# Matrix Difference Equations and a Nested Bethe Ansatz

H. Babujian[1,2,3], M. Karowski[4] and A. Zapletal[5,6]

*Institut für Theoretische Physik*

*Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany*

August 27, 1998

**Abstract**

Systems of $SU(N)$- and $U(N)$-matrix difference equations are solved by means of a nested version of a generalized Bethe Ansatz, also called "off shell" Bethe Ansatz [1]. To solve this new Bethe Ansatz in the algebraic language analogous to the conventional case, a new type of monodromy matrices is introduced. They fulfill a new type of Yang-Baxter equations which simplify the proofs. Similar as for the conventional nested Bethe Ansatz the problem is solved iteratively. The vanishing of the "unwanted terms" of the first level Ansatz is equivalent to a set of second level difference equations. The solutions are obtained as sums over "off shell" Bethe vectors. These sums are "Jackson-type Integrals". The highest weight property of the solutions is proved. For several examples of $SU(N)$- and $U(N)$-representations the solutions are calculated explicitly.

## 1 Introduction

Difference equations play a role in various contexts of mathematical physics (see e.g. [2] and references therein). We are interested in the application to the form factor program in the exact integrable 1+1 dimensional field theory, which was formulated in 1978 by one

[1]Permanent address: Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan, 375036 Armenia.

[2]Partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forschung und Technologie and through the VW project 'Cooperation with scientists from CIS'.

[3]e-mail: babujian@lx2.yerphi.am

[4]e-mail: karowski@physik.fu-berlin.de

[5]Supported by DFG, Sonderforschungsbereich 288 'Differentialgeometrie und Quantenphysik'

[6]e-mail: zapletal@physik.fu-berlin.de

of the authors (M.K.) and Weisz [3]. Form factors are matrix elements of local operators $\mathcal{O}(x)$

$$F(i\pi - \theta) = \langle\, p' \,|\, \mathcal{O}(0) \,|\, p \,\rangle$$

where $p'p = m^2 \cosh\theta$. Difference equations for these functions are obtained by Watson's equations [4]

$$F(\theta) = S(\theta)\, F(-\theta)\ , \quad F(i\pi - \theta) = F(i\pi + \theta) \tag{1.1}$$

where $S$ is the $S$-matrix. For several models these equations have been solved in [3] and many later publications (see e.g. [5, 6] and references therein). Generalized form factors are matrix elements for many particle states. For generalized form factors Watson's equations lead typically to matrix difference equations, which can be solved by a generalized Bethe Ansatz, also called "off-shell Bethe Ansatz". The conventional Bethe Ansatz introduced by Bethe [7] is used to solve eigenvalue problems. The algebraic formulation, which is also used in this article, was worked out by Faddeev and coworkers (see e.g. [8]). The "off-shell Bethe Ansatz" has been introduced by one of the authors (H.B.) to solve the Knizhnik-Zamolodchikov equations which are differential equations. In [9] a variant of this technique has been formulated to solve matrix difference equations of the form

$$f(x_1, \ldots, x_i + 2, \ldots, x_n) = Q(x_1, \ldots, x_n, ; i)\, f(x_1 \ldots, x_i, \ldots, x_n)\ , \quad (i = 1, \ldots, n)$$

where $f(\underline{x})$ is a vector valued function and the $Q(\underline{x}, i)$ are matrix valued functions to be specified later. (For further applications of this technique see also [2, 11].) For higher rank internal symmetry groups the nested version of this Bethe Ansatz has to be applied. The nested Bethe Ansatz to solve eigenvalue problems was introduced by Yang [10] and further developed by Sutherland [12] (see also [13] for the algebraic formulation and for new applications [11]). The very interesting generalization of this technique, which is applicable to difference equations, is developed in this article for the $SU(N)$ symmetry group. This generalization demonstrates the power of the Bethe Ansatz even more beautifully than the conventional applications. In addition we solve difference equations for the $U(N)$ case. It turns out that this problem is much more involved because of the more complicated Bethe Ansatz pseudo ground state. However, we need these solution to solve the form factor problem for the $SU(N)$ chiral Gross-Neveu model in [17]. Other methods to solve matrix difference equations have been discussed in [5, 14, 15, 16].

The article is organized as follows. In Section 2 we recall some well known results concerning the $SU(N)$ and $U(N)$ R-matrices, the monodromy matrix and some commutation rules. In Section 3 we introduce the nested generalized Bethe Ansatz to solve a system of matrix difference equations and present the solutions in terms of "Jackson-type Integrals". The proof of the main theorem avoids the decomposition of the monodromy

matrix, as used e.g. in [9]. Instead we introduce a new type of monodromy matrix fulfilling a new type of Yang-Baxter relation and which is adapted to the difference problem. In particular this yields an essential simplification of the proof of the main theorem. In Section 4 we prove the highest weight property of the solutions and calculate the weights. Section 5 contains some examples of solutions of the matrix difference equations. As already mentioned compared to the $SU(N)$ case the $U(N)$ case is much more involved. Therefore we separated the treatment of both cases in each section, such that it is possible to skip the $U(N)$ parts but to read and to understand only the $SU(N)$ parts.

# 2 The R-matrix

We consider the vector representations of the spectral parameter depending R-matrices as rational solutions of the Yang-Baxter equations.

## 2.1 The $SU(N)$ - case

Let $V^{1\dots n}$ be the tensor product space

$$V^{1\dots n} = V_1 \otimes \dots \otimes V_n \tag{2.1}$$

where the vector spaces $V_i \cong \mathbf{C}^N$, $(i = 1, \dots, n)$ are considered as fundamental (vector) representation spaces of $SU(N)$. It is straightforward to generalize the results of this paper to the case where the $V_i$ are vector spaces for other representations. We denote the canonical basis vectors by

$$|\, \alpha_1 \dots \alpha_n \,\rangle \in V^{1\dots n}, \quad (\alpha_i = 1, \dots, N). \tag{2.2}$$

A vector $v^{1\dots n} \in V^{1\dots n}$ is given in term of its components by

$$v^{1\dots n} = \sum_{\underline{\alpha}} |\, \alpha_1 \dots \alpha_n \,\rangle\, v^{\alpha_1,\dots,\alpha_n}. \tag{2.3}$$

A matrix acting in $V^{1\dots n}$ by is denoted by

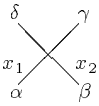
$$A_{1\dots n} \;:\; V^{1\dots n} \to V^{1\dots n}. \tag{2.4}$$

The $SU(N)$ spectral parameter dependent R-matrix [18] acts on the tensor product of two (fundamental) representation spaces of $SU(N)$. It may be written and depicted as

$$R_{12}(x_1 - x_2) = b(x_1 - x_2)\, \mathbf{1}_{12} + c(x_1 - x_2)\, P_{12} \quad = \quad \raisebox{-0.5ex}{\includegraphics[height=2.5ex]{}} \quad : V^{12} \to V^{21}, \tag{2.5}$$

where $P_{12}$ is the permutation operator. Here and in the following we associate a variable (spectral parameter) $x_i \in \mathbf{C}$ to each space $V_i$ which is graphically represented by a line labeled by $x_i$ (or simply by $i$). The components of the R-matrix are

$$R^{\delta\gamma}_{\alpha\beta}(x_1 - x_2) = \delta_{\alpha\gamma}\delta_{\beta\delta}\, b(x_1 - x_2) + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c(x_1 - x_2) \quad = \quad \tag{2.6}$$

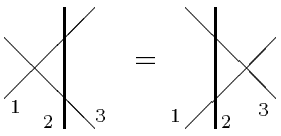



and the functions

$$b(x) = \frac{x}{x - 2/N}, \quad c(x) = \frac{-2/N}{x - 2/N} \tag{2.7}$$

are obtained as the rational solution of the Yang-Baxter equation which reads and may be depicted as

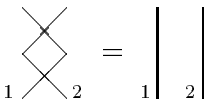
$$R_{12}(x_1 - x_2)\, R_{13}(x_1 - x_3)\, R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3)\, R_{13}(x_1 - x_3)\, R_{12}(x_1 - x_2)$$




(2.8)

where we have employed the usual notation [10]. The "unitarity" of the R-matrix reads and may depicted as

$$R_{21}(x_2 - x_1)\, R_{12}(x_1 - x_2) = 1 \quad : \quad$$




As usual we define the monodromy matrix (with $\underline{x} = x_1, \ldots, x_n$)

$$T_{1\ldots n,0}(\underline{x}, x_0) = R_{10}(x_1 - x_0)\, R_{20}(x_2 - x_0) \ldots R_{n0}(x_n - x_0) = \tag{2.9}$$

as a matrix acting in the tensor product of the "quantum space" $V^{1\ldots n}$ and the "auxiliary space" $V_0$ (all $V_i \cong \mathbf{C}^N$). The Yang-Baxter algebra relations

$$T_{1\ldots n,a}(\underline{x}, x_a)\, T_{1\ldots n,b}(\underline{x}, x_b)\, R_{ab}(x_a - x_b) = R_{ab}(x_a - x_b)\, T_{1\ldots n,b}(\underline{x}, x_b)\, T_{1\ldots n,a}(\underline{x}, x_a) \tag{2.10}$$

imply the basic algebraic properties of the sub-matrices w.r.t the auxiliary space defined by

$$T_{1\ldots n}{}^{\alpha}_{\beta}(\underline{x}, x) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{x}, x) & B_{1\ldots n\beta}(\underline{x}, x) \\ C_{1\ldots n}{}^{\alpha}(\underline{x}, x) & D_{1\ldots n}{}^{\alpha}_{\beta}(\underline{x}, x) \end{pmatrix} . \tag{2.11}$$

