ or $v \to \pm i\infty$) in the Yang-Baxter relation for the monodromy T (2.6) we get the spectral parameter independent Yang-Baxter relations

$$
S_{+}{}_{\alpha'\beta'}^{\alpha''\beta''} T_{+}{}_{\alpha}^{\alpha'} T_{+}{}_{\beta}^{\beta'} = T_{+}{}_{\beta'}^{\beta''} T_{+}{}_{\alpha'}^{\alpha''} S_{+}{}_{\alpha\ \beta}^{\alpha'\beta'}, \tag{4.8}
$$

$$
S_{+}{}_{\alpha'\beta'}^{\alpha''\beta''} T_{+}{}_{\alpha}^{\alpha'} T_{-}{}_{\beta}^{\beta'} = T_{-}{}_{\beta'}^{\beta''} T_{+}{}_{\alpha'}^{\alpha''} S_{+}{}_{\alpha\ \beta}^{\alpha'\beta'}, \tag{4.9}
$$

$$
S_{-}{}_{\alpha'\beta'}^{\alpha''\beta''} T_{-}{}_{\alpha}^{\alpha'} T_{-}{}_{\beta}^{\beta'} = T_{-}{}_{\beta'}^{\beta''} T_{-}{}_{\alpha'}^{\alpha''} S_{-}{}_{\alpha\ \beta}^{\alpha'\beta'}, \tag{4.10}
$$

Using the definitions of S_+, S_-, T_+ and T_- given by eqs. (4.2 - 4.5) together with the above relations for special values of the free indices (α'',β'', α and β) we obtain the following q-(anti)commutation rules and cubic q-Serre relations (with $H_1 = W_1 - W_2$ and $H_2 = W_2 + W_3$)

$$
\begin{aligned}
q^{H_1} q^{H_2} &= q^{H_2} q^{H_1}, \\
q^{H_i} F_j q^{-H_i} &= q^{a_{ij}} F_j, \\
q^{H_i} E_j q^{-H_i} &= q^{-a_{ij}} E_j,
\end{aligned} \tag{4.11}
$$

$$
[F_1, E_1] = \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}, \quad [F_1, E_2] = 0,
$$

$$[F_2, E_2]_+ = \frac{q^{H_2} - q^{-H_2}}{q - q^{-1}}, \quad [F_2, E_1] = 0, \tag{4.12}$$
$$E_2{}^2 = F_2{}^2 = 0,$$

$$\Big(F_1\Big)^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 \Big(F_1\Big)^2 = 0,$$
$$\Big(E_1\Big)^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 \Big(E_1\Big)^2 = 0 \tag{4.13}$$

Here a_{ij} is an element of the "graded" Cartan matrix given by $a_{11} = 2$, $a_{12} = a_{21} = -1$ and $a_{22} = 0$. These relations define a deformation of the $spl(2,1)$ superalgebra since in the isotropic limit $\gamma \to 0$ the usual $spl(2,1)$ (anti-)commutators and cubic Serre relations [20,26] is recovered. It defines the $spl_q(2,1)$ quantum group. Note that the complicated commutation rules (4.11) – (4.13) are equivalent to the beautiful spectral parameter independent Yang-Baxter relations (4.8) – (4.10) which follow from the general Yang-Baxter equation (2.6).

Let us now consider the "doubled" monodromy matrix $\mathcal{U}$. From eqs.(2.7) and (2.12) we can write the elements of $\mathcal{U}$ in terms of the row-to-row monodromy T (2.5) and its inverse T^{-1}