The indices $\alpha, \beta$ on the left hand side run from 1 to $N$ and on the right hand side from 2 to $N$. The commutation rules which we will need later are

$$B_{1\ldots n\alpha}(\underline{x}, x')\, B_{1\ldots n\beta}(\underline{x}, x) = B_{1\ldots n\beta'}(\underline{x}, x)\, B_{1\ldots n\alpha'}(\underline{x}, x')\, R^{\alpha'\beta'}_{\beta\alpha}(x - x'), \tag{2.12}$$

$$A_{1\dots n}(\underline{x},x')\,B_{1\dots n\beta}(\underline{x},x) \;=\; \frac{1}{b(x'-x)}\,B_{1\dots n\beta}(\underline{x},x)\,A_{1\dots n}(\underline{x},x') - \frac{c(x'-x)}{b(x'-x)}\,B_{1\dots n\beta}(\underline{x},x')\,A_{1\dots n}(\underline{x},x) \tag{2.13}$$

and

$$D_{1\dots n\gamma}^{\ \ \gamma'}(\underline{x},x')\,B_{1\dots n\beta}(\underline{x},x) \;=\; \frac{1}{b(x-x')}\,B_{1\dots n\beta'}(\underline{x},x)\,D_{1\dots n\gamma''}^{\ \ \gamma'}(\underline{x},x')\,R_{\beta\gamma}^{\gamma''\beta'}(x-x') - \frac{c(x-x')}{b(x-x')}\,B_{1\dots n\gamma}(\underline{x},x')\,D_{1\dots n\beta}^{\ \ \gamma'}(\underline{x},x). \tag{2.14}$$

## 2.2 The $U(N)$ - case

Let $V^{1\dots n}$ be the tensor product space

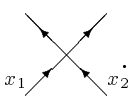
$$V^{1\dots n} = V_1 \otimes \dots \otimes V_n\ , \quad V_i = (V \oplus \bar{V})_i. \tag{2.15}$$

Here $V \cong \mathbf{C^N}$ is considered as a fundamental representation space of $U(N)$ and $\bar{V} \cong \mathbf{C^N}$ as the conjugate representation space. The vectors of $V$ are sometimes called "particles" with positive $U(1)$ charge and those of $\bar{V}$ "anti-particles" with negative $U(1)$ charge. We will also use instead of $1,2,\dots$ the indices $a,b$, etc. to refer to the spaces $V_a, V_b$, etc. It is straightforward to generalize the results of this paper to the case where the $V_i$ are vector spaces for other representations. We denote the canonical basis vectors by

$$|\,\alpha_1 \dots \alpha_n\,\rangle \in V^{1\dots n}, \quad \Big(\alpha_i = (1,+),\dots,(N,+),(1,-),\dots,(N,-)\Big). \tag{2.16}$$

We will later also use the simpler notation $(\alpha,+) \equiv \alpha$ for particles and $(\alpha,-) \equiv \bar{\alpha}$ for anti-particles. A vector in $V^{1\dots n}$ is denoted by $v^{1\dots n}$ and a matrix acting in $V^{1\dots n}$ by $A_{1\dots n} \in End(V^{1\dots n})$.

The $U(N)$ spectral parameter depending R-matrix $R_{12}(x_1 - x_2)$ [18] acting on the tensor product of two particle spaces $V^{12} = V \otimes V$ coincides with that of $SU(N)$ used in Subsection 2.1 given by eq. (2.5). Here it will be depicted as

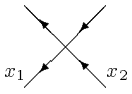



(2.17)

Here and in the following we associate a variable (spectral parameter) $x_i \in \mathbf{C}$ to each space $V_i$ which is graphically represented by a line labeled by $x_i$ (or simply by $i$). For the $U(N)$-case in addition an arrow on the lines denotes the $U(1)$-charge flow. (If we do not want to specify the direction of charge flow, we draw no arrow.) For $U(N)$ there is

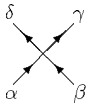
in addition the R-matrix acting on the tensor product of the anti-particle particle spaces $V^{\bar{1}2} = \bar{V} \otimes V$

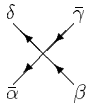
$$R_{\bar{1}2}(x_1 - x_2) = \mathbf{1}_{\bar{1}2} + d(x_1 - x_2)\, K_{\bar{1}2} \quad = \quad : V^{\bar{1}2} \to V^{2\bar{1}}, \tag{2.18}$$




where $K_{\bar{1}2}$ is the annihilation-creation matrix. There is no particle-anti-particle "backward scattering" (see e.g. [18]). The components of the R-matrices are

$$R^{\delta\gamma}_{\alpha\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\, b + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c \quad = \quad , \quad R^{\delta\bar{\gamma}}_{\bar{\alpha}\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\beta}\delta_{\delta\gamma}\, d \quad = \tag{2.19}$$



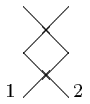



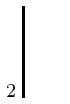

with $\alpha = (\alpha, +)$ etc. and $\bar{\alpha} = (\alpha, -)$ etc. The functions

$$b(x) = \frac{x}{x - 2/N}, \quad c(x) = \frac{-2/N}{x - 2/N}, \quad d(x) = \frac{2/N}{x - 1}. \tag{2.20}$$

belong to the rational solution of the Yang-Baxter equations (2.8) [18]. which holds here for all possible charge flows. The matrices $R_{1\bar{2}}$ and $R_{\bar{1}\bar{2}}$ have the same matrix elements as $R_{\bar{1}2}$ and $R_{12}$, respectively. The inversion ("unitarity") relation of the R-matrix reads and may be depicted as

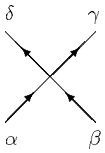
$$R_{21}(x_2 - x_1)\, R_{12}(x_1 - x_2) = 1 \quad : \quad = \tag{2.21}$$



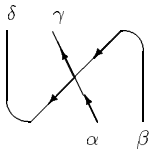




again for all possibilities charge flows. A further property of the $U(N)$ R-matrices is crossing and may be written and depicted as

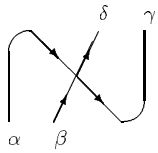
$$R^{\delta\gamma}_{\alpha\beta}(x_1 - x_2) = b(x_1 - x_2)\, R^{\gamma\bar{\beta}}_{\bar{\delta}\alpha}((x_2 + 1) - x_1) = b(x_1 - x_2)\, R^{\bar{\alpha}\delta}_{\beta\bar{\gamma}}(x_2 - (x_1 - 1))$$



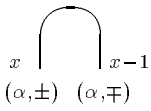

$= b$

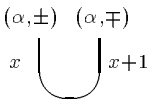



$= b$




where again we have used the notation $\alpha = (\alpha, +)$ etc. and $\bar{\alpha} = (\alpha, -)$ etc. We have introduced the graphical rule that a line changing the "time direction" also interchanges particles and anti-particles and changes $x \to x \pm 1$ as follows

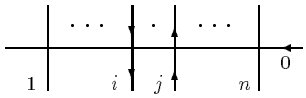



. (2.22)

Similar as above we introduce a monodromy matrix

$$T_{1\ldots n,0}(\underline{x}; x_0) = R_{10}(x_1 - x_0) \ldots R_{n0}(x_n - x_0) \tag{2.23}$$

$=$ 


as a matrix acting in the tensor product of the "quantum space" $V^{1\dots n}$ and the "auxiliary space" $V_0 \cong V \cong \mathbf{C}^N$ as a particle space. Since there is no "charge reflection" the positions of the particles and the anti-particles will not change under the application of $T_{1\dots n,0}$ to a state in $V^{1\dots n}$. (The construction of the Bethe states will be not symmetric with respect to particles and anti-particles, because we only use the monodromy matrix with particles for the auxiliary space.)

The Yang-Baxter algebra relations (2.10) also hold for the $U(N)$-monodromy matrix and the resulting commutation rules (2.12) - (2.14) of the $A, B, C, D$ matrices are the same as for the $SU(N)$ case of Subsection 2.1.

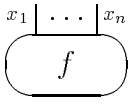
# 3 The matrix difference equation and "generalized Bethe vectors".

In this section we formulate the matrix difference equations for vector valued functions and solve them by means of the nested version of the Bethe Ansatz, a variant of the Bethe Ansatz also called "off-shell Bethe Ansatz". We will call the solutions "generalized Bethe vectors". The conventional Bethe Ansatz is used to solve eigen value problems and leads to the "Bethe Ansatz equations". This system of equations can usually be solved only for particular cases. Here the Bethe Ansatz leads to some simple functional equations which can be solved easily and the solutions of the difference equations are given in terms of infinite sums called "Jackson-type integrals".

## 3.1 The $SU(N)$ - case

Let

$$f^{1\dots n}(\underline{x}) = \quad \in V^{1\dots n}$$




be a vector valued function of $\underline{x} = (x_1, \dots, x_n) \in \mathbf{C}^n$ with values in $V^{1\dots n}$. The components of this vector are denoted by

$$f^{\alpha_1\dots\alpha_n}(\underline{x}) \ , \quad (1 \le \alpha_i \le N).$$

**Conditions 3.1** *The following symmetry and periodicity conditions of the vector valued function $f^{1\dots n}(\underline{x})$ are supposed to be valid:*

(i) *The symmetry property under the exchange of two neighboring spaces $V_i$ and $V_j$ and the variables $x_i$ and $x_j$, at the same time, is given by*

$$f^{\dots ji\dots}(\dots, x_j, x_i, \dots) = R_{ij}(x_i - x_j)\, f^{\dots ij\dots}(\dots, x_i, x_j, \dots). \tag{3.1}$$

(ii) *The* **system of matrix difference equations** *holds*

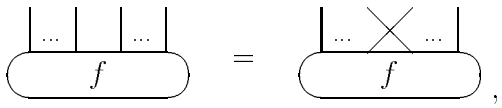
$$\boxed{f^{1\dots n}(\dots, x_i + 2, \dots) = Q_{1\dots n}(\underline{x}; i)\, f^{1\dots n}(\dots, x_i, \dots)\ , \quad (i = 1, \dots, n)} \tag{3.2}$$

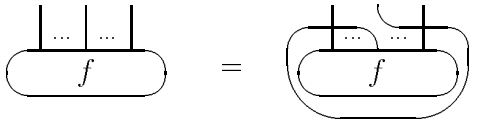
*where the matrices $Q_{1\dots n}(\underline{x}; i) \in End(V^{1\dots n})$ are defined by*

$$Q_{1\dots n}(\underline{x}; i) = R_{i+1i}(x_{i+1} - x_i') \dots R_{ni}(x_n - x_i')\, R_{1i}(x_1 - x_i) \dots R_{i-1i}(x_{i-1} - x_i) \tag{3.3}$$

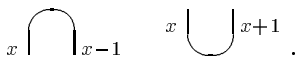
*with $x_i' = x_i + 2$.*

The Yang-Baxter equations for the R-matrix guarantee that these conditions are compatible. The shift of 2 in eq. (3.2) could be replaced by an arbitrary $\kappa$. For the application to the form factor problem, however, it is fixed to 2 because of crossing symmetry. Conditions 3.1 (i) and (ii) may be depicted as

(i) 


(ii) 


with the graphical rule that a line changing the "time direction" changes the spectral parameters $x \to x \pm 1$ as follows




The $Q_{1\dots n}(\underline{x}; i)$ fulfill the commutation rules

$$\begin{aligned} &Q_{1\dots n}(\dots x_i \dots x_j + 2 \dots; i)\, Q_{1\dots n}(\dots x_i \dots x_j \dots; j) \\ &\quad = Q_{1\dots n}(\dots x_i + 2 \dots x_j \dots; j)\, Q_{1\dots n}(\dots x_i \dots x_j \dots; i). \end{aligned} \tag{3.4}$$

The following Proposition is obvious

**Proposition 3.2** *Let the vector valued function $f^{1\dots n}(\underline{x}) \in V^{1\dots n}$ fulfill Condition 3.1 (i), then Conditions 3.1 (ii) for all $i = 1, \dots, n$ are equivalent to each other and also equivalent to the following periodicity property under cyclic permutation of the spaces and the variables*

$$f^{12\dots n}(x_1, x_2, \dots, x_n + 2) = f^{n1\dots n-1}(x_n, x_1, \dots, x_{n-1}). \tag{3.5}$$

**Remark 3.3** *The equations (3.1,3.5) imply Watson's (1.1) equations for the form factors [17].*

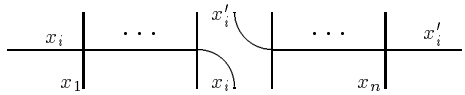
For later convenience we write the matrices

$$Q_{1\dots n}(\underline{x};i) = \mathrm{tr}_0\, T^Q_{1\dots n,0}(\underline{x};i) \tag{3.6}$$

as the trace of of a new type of monodromy matrices where to the horizontal line two different spectral parameters are associated, namely one to the right hand side and the other one to the left hand side. However, both are related to a spectral parameter of one of the vertical lines. This new monodromy matrix is given by the following

**Definition 3.4** *For* $i = 1,\dots,n$

$$T^Q_{1\dots n,0}(\underline{x};i) = R_{10}(x_1 - x_i)\dots R_{i-10}(x_{i-1} - x_i)\, P_{i0}\, R_{i+10}(x_{i+1} - x'_i)\dots R_{n0}(x_n - x'_i) \tag{3.7}$$

$=$ 


*with* $x'_i = x_i + 2$.

Note that for $i = n$ one has simply $T^Q_{1\dots n,0}(\underline{x};n) = T_{1\dots n,0}(\underline{x},x_n)$ since $R(0)$ is the permutation operator $P$.

The new type of monodromy matrix fulfills a new type of Yang-Baxter relation. Instead of eq. (2.10) we have for $i = 1,\dots,n$

$$T^Q_{1\dots n,a}(\underline{x};i)\, T_{1\dots n,b}(\underline{x},u)\, R_{ab}(x'_i - u) = R_{ab}(x_i - u)\, T_{1\dots n,b}(\underline{x}',u)\, T^Q_{1\dots n,a}(\underline{x};i) \tag{3.8}$$

with $\underline{x}' = x_1,\dots,x'_i,\dots,x_n$ and $x'_i = x_i + 2$. This relation follows from the Yang-Baxter equation for the R-matrix and the obvious relation for the permutations operator $P$

$$P_{ia}\, R_{ib}(x_i - u)\, R_{ab}(x'_i - u) = R_{ab}(x_i - u)\, R_{ib}(x'_i - u)\, P_{ia}.$$

Correspondingly to eq. (2.11) we introduce (suppressing the indices $1\dots n$)

$$T^{Q\,\alpha}{}_{\beta}(\underline{x};i) \equiv \begin{pmatrix} A^Q(\underline{x};i) & B^Q{}_{\beta}(\underline{x};i) \\ C^{Q\,\alpha}(\underline{x};i) & D^{Q\,\alpha}{}_{\beta}(\underline{x};i) \end{pmatrix}. \tag{3.9}$$

with the commutation rules with respect to the usual $A, B, C, D$

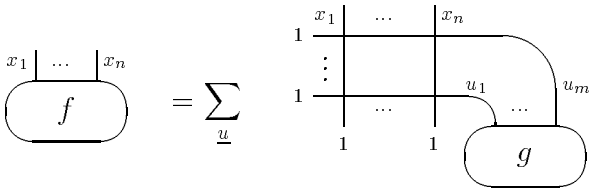
$$A^Q(\underline{x};i)\, B_b(\underline{x},u) = \frac{1}{b(x'_i - u)}\, B_b(\underline{x}',u)\, A^Q(\underline{x};i) - \frac{c(x'_i - u)}{b(x'_i - u)}\, B^Q{}_b(\underline{x};i)\, A(\underline{x},u), \tag{3.10}$$

$$D^{Q}{}_{a}(\underline{x};i)\,B_b(\underline{x},u) \;=\; \frac{1}{b(u-x_i)}\,B_b(\underline{x}',u)\,D^{Q}{}_{a}(\underline{x};i)\,R_{ba}(u-x_i') - \frac{c(u-x_i)}{b(u-x_i)}\,B^{Q}{}_{b}(\underline{x};i)\,D_a(\underline{x},u)\,P_{ab}. \tag{3.11}$$

The system of difference equations (3.2) can be solved by means of a generalized ("off-shell") nested Bethe Ansatz. The first level is given by the

**BETHE ANSATZ 3.5**

$$f^{1\ldots n}(\underline{x}) = \sum_{\underline{u}}\; B_{1\ldots n\,\beta_m}(\underline{x},u_m)\ldots B_{1\ldots n\,\beta_1}(\underline{x},u_1)\,\Omega^{1\ldots n}\; g^{\beta_1\ldots\beta_m}(\underline{x},\underline{u}) \tag{3.12}$$