$$\mathcal{A}(x) = A(x)\tilde{A}(1/x) + \sum_{\alpha=2}^{3} B_\alpha(x)\tilde{C}_\alpha(1/x),$$
$$\mathcal{B}_\alpha(x) = A(x)\tilde{B}_\alpha(1/x) + \sum_{\beta=2}^{3} B_\beta(x)\tilde{D}^\beta_\alpha(1/x), \tag{4.14}$$
$$\mathcal{C}_\alpha(x) = C_\alpha(x)\tilde{A}(1/x) + \sum_{\beta=2}^{3} D^\alpha_\beta(x)\tilde{C}_\beta(1/x),$$
$$\mathcal{D}^\alpha_\beta(x) = C_\alpha(x)\tilde{B}_\beta(1/x) + \sum_{\gamma=2}^{3} D^\alpha_\gamma(x)\tilde{D}^\gamma_\beta(1/x),$$

where the indices α,β assume the values 2 or 3. Then, taking the limit $x \to \infty, 0$ of the "doubled" monodromy matrix $\mathcal{U}$ we find that the leading terms also yield the generators of $spl_q(2,1)$. We list below just the relevant ones for next calculations

$$\mathcal{A}(x \to \infty) \sim q^{-L} q^{2W_1},$$

$$\mathcal{D}^3_3(x \to 0) \sim q^{L} q^{2W_3},$$

$$\begin{aligned}
\mathcal{B}_2(x\to\infty) &\sim \alpha_- q^{-\frac{L}{2}} q^{W_1} E_1 q^{-\frac{1}{2}W_3},\\
\mathcal{C}_2(x\to\infty) &\sim \alpha_- q^{-\frac{L}{2}} q^{-\frac{1}{2}W_3} F_1 q^{W_1},\\
\mathcal{D}_3^2(x\to 0) &\sim -\alpha_+ q^{\frac{L}{2}} E_2 q^{\frac{1}{2}W_1+2W_3}\tilde{\sigma},\\
\mathcal{D}_2^3(x\to 0) &\sim -\alpha_+ q^{\frac{L}{2}} \tilde{\sigma} q^{\frac{1}{2}W_1+2W_3} F_2.
\end{aligned} \tag{4.15}$$

This means the $spl_q(2,1)$ generators can also be obtained from $\mathcal{U}$ and using the Yang-Baxter relation for the "doubled" monodromy $\mathcal{U}$ we also get the q-(anti)commutation and cubic q-Serre relations (4.11) – (4.13). It should be stressed that the transfer matrix constructed from T with periodic boundary conditions is not "quantum group" invariant since it does not commute with the $spl_q(2,1)$ generators. In fact, this feature also appears in the six vertex model with anisotropy [11]. In addition, the eigenvalues of this transfer matrix are not real. In order to get quantum group invariance, we shall work with the "doubled" monodromy matrix $\mathcal{U}$ (associated with open boundary conditions). This can be shown as follows. Letting $x' \to \infty, 0$ in eq.(3.2) we can evaluate the relevant commutation rules between $\mathcal{A}(x)$,$\mathcal{D}_2^2(x)$ and $\mathcal{D}_3^3(x)$ and the $spl_q(2,1)$ generators. After some calculations we obtain

$$[\mathcal{A}(x), E_1] = -q^2[\mathcal{D}_2^2(x), E_1] = -q^{-\frac{1}{2}-\frac{L}{2}} q^{-W_1} \mathcal{B}_2(x) q^{2W_1} q^{\frac{1}{2}W_3}, \tag{4.16}$$

$$[\mathcal{D}_3^3(x), E_1] = 0,$$

$$[\mathcal{D}_2^2(x), E_2] = [\mathcal{D}_3^3(x), E_2] = -q^{\frac{1}{2}+\frac{L}{2}} \mathcal{D}_3^2(x) q^{-\frac{1}{2}W_1}, \tag{4.17}$$

$$[\mathcal{A}(x), E_2] = 0,$$

$$[\mathcal{A}(x), F_1] = -q^2[\mathcal{D}_2^2(x), F_1] = q^{\frac{3}{2}-\frac{L}{2}} q^{\frac{1}{2}W_3} \mathcal{C}_2(x) q^{W_1}, \tag{4.18}$$