*where summation over* $\beta_1,\ldots,\beta_m$ *is assumed and* $\Omega^{1\ldots n} \in V^{1\ldots n}$ *is the reference state defined by* $C_{1\ldots n}{}^{\beta}\,\Omega^{1\ldots n} = 0$ *for* $1 < \beta \le N$. *The summation over* $\underline{u}$ *is specified by*

$$\underline{u} = (u_1,\ldots,u_m) = (\tilde{u}_1 - 2l_1,\ldots,\tilde{u}_m - 2l_m)\;, \quad l_i \in \mathbf{Z}, \tag{3.13}$$

*where the* $\tilde{u}_i$ *are arbitrary constants.*

The reference state is

$$\Omega^{1\ldots n} = |\,1\ldots1\,\rangle, \tag{3.14}$$

a basis vector with components $\Omega^{\alpha_1\ldots\alpha_n} = \prod_{i=1}^{n}\delta_{\alpha_i 1}$. It is an eigenstate of $A_{1\ldots n}$ and $D_{1\ldots n}$

$$A_{1\ldots n}(\underline{x},u)\Omega^{1\ldots n} = \Omega^{1\ldots n}\;, \quad D_{1\ldots n\,\beta}^{\;\alpha}(\underline{x},u)\Omega^{1\ldots n} = \Omega^{1\ldots n}\,\delta_{\alpha\beta}\prod_{i=1}^{n} b(x_i-u). \tag{3.15}$$

The sums (3.12) are also called "Jackson-type integrals" (see e.g. [9] and references therein). Note that the summations over $\beta_i$ run only over $1 < \beta_i \le N$. Therefore the $g^{\beta_1\ldots\beta_m}$ are the components of a vector $g^{b_1\ldots b_m}$ in the tensor product of smaller spaces

$$g^{b_1\ldots b_m} \in V^{(1)\,b_1\ldots b_m} = V_1^{(1)}\otimes\ldots\otimes V_m^{(1)} \tag{3.16}$$

with

$$V_i^{(1)} = \{\,|\,\beta\,\rangle; 1 < \beta \le N\} \cong \mathbf{C}^{N-1}. \tag{3.17}$$

We define a new vector valued function $f^{(1)\,b_1\ldots b_m}(\underline{u}) \in V^{(1)\,b_1\ldots b_m}$ by

**Definition 3.6** *Let the vector valued function* $g^{b_1\dots b_m}(\underline{u}) \in V^{(1)^{b_1\dots b_m}}$ *be given by*

$$g^{b_1\dots b_m}(\underline{x},\underline{u}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\psi(x_i-u_j)\prod_{1\le i<j\le m}\tau(u_i-u_j)\,f^{(1)^{b_1\dots b_m}}(\underline{u}) \tag{3.18}$$

*where the functions* $\psi(x)$ *and* $\tau(x)$ *fulfill the functional equations*

$$b(x)\,\psi(x) = \psi(x-2)\ , \quad \frac{\tau(x)}{b(x)} = \frac{\tau(x-2)}{b(2-x)}. \tag{3.19}$$

Using the definition of $b(x)$ (2.7) we get the solutions of eqs. (3.19)

$$\psi(x) = \frac{\Gamma(1-\frac{1}{N}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}, \quad \tau(x) = \frac{x\,\Gamma(\frac{1}{N}+\frac{x}{2})}{\Gamma(1-\frac{1}{N}+\frac{x}{2})} \tag{3.20}$$

where the general solutions are obtained by multiplication with arbitrary periodic functions with period 2. Just as $g^{1\dots m}(\underline{x},\underline{u})$ also the vector valued function $f^{(1)^{b_1\dots b_m}}(\underline{u})$ is an element of the tensor product of the smaller spaces $V_i \cong \mathbf{C}^N$

$$f^{(1)^{b_1\dots b_m}}(\underline{u}) \in V^{(1)^{b_1\dots b_m}}.$$

We say $f^{(1)^{b_1\dots b_m}}(\underline{u})$ fulfills Conditions 3.1 (i)$^{(1)}$ and (ii)$^{(1)}$ if eqs. (3.1) and (3.2) hold in this space, which means that everywhere in eqs. (3.1) and (3.3) the matrix $R$ has to be replaced by

$$R^{(1)} \equiv R \text{ restrictedto } V^{(1)}\otimes V^{(1)} \tag{3.21}$$

(cf eq. (3.17)). We are now in a position to formulate the main theorem of this paper.

**Theorem 3.7** *Let the vector valued function* $f^{1\dots n}(\underline{x})$ *be given by the Bethe Ansatz 3.5 and let* $g^{b_1\dots b_m}(\underline{x},\underline{u})$ *be of the form of Definition 3.6. If in addition the vector valued function* $f^{(1)^{b_1\dots b_m}}(\underline{u}) \in V^{(1)^{b_1\dots b_m}}$ *fulfills the Conditions 3.1* (i)$^{(1)}$ *and* (ii)$^{(1)}$*, then* $f^{1\dots n}(\underline{x}) \in V^{1\dots n}$ *fulfills the Conditions 3.1* (i) *and* (ii)*, i.e.* $f^{1\dots n}(\underline{x})$ *is a solution of the set of difference equations (3.2).*

**Remark 3.8** *For* $SU(2)$ *(see e.g.[9]) the problem is already solved by 3.18 since then* $f^{(1)}$ *is a constant.*

**Proof:** Condition 3.1 (i) follows directly from the definition and the normalization of the R-matrix (2.5)

$$R_{ij}(x_i-x_j)\,\Omega^{\dots ij\dots} = \Omega^{\dots ij\dots},$$

the symmetry of $g^{1\dots m}(\underline{x},\underline{u})$ given by eq. (3.18) under the exchange of $x_1,\dots,x_n$ and

$$B_{\dots ji\dots,\beta}(\dots x_j,x_i\dots,u)\,R_{ij}(x_i-x_j) = R_{ij}(x_i-x_j)\,B_{\dots ij\dots,\beta}(\dots x_i,x_j\dots,u)$$

which is a consequence of the Yang-Baxter relations for the R-matrix.

Because of Proposition 3.2 it is sufficient to prove Condition 3.1 (ii) only for $i = n$

$$Q(\underline{x};n)\, f(\underline{x}) = \mathrm{tr}_a\, T_a^Q(\underline{x};n)\, f(\underline{x}) = f(\underline{x}')\ , \quad (\underline{x}' = x_1, \ldots, x_n' = x_n + 2).$$

where the indices $1 \ldots n$ have been suppressed. For the first step we apply a technique quite analogous to that used for the conventional algebraic Bethe Ansatz which solves eigenvalue problems. We apply the trace of $T_a^Q(\underline{x};n)$ to the vector $f(\underline{x})$ as given by eq. (3.12) and push $A^Q(\underline{x};n)$ and $D_a^Q(\underline{x};n)$ through all the $B$'s using the commutation rules (3.10) and (3.11). Again with $\underline{x}' = x_1, \ldots, x_n' = x_n + 2$ we obtain

$$A^Q(\underline{x};n)\, B_{b_m}(\underline{x},u_m) \ldots B_{b_1}(\underline{x},u_1) = B_{b_m}(\underline{x}',u_m) \ldots B_{b_1}(\underline{x}',u_1) \prod_{j=1}^{m} \frac{1}{b(x_n' - u_j)} \times A^Q(\underline{x};n) + \mathrm{uw}_A,$$

$$D_a^Q(\underline{x};n)\, B_{b_m}(\underline{x},u_m) \ldots B_{b_1}(\underline{x},u_1) = B_{b_m}(\underline{x}',u_m) \ldots B_{b_1}(\underline{x}',u_1) \prod_{j=1}^{m} \frac{1}{b(u_j - x_n)} \times D_a^Q(\underline{x};n)\, R^{(1)}_{b_1 a}(u_1 - x_n') \ldots R^{(1)}_{b_m a}(u_m - x_n') + \mathrm{uw}_{D_a},$$

where $R^{(1)}$ is defined by (3.21). The "wanted terms" written explicitly originate from the first term in the commutations rules (3.10) and (3.11); all other contributions yield the "unwanted terms". If we insert these equations into the representation (3.12) of $f(\underline{x})$ we find that the wanted contribution from $A^Q$ already gives the desired result. The wanted contribution from $D^Q$ applied to $\Omega$ gives zero. The unwanted contributions can be written as as difference which vanishes after summation over the $u$'s. These three facts can be seen as follows. We have

$$A^Q(\underline{x};n)\, \Omega = \Omega\ , \quad D_a^Q(\underline{x};n)\, \Omega = 0$$

which follow from eq. (3.15) since $T^Q(\underline{x};n) = T(\underline{x},x_n)$ and $b(0) = 0$. The defining relation of $\psi(x)$ (3.19) implies that the wanted term from $A$ yields $f(\underline{x}')$. The commutation relations (3.10), (3.11), (2.13) and (2.14) imply that the unwanted terms are proportional to a product of $B$-operators, where exactly one $B_{b_j}(\underline{x},u_j)$ is replaced by $B_{b_j}^Q(\underline{x};n)$. Because of the commutation relations of the $B$'s (2.12) and the symmetry property given by Condition 3.1 (i)$^{(1)}$ of $g^{1\ldots m}(\underline{x},\underline{u})$ it is sufficient to consider only the unwanted terms for $j = m$ denoted by $\mathrm{uw}_A^m$ and $\mathrm{uw}_D^m$. They come from the second term in (3.10) if $A^Q(\underline{x};n)$ is commuted with $B_{b_m}(\underline{x},u_m)$ and then the resulting $A(\underline{x},u_m)$ pushed through the other $B$'s taking only the first terms in (2.13) into account and correspondingly for $D_a^Q(\underline{x};u_m)$.

$$\mathrm{uw}_A^m = -\frac{c(x_n' - u_m)}{b(x_n' - u_m)}\, B_{b_m}^Q(\underline{x};m) \ldots B_{b_1}(\underline{x},u_1) \prod_{j<m} \frac{1}{b(u_m - u_j)}\, A(\underline{x},u_m)$$

$$\mathrm{uw}^m_{D_a} = -\frac{c(u_m - x_n)}{b(u_m - x_n)}\, B^Q_{b_m}(\underline{x}; m) \ldots B_{b_1}(\underline{x}, u_1) \prod_{j<m} \frac{1}{b(u_j - u_m)}\, D_a(\underline{x}, u_m)\, T^{Q(1)}_{b_1 \ldots b_m, a}(\underline{u}; m)$$

where $T^{Q(1)}$ is the new type of monodromy matrix

$$T^{Q(1)}_{b_1 \ldots b_m, a}(\underline{u}; m) = R^{(1)}_{b_1 a}(u_1 - u_m) \ldots R^{(1)}_{b_{m-1} a}(u_{m-1} - u_m)\, P_{b_m a} \tag{3.22}$$

analogous to (3.6) whose trace over the auxiliary space $V^{(1)}_a$ yields the shift operator $Q^{(1)}(\underline{u}; m)$. With $D_a(\underline{x}, u_m)\Omega = \mathbf{1}_a \prod_{i=1}^n b(x_i - u_m)\,\Omega$ (see (3.15)), by the assumption

$$Q^{(1)}(\underline{u}; m)\, f^{(1)}(\underline{u}) = f^{(1)}(\underline{u}') \ , \quad (\underline{u}' = u_1, \ldots, u'_m = u_m + 2)$$

and the defining relations (3.19) of $\psi(x)$ and $\tau(x)$, we obtain

$$\mathrm{tr}_a\, \mathrm{uw}^m_{D_a}(\underline{u})\, \Omega\, g(\underline{x}, \underline{u}) = -\mathrm{uw}^m_A(\underline{u}')\, \Omega\, g(\underline{x}, \underline{u}')$$

where $c(-x)/b(-x) = -c(x)/b(x)$ has been used. Therefore the sum of all unwanted terms yield a difference analog of a total differential which vanishes after summation over the $u$'s.

Iterating Theorem 3.7 we obtain the nested generalized Bethe Ansatz with levels $k = 1, \ldots, N-1$. The Ansatz of level $k$ reads

$$f^{(k-1)^{1 \ldots n_{k-1}}}\left(\underline{x}^{(k-1)}\right) = \sum_{\underline{x}^{(k)}} B^{(k-1)}_{1 \ldots n_{k-1} \beta_{n_k}}\left(\underline{x}^{(k-1)}, x^{(k)}_{n_k}\right) \ldots \tag{3.23}$$

$$\ldots B^{(k-1)}_{1 \ldots n_{k-1} \beta_1}\left(\underline{x}^{(k-1)}, x^{(k)}_1\right)\, \Omega^{(k-1)^{1 \ldots n_{k-1}}}\, g^{(k-1)^{\beta_1 \ldots \beta_{n_k}}}\left(\underline{x}^{(k-1)}, \underline{x}^{(k)}\right)$$

The functions $f^{(k)}$ and $g^{(k)}$ are vectors with

$$f^{(k)^{1 \ldots n_k}}, g^{(k-1)^{1 \ldots n_k}} \in V^{(k)^{1 \ldots n_k}} = V^{(k)}_1 \otimes \ldots \otimes V^{(k)}_{n_k} \ , \quad (V^{(k)}_i \cong \mathbf{C}^{N-k}).$$

The basis vectors of these spaces are $|\, \alpha_1 \ldots \alpha_{n_k}\, \rangle^{(k)} \in V^{(k)^{1 \ldots n_k}}$ and $k < \alpha_i \leq N$.

Analogously to Definition 3.6 we write

$$g^{(k-1)^{1 \ldots n_k}}\left(\underline{x}^{(k-1)}, \underline{x}^{(k)}\right) = \prod_{i=1}^{n_{k-1}} \prod_{j=1}^{n_k} \psi\left(x^{(k-1)}_i - x^{(k)}_j\right) \prod_{1 \leq i < j \leq n_k} \tau\left(x^{(k)}_i - x^{(k)}_j\right)\, f^{(k)^{1 \ldots n_k}}\left(\underline{x}^{(k)}\right) \tag{3.24}$$

where the functions $\psi(x)$ and $\tau(x)$ fulfill the functional equations (3.19) with the solutions (3.20). Then the start of the iteration is given by a $k_{max} \leq N$ with

$$f^{(k_{max}-1)^{1 \ldots n_{n_{k_{max}-1}}}} = |\, k_{max} \ldots k_{max}\, \rangle \ , \quad \text{and} \quad n_k = 0 \ \text{ for } k \geq k_{max} \tag{3.25}$$

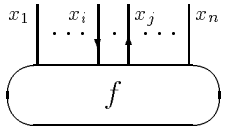
which is the reference state of level $k_{max} - 1$ and trivially fulfills the Conditions 3.1.

**Corollary 3.9** *The system of $SU(N)$ matrix difference equations (3.2) is solved by the nested Bethe Ansatz (3.23) with (3.24), (3.25) and $f^{1 \ldots n}(\underline{x}) = f^{(0)^{1 \ldots n}}(\underline{x})$.*

## 3.2 The $U(N)$ - case

Let again

$$f^{1\dots n}(\underline{x}) = \quad \in V^{1\dots n}$$




be a vector valued function of $\underline{x} = x_1, \dots, x_n$ with values in $V^{1\dots n}$ which is now given by eq. (2.15).

Analogously to the $SU(N)$-case in Subsection 3.1 we constrain the function $f^{1\dots n}$ by the Conditions 3.1. The $Q_{1\dots n}(\underline{x}; i)$ fulfill the same commutation rules (3.4) as in Subsection 3.1. Also Proposition 3.2 and Remark 3.3 hold for the $U(N)$ case. We write the Q-matrices again as the trace (3.6) of the new type of monodromy matrices analogous to those given by Definition 3.4. We use this monodromy matrix if the index $i$ is associated to a particle. If the index $i$ is associated to an anti-particle, we will use an additional monodromy matrix of a new type which yields the inverse of the Q-operators.

$$Q^{-1}_{1\dots n}(\underline{x}; i) = \tilde{Q}_{1\dots n}(\underline{x}; i) = \mathrm{tr}_{\bar{0}}\, \tilde{T}^{Q}_{\bar{0},1\dots n}(\underline{x}; i). \tag{3.26}$$

It is given by the following

**Definition 3.10** *For* $i = 1, \dots, n$

$$\tilde{T}^{Q}_{\bar{0},1\dots n}(\underline{x}; i) = R_{\bar{0}1}(x_i - x_1) \dots R_{\bar{0}i-1}(x_i - x_{i-1})\, P_{\bar{0}\bar{i}}\, R_{\bar{0}i+1}(x'_i - x_{i+1}) \dots R_{\bar{0}n}(x'_i - x_n) \tag{3.27}$$

*with the auxiliary space* $V_{\bar{0}} = \bar{V}$ *and* $x'_i = x_i + 2$.

The "unitarity" of the R-matrix (2.21) implies (3.26).