$$[\mathcal{D}_3^3(x), F_1] = 0,$$

$$[\mathcal{D}_2^2(x), F_2] = [\mathcal{D}_3^3(x), F_2] = -q^{-\frac{3}{2}+\frac{L}{2}} q^{-\frac{1}{2}W_1-2W_3} \tilde{\sigma} \mathcal{D}_2^3(x) q^{2W_3}, \tag{4.19}$$

$$[\mathcal{A}(x), F_2] = 0,$$

$$[\mathcal{A}(x), q^{W_i}] = [\mathcal{D}_2^2(x), q^{W_i}] = [\mathcal{D}_3^3(x), q^{W_i}] = 0, \quad i = 1,2,3. \tag{4.20}$$

Using theses commutations relations and the definition of the matrix $\tau = tr(K\mathcal{U})$ in eq.(2.15) we easily find that τ commutes with the $spl_q(2,1)$ generators, i.e.,

$$
\begin{aligned}
&[\tau(x), E_1] = [\tau(x), E_2] = 0 \\
&[\tau(x), F_1] = [\tau(x), F_2] = 0 \\
&[\tau(x), q^{H_i}] = 0, \quad i = 1, 2.
\end{aligned}
\tag{4.21}
$$

Therefore, the transfer matrix τ is $spl_q(2,1)$ invariant and consequently the eigenstates of the deformed t-J hamiltonian (2.18) can be classified by the $spl_q(2,1)$ superalgebra.

The question of the completeness of the Bethe states for the deformed t-J model was not considered so far. In fact, this does not to seem to be an easy task, since it was not proved yet even for the Heisenberg H_{XXZ} model. There is however some hope that the quantum group invariance could bring some insight on the problem of the completeness.

5. SUMMARY

In this paper we introduce a new integrable vertex model, namely, the "graded" 15-vertex model with anisotropy. It is a vertex model with three states, which can be bosonic or fermionic. A general procedure using Markov traces is proposed for the construction of quantum group invariant integrable models. We derive the Bethe ansatz equations of the model. Moreover, we show that this vertex model can be related to a deformed one-dimensional supersymmetric t-J model.

The underlying quantum group structure is also investigated. We find that a deformation of the "graded" Lie algebra $spl(2,1)$ follows from the Yang Baxter algebra in the limit $v \to \pm i\infty$. Finally, we prove that this model is $spl_q(2,1)$ quantum group invariant.

APPENDIX A : Superalgebra $spl(2,1)$

The classical simple graded Lie algebra $spl(2,1)$ is defined as the associative algebra over $\mathbf{C}$ generated by h_i, f_i, $(i = 1,2)$ (in the "graded" Cartan-Chevalley basis [20]), where h_i, e_1, f_1 (e_2, f_2) are even (odd) generators, which satisfy the (anti) commutation relations

$$[h_1, h_2] = 0, \tag{A.1}$$

$$[h_i, f_j] = a_{ij} f_j, \quad [h_i, e_j] = -a_{ij} e_j,$$

$$[f_1, e_1] = h_1, \quad [f_1, e_2] = 0,$$

$$[f_2, e_2]_+ = h_2, \quad [f_2, e_1] = 0,$$

$${f_2}^2 = {e_2}^2 = 0,$$

together with the cubic Serre relations

$$(f_1)^2 f_2 - 2 f_1 f_2 f_1 + f_2 (f_1)^2 = 0 \tag{A.2}$$

$$(e_1)^2 e_2 - 2 e_1 e_2 e_1 + e_2 (e_1)^2 = 0$$

In the relations above a_{ij} is an element of the "graded" Cartan matrix A given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}. \tag{A.3}$$

The fundamental representation of the generators is given below

$$h_1 \equiv w_1 - w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_2 \equiv w_2 + w_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.4}$$

$$f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The $spl(2,1)$ graded Lie Algebra can also be defined in the "graded" Cartan-Weyl basis, as in ref. [26]

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