The new type of monodromy matrix $\tilde{T}^{Q}_{\bar{0},1\dots n}$ fulfills a new type of Yang-Baxter relation of the form

$$\tilde{T}^{Q}_{\bar{a},1\dots n}(\underline{x}; i)\, R_{\bar{a}b}(x_i - u)\, T_{1\dots n,b}(\underline{x}', u) = T_{1\dots n,b}(\underline{x}, u)\, R_{\bar{a}b}(x'_i - u)\, \tilde{T}^{Q}_{\bar{a},1\dots n}(\underline{x}; i) \tag{3.28}$$

with $\underline{x}' = x_1, \dots, x'_i, \dots, x_n$ and $x'_i = x_i + 2$. In addition to the commutation rules (3.10) and (3.11) for the case that the indices $i$ and $a$ belong to particles, we have the commutation rules for the case that the indices $i$ and $a$ correspond to anti-particles

$$\tilde{A}^{Q}(\underline{x}; i)\, B_b(\underline{x}', u) = \frac{1 - d^2(x'_i - u)}{1 + d(x_i - u)}\, B_b(\underline{x}', u)\, \tilde{A}^{Q}(\underline{x}; i) + d(x'_i - u)\, \tilde{B}^{Q}_{\bar{b}}(\underline{x}; i)\, A(\underline{x}', u) \tag{3.29}$$

$$\tilde{D}^{Q}_{\bar{a}}(\underline{x}; i)\, B_b(\underline{x}', u) = B_b(\underline{x}, u)\, R_{\bar{a}b}(x'_i - u)\, \tilde{D}^{Q}_{\bar{a}}(\underline{x}; i) - d(x_i - u)\, \tilde{B}^{Q}_{\bar{a}}(\underline{x}; i)\, K_{\bar{a}b}\, D_b(\underline{x}', u). \tag{3.30}$$

The system of difference equations (3.2) can be solved by means of a generalized ("off-shell") nested Bethe Ansatz. The first level is given by the

**BETHE ANSATZ 3.11**

$$f^{1\dots n}(\underline{x}) = \sum_{\underline{u}} B_{1\dots n\beta_m}(\underline{x}, u_m)\dots B_{1\dots n\beta_1}(\underline{x}, u_1)\, g^{1\dots n\beta_1\dots\beta_m}(\underline{x}, \underline{u}) \tag{3.31}$$

*where summation over* $1 < \beta_1, \dots, \beta_m \leq N$ *is assumed. The summation over* $\underline{u}$ *is specified by*

$$\underline{u} = (u_1, \dots, u_m) = (\tilde{u}_1 - 2l_1, \dots, \tilde{u}_m - 2l_m)\ , \quad l_i \in \mathbf{Z}, \tag{3.32}$$

*where the* $\tilde{u}_i$ *are arbitrary constants. For a fixed set of indices* $\underline{\beta} = \beta_1, \dots, \beta_m$ $(1 < \beta_i \leq N)$ *the reference state vector* $g^{1\dots n\underline{\beta}} \in V^{1\dots n}$ *fulfills*

$$C_{1\dots n}{}^{\gamma}\, g^{1\dots n\underline{\beta}} = 0 \quad (1 < \gamma \leq N) \tag{3.33}$$

Note that we have denoted here by $g$, what was called $\Omega\, g$ in Subsection 3.1. This notation is more convenient, since in contrast to the $SU(N)$ case the space of reference states is higher dimensional for $U(N)$. This space $V_{ref}$ of states fulfilling (3.33) is spanned by all basis vectors of the form

$$|\, \alpha_1, \dots, \alpha_n \,\rangle\ , \quad \alpha_i = \begin{cases} (1, +) & \text{for particles} \\ (\alpha_i, -)\ , \quad (1 < \alpha_i \leq N) & \text{for antiparticles.} \end{cases}$$

The $U(1)$-charge is left invariant by operators like the monodromy matrices $T_{1\dots n,0}$, $T^{Q}_{1\dots n,0}$, $\tilde{T}^{Q}_{\bar{0},1\dots n}$ and also by the operations of Conditions 3.1. Therefore the space $V^{1\dots n}$ decomposes into invariant subspaces of fixed charge, i.e. fixed numbers $n_-, n_+$ $(n_- + n_+ = n)$ of anti-particles and particles, respectively. Moreover because of the symmetry property (i) it is sufficient to consider the even smaller subspaces $V^{I_-I_+}$ where the anti-particles and the particles are sitting at fixed places $I_-$ and $I_+$, respectively

$$V^{I_-I_+} = V^{I_-} \otimes V^{I_+}\ , \qquad V^{I_-} = \bigotimes_{i\in I_-} \bar{V}_i\ , \qquad V^{I_+} = \bigotimes_{i\in I_+} V_i, \tag{3.34}$$

such that

$$V^{1\dots n} = \bigoplus_{I_-\cup I_+ = \{1,\dots,n\}} V^{I_-I_+}. \tag{3.35}$$

In the following we consider the case were the anti-particles are sitting at the first $n_-$ places, i.e. $I_- = \{1, \dots, n_-\}$. Then the space of reference states in the subspace $V^{I_-I_+}$ may be written as

$$V_{ref} \cap V^{I_-I_+} \cong \bar{V}_1^{(1)} \otimes \dots \otimes \bar{V}_{n_-}^{(1)} \otimes \Omega^{I_+} \tag{3.36}$$

where the spaces $\bar{V}_i^{(1)} \cong \mathbf{C}^{N-1}$ contain only anti-particle states with $\bar{\alpha} > 1$ and the vector $\Omega^{I_+}$ consists of particles with $\alpha = 1$

$$\Omega^{I_+} = |\, 1 \dots 1 \,\rangle \in V^{I_+}$$

Therefore, similar as in (3.16) $g$ may also be considered as a vector

$$g^{1\ldots nb_1\ldots b_m} \in {V^{(1)}}^{I_- b_1\ldots b_m} \otimes \Omega^{I_+}$$

with

$${V^{(1)}}^{I_- b_1\ldots b_m} = \bigotimes_{i\in I_-} \bar{V}_i^{(1)} \otimes V_{b_1}^{(1)} \otimes \ldots \otimes V_{b_m}^{(1)} \tag{3.37}$$

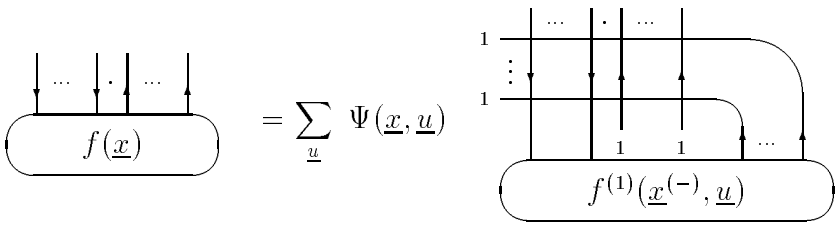
where the $\bar{V}_i^{(1)} \cong \mathbf{C}^{N-1}$ are the smaller anti-particle spaces and the $V_{b_i}^{(1)} \cong \mathbf{C}^{N-1}$ are smaller particle spaces. Again as in Subsection 3.1 we define a new vector valued function ${f^{(1)}}^{I_- b_1\ldots b_m} \in {V^{(1)}}^{I_- b_1\ldots b_m}$ by

**Definition 3.12** *Let $g^{1\ldots nb_1\ldots b_m}(\underline{x},\underline{u})$ be given in the subspace with fixed positions of anti-particles $I_-$ and particles $I_+$ as*

$$g^{1\ldots nb_1\ldots b_m}(\underline{x},\underline{u}) = \prod_{i\in I_+}\prod_{j=1}^{m} \psi(x_i-u_j) \prod_{1\le i<j\le m} \tau(u_i-u_j)\, {f^{(1)}}^{I_- b_1\ldots b_m}\left(\underline{x}^{(-)},\underline{u}\right) \otimes \Omega^{1\ldots n_+} \tag{3.38}$$

*where $\underline{x}^{(-)} = \{x_i\,;\, i\in I_-\}$ are the spectral parameters of the anti-particles only.*

The functions $\psi(x)$ and $\tau(x)$ are the same as in Subsection 3.1 given by eqs. (3.20). For the case that the anti-particles are sitting at the first $n_-$ places the BETHE ANSATZ 3.11 may be depicted as




where $\Psi(\underline{x},\underline{u})$ is given by the products of $\psi$'s and $\tau$'s of eq. (3.38). The main theorem of this article is analogous to that one of Subsection 3.1.

**Theorem 3.13** *Let the vector valued function $f^{1\ldots n}(\underline{x})$ be given by the BETHE ANSATZ 3.11 where $g^{1\ldots m}(\underline{x},\underline{u})$ is of the form of Definition 3.12. If in addition the vector valued function ${f^{(1)}}^{b_1\ldots b_m}(\underline{x}^{(-)},\underline{u}) \in {V^{(1)}}^{b_1\ldots b_m}$ fulfills the Conditions 3.1* (i)$^{(1)}$ *and* (ii)$^{(1)}$*, then $f^{1\ldots n}(\underline{x}) \in V^{1\ldots n}$ fulfills the Conditions 3.1* (i) *and* (ii)*, i.e. $f^{1\ldots n}(\underline{x})$ is a solution of the set of difference equations (3.2).*

**Proof:** Condition 3.1 (i) follows as for the $SU(N)$ case. Again it is sufficient to prove Condition 3.1 (ii) for $i = n$

$$Q(\underline{x};n)\, f(\underline{x}) = f(\underline{x}') \ , \quad (\underline{x}' = x_1,\ldots,x_n' = x_n+2).$$

where the indices $1 \dots n$ have been suppressed. For the case that the index $n$ belongs to a particle the proof is similar as in Subsection 3.1. The only difference is that here the next level Q-matrix is more complicated. Instead of eq. (3.22) we have here

$$T^{Q(1)}_{\underline{b}a}\left(\underline{x}^{(-)}, \underline{u}; b_m\right) = \prod_{i \in I_-} R^{(1)}_{ia}(x_i - u_m)\, R^{(1)}_{b_1 a}(u_1 - u_m) \dots R^{(1)}_{b_{m-1} a}(u_{m-1} - u_m)\, P_{b_m a}\,. \quad (3.39)$$

For the case that the index $n$ belongs to an anti-particle we will prove the inverse of (ii) using $\tilde{T}^{Q}_{\bar{a},1\dots n}$. We apply $Q^{-1}(\underline{x}; n) = \tilde{Q}(\underline{x}; n)$ given by the trace of $\tilde{T}^{Q}_{\bar{a},1\dots n}(\underline{x}; n)$ (cf. eq. (3.27)) to the vector $f^{1\dots n}(\underline{x}')$ as given by eq. (3.31) and push $\tilde{A}^{Q}(\underline{x}; n)$ and $\tilde{D}^{Q}_{\bar{a}}(\underline{x}; n)$ through all the $B$'s using the commutation rules (3.29) and (3.30). Again with $\underline{x}' = x_1, \dots, x'_n = x_n + 2$ we obtain

$$\begin{aligned}\tilde{A}^{Q}(\underline{x}; n)\, B_{b_m}(\underline{x}', u_m) \dots B_{b_1}(\underline{x}', u_1) &= B_{b_m}(\underline{x}, u_m) \dots B_{b_1}(\underline{x}, u_1) \\ &\quad \times \prod_{j=1}^{m} \frac{1 - d^2(x'_n - u)}{1 + d(x_n - u)}\, \tilde{A}^{Q}(\underline{x}; n) + \mathrm{uw}_A\end{aligned}$$

$$\begin{aligned}\tilde{D}^{Q}_{\bar{a}}(\underline{x}; \bar{n})\, B_{b_m}(\underline{x}', u_m) \dots B_{b_1}(\underline{x}', u_1) &= B_{b_m}(\underline{x}, u_m) \dots B_{b_1}(\underline{x}, u_1) \\ &\quad \times R^{(1)}_{\bar{a} b_1}(x'_n - u_1) \dots R^{(1)}_{\bar{a} b_m}(x'_n - u_m)\, \tilde{D}^{Q}_{\bar{a}}(\underline{x}; \bar{n}) + \mathrm{uw}_{D_a}\end{aligned}$$

The "wanted" and "unwanted" terms have origins analogous to the ones for the $SU(N)$ case above. If we insert these equations into the representation (3.31) of $f(\underline{x}')$ we find however, that the wanted contribution from $A^Q$ vanishes and the wanted contribution from $D^Q$ already gives the desired result. By definition of $\tilde{T}^Q$ and the reference state vector $g$ we have

$$\tilde{A}^{Q}(\underline{x}; n)\, g = 0\ , \quad \tilde{D}^{Q}_{\bar{a}}(\underline{x}; n)\, g = \prod_{i \in I_-} R^{(1)}_{\bar{a} i}(x_n - x_i)\, P_{\bar{a} n}\, g$$

if the index $n$ correspond to an anti-particle. With

$$Q^{(1)^{-1}}(\underline{x}^{(-)}, \underline{u}; n) = \mathrm{tr}_{\bar{a}} \left( \prod_{i \in I_-} R^{(1)}_{\bar{a} i}(x_n - x_i)\, P_{\bar{a}\bar{n}}\, R^{(1)}_{\bar{a} b_1}(x'_n - u_1) \dots R^{(1)}_{\bar{a} b_m}(x'_n - u_m) \right)$$

and the assumption

$$\tilde{Q}^{-1}\left(\underline{x}^{(-)}, \underline{u}; n\right)\ f^{(1)}\left(\underline{x}^{(-)'}, \underline{u}\right) = f^{(1)}\left(\underline{x}^{(-)}, \underline{u}\right)$$

it follows that the wanted term from $\tilde{D}^Q$ yields $f(\underline{x})$. Again as for the $SU(N)$ case the unwanted contributions can be written as differences which vanish after summation over the $u$'s. Because of the commutation relations of the $B$'s (2.12) and the symmetry

property (i)$^{(1)}$ of $g^{1\dots n b_1\dots b_m}(\underline{x}^{(-)},\underline{u})$ it is sufficient to consider only the unwanted terms for $j=m$ denoted by $\mathrm{uw}_A^m$ and $\mathrm{uw}_D^m$.

$$\begin{aligned}
\mathrm{uw}_A^m(\underline{x},\underline{u}) &= d(x'_n-u_m)\,\tilde{B}^Q_{\bar{b}_m}(\underline{x};m)\dots B_{b_1}(\underline{x},u_1)\prod_{j<m}\frac{1}{b(u_m-u_j)}\,A(\underline{x},u_m)\\
\mathrm{uw}_{D\bar{a}}^m(\underline{x},\underline{u}) &= -d(x_n-u_m)\,\tilde{B}^Q_{\bar{a}}(\underline{x};m)\dots B_{b_1}(\underline{x},u_1)\prod_{j<m}\frac{1}{b(u_j-u_m)}\\
&\quad\times D_{b_m}(\underline{x}',u_m)\,K_{\bar{a}b_m}\,R^{(1)}_{b_1b_m}(u_1-u_m)\dots R^{(1)}_{b_{m-1}b_m}(u_{m-1}-u_m)
\end{aligned}$$

With

$$D_a(\underline{x},u)\,g(\underline{x},\underline{u})=\prod_{i\in I_+}b(x_i-u)\prod_{i\in I_-}R^{(1)}_{ia}(x_i-u)\,g(\underline{x},\underline{u}),$$

$$Q^{(1)}\left(\underline{x}^{(-)},\underline{u};b_m\right)=\mathrm{tr}_a\,T^{Q(1)}_a\left(\underline{x}^{(-)},\underline{u};b_m\right)$$

(see 3.39)), the assumption (ii)$^{(1)}$, in particular the relation

$$Q^{(1)}\left(\underline{x}^{(-)},\underline{u};b_m\right)\,f^{(1)}\left(\underline{x}^{(-)},\underline{u}\right)=f^{(1)}\left(\underline{x}^{(-)},\underline{u}'\right)\quad,\quad(\underline{u}'=u_1,\dots,u'_m=u_m+2)$$

and the defining relations of the functions $\psi(x)$ and $\tau(x)$ (3.19) follows

$$\mathrm{uw}_A^m(\underline{x},\underline{u})\,g(\underline{x},\underline{u})=-\mathrm{tr}_{\bar{a}}\;\mathrm{uw}^m_{D\bar{a}}(\underline{x},\underline{u}')\,g(\underline{x},\underline{u}')$$

which concludes the proof.

As in Subsection 3.1 we can iterate Theorem 3.13 to get the nested generalized Bethe Ansatz with levels $k=1,\dots,N-1$. To simplify the notation we introduce as an extension of $I_\pm$ the index sets $I_k$ with $n_k=|I_k|$ elements for $k=0,\dots,N-1$ and as an extension of eqs. (3.35) and (3.37) the spaces

$$V^{(k)^{I_-I_k}}=\bigotimes_{i\in I_-}\bar{V}_i^{(k)}\bigotimes_{i\in I_k}V_i^{(k)} \tag{3.40}$$

with basis vectors $|\,\bar{\alpha}_1\dots\bar{\alpha}_{n_-}\alpha_1\dots\alpha_{n_k}\,\rangle$ $(k<\bar{\alpha}_i,\alpha_i\le N)$. In terms of the previous notation we identify $I_+=I_0$, $n_+=n_0$, $V^{1\dots n}=V^{(0)^{I_-I_0}}$ and $V^{(1)^{b_1\dots b_m}}=V^{(1)^{I_-I_1}}$. The Ansatz of level $k$ reads

$$\begin{aligned}
f^{(k-1)^{I_-I_{k-1}}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)}\right)&=\sum_{\underline{x}^{(k)}}B^{(k-1)}_{I_-I_{k-1}\beta_{n_k}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},x^{(k)}_{n_k}\right)\dots \qquad (3.41)\\
&\dots B^{(k-1)}_{I_-I_{k-1}\beta_1}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},x^{(k)}_1\right)\,g^{(k-1)^{I-I_{k-1}\beta_1\dots\beta_{n_k}}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},\underline{x}^{(k)}\right)
\end{aligned}$$

where for $k<N-1$ analogously to Definition 3.12

$$\begin{aligned}
g^{(k-1)^{I_-I_{k-1}I_k}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},\underline{x}^{(k)}\right)&=\prod_{i=1}^{n_{k-1}}\prod_{j=1}^{n_k}\psi\left(x_i^{(k-1)}-x_j^{(k)}\right) \qquad (3.42)\\
&\times\prod_{1\le i<j\le n_k}\tau\left(x_i^{(k)}-x_j^{(k)}\right)\,f^{(k)^{I_-I_k}}\left(\underline{x}^{(-)},\underline{x}^{(k)}\right)\otimes\Omega^{(k-1)^{I_{k-1}}}.
\end{aligned}$$

For $k=0$ with $f^{(0)}=f$ and $\underline{x}^{(0)}=\underline{x}^{(+)}$ we have to replace $\prod_{i=1}^{n_0}$ by $\prod_{i\in I_+}$. The start of the iteration is given by a $k_{max}$ ($1\le k_{max}\le N$) such that all $n_k=0$ for $k\ge k_{max}$. We have to construct a vector valued function proportional to a fixed vector which fulfills the assumptions of Theorem 3.13. This is given by

**Lemma 3.14** *The vector valued functions for $k_{max}<N$*

$$f^{(k_{max}-1)^{I_-I_{k_{max}-1}}} = |\,\bar{N}\ldots\bar{N}k_{max}\ldots k_{max}\,\rangle \tag{3.43}$$

*and for $k_{max}=N$*

$$f^{(N-1)^{I_-I_{N-1}}}\Big(\underline{x}^{(-)},\underline{x}^{(N-1)}\Big)=\prod_{i\in I_-}\prod_{j\in I_{N-1}}\bar{\psi}_{N-1}\Big(x_i^{(-)}-x_j^{(N-1)}\Big)\;|\,\bar{N}\ldots\bar{N}N\ldots N\,\rangle \tag{3.44}$$

*fulfill the Conditions 3.1. The function $\bar{\psi}_{N-1}(x)$ has to obey the functional equation*

$$(1+d(x))\,\bar{\psi}_{N-1}(x)=\bar{\psi}_{N-1}(x-2) \tag{3.45}$$

*which is solved by*

$$\bar{\psi}_{N-1}(x)=\frac{\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{N}+\frac{x}{2}\right)}. \tag{3.46}$$

*Again the general solution is obtained by multiplication with an arbitrary periodic function with period 2.*

**Proof:** Condition 3.1 (i) is fulfilled because of the symmetry with respect to the particles and anti-particles among themselves. Conditions 3.1 (ii) follows from the definition (3.3) of the $Q$-matrix and (2.18) of the R-matrix. For $k_{max}<N$ Condition 3.1 (ii) follows since the Q-operators act as the unit operator on the state $f^{(k_{max}-1)^{I_-I_{k_{max}-1}}}$. For $k_{max}=N$ we have to take into account annihilation-creation contributions of the anti-particle-particle R-matrix (2.18). For the last index in $I_-$ we have

$$\tilde{Q}^{(N-1)\,I_-I_{N-1}}\Big(\underline{x}^{(-)},\underline{x}^{(N-1)};n_-\Big)=\prod_{j\in I_{N-1}}\Big(1+d\Big(x_{n_-}^{(-)}+2-x_j^{(N-1)}\Big)\Big)\,\mathbf{1}$$

and for the last index in $I_{N-1}$

$$Q^{(N-1)\,I_-I_{N-1}}\Big(\underline{x}^{(-)},\underline{x}^{(N-1)};n_{N-1}\Big)=\prod_{j\in I_-}\Big(1+d\Big(x_j^{(-)}-x_{n_{N-1}}^{(N-1)}\Big)\Big)\,\mathbf{1}$$

which, together with eq. (3.45) implies the difference equations (ii) for (3.44). The solution (3.46) of eq. (3.45) follows since $1+d(x)=(x-1+\frac{2}{N})/(x-1)$.

**Corollary 3.15** *The system of $U(N)$ matrix difference equations (3.2) is solved by the generalized nested Bethe Ansatz (3.41) with (3.42-3.45).*

# 4 Weights of generalized Bethe vectors

In this section we analyze some group theoretical properties of generalized Bethe states. We calculate the weights of the states and show that they are highest weight states. The first result does not depend on any restriction to the states. On the other hand the second result is not only true for the conventional Bethe Ansatz, which solves an eigenvalue problem and which is well known, but also, as we will show, for the generalized one which solves a difference equation (or a differential equation).

## 4.1 The $SU(N)$ - case

By asymptotic expansion of the $R$-matrix and the monodromy matrix $T$ (cf. eqs.(2.5) and(2.9)) we get for $u \to \infty$

$$R_{ab}(u) = \mathbf{1}_{ab} - \frac{2}{Nu} P_{ab} + O(u^{-2}) \tag{4.1}$$

$$T_{1\ldots n,a}(\underline{x}, u) = \mathbf{1}_{1\ldots n,a} + \frac{2}{Nu} M_{1\ldots n,a} + O(u^{-2}). \tag{4.2}$$

Explicitly we get from eq. (2.9)

$$M_{1\ldots n,a} = P_{1a} + \ldots + P_{na} \tag{4.3}$$

where the $P$'s are the permutation operators. The matrix elements of $M_{1\ldots n,a}$ as a matrix in the auxiliary space are the $su(N)$ Lie algebra generators. In the following we will consider only operators acting in the fixed tensor product space $V = V^{1\ldots n}$ of (2.1); therefore we will omit the indices $1\ldots n$. In terms of matrix elements in the auxiliary space $V_a$ the generators act on the basis states as

$$M_\alpha^{\alpha'} \,|\, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n \,\rangle = \sum_{i=1}^{n} \delta_{\alpha'\alpha_i} \,|\, \alpha_1, \ldots, \alpha, \ldots, \alpha_n \,\rangle. \tag{4.4}$$

The Yang-Baxter relations (2.10) yield for $x_a \to \infty$

$$[M_a + P_{ab}, T_b(x_b)] = 0 \tag{4.5}$$

and if additionally $x_b \to \infty$

$$[M_a + P_{ab}, M_b] = 0 \tag{4.6}$$

or for the matrix elements

$$[M_\alpha^{\alpha'}, T_\beta^{\beta'}(u)] = \delta_{\alpha'\beta} T_\alpha^{\beta'}(u) - \delta_{\alpha\beta'} T_\beta^{\alpha'}(u) \tag{4.7}$$

$$[M_\alpha^{\alpha'}, M_\beta^{\beta'}] = \delta_{\alpha'\beta} M_\alpha^{\beta'} - \delta_{\alpha\beta'} M_\beta^{\alpha'}. \tag{4.8}$$

Equation (4.8) represents the structure relations of the $su(N)$ Lie algebra and (4.7) the $SU(N)$-covariance of $T$. In particular the transfer matrix is invariant

$$[M_\alpha^{\alpha'}, \operatorname{tr} T(u)] = 0. \tag{4.9}$$

We now investigate the action of the lifting operators $M_\alpha^{\alpha'}$ $(\alpha' > \alpha)$ to generalized Bethe vectors.

**Lemma 4.1** *Let $F[g](\underline{x}) \in V$ be a Bethe Ansatz vector given in terms of a vector $g(\underline{x}, \underline{u}) \in V^{(1)} \cong \mathbf{C}^{(N-1)\otimes m}$ by*

$$F[g](\underline{x}, \underline{u}) = B_{\beta_m}(\underline{x}, u_m) \dots B_{\beta_1}(\underline{x}, u_1)\, \Omega\; g^{\underline{\beta}}(\underline{x}, \underline{u}) \tag{4.10}$$

*with $\underline{\beta} = \beta_1, \dots, \beta_m$. Then $M_\alpha^{\alpha'} F[g]$ is of the form*

$$M_\alpha^{\alpha'} F[g] = \begin{cases} \sum_{j+1}^{m} B_{\beta_m} \dots \delta_{\alpha'\beta_m} \dots B_{\beta_1}\, \Omega\; G_j^{\underline{\beta}}(\underline{x}, \underline{u}) & \text{for } \alpha' > \alpha = 1 \\ F[M^{(1)}{}_\alpha^{\alpha'}\, g] & \text{for } \alpha' > \alpha > 1, \end{cases} \tag{4.11}$$

*where the $M^{(1)}{}_\alpha^{\alpha'}$ are the $su(N-1)$ generators represented in $V^{(1)}$ (analogously to (4.3)) and*

$$G_m(\underline{x}, \underline{u}) = \left( \frac{1}{\prod_{j=1}^{m} b(u_m - u_j)} - \frac{\prod_{i=1}^{n} b(x_i - u_m)}{\prod_{j=1}^{m} b(u_j - u_m)}\, Q^{(1)}(\underline{u}; m) \right) g(\underline{x}, \underline{u}). \tag{4.12}$$

*The operator $Q^{(1)}(\underline{u}; m) \in End(V^{(1)})$ is a next level Q-matrix given by the trace*

$$Q^{(1)}(\underline{u}; m) = \operatorname{tr}_a T_a^{Q(1)}(\underline{u}; m) \tag{4.13}$$

*(see eq. (3.22)). The other $G_j$ are obtained by Yang-Baxter relations.*

**Proof:** First we consider the case $\alpha = 1$. The commutation rule (4.7) reads for $\beta' = 1$ and $\alpha' \to \alpha$

$$[M_1^\alpha, B_\beta(u)] = \delta_{\alpha\beta}\, A(u) - D_\beta^\alpha(u).$$

We commute $M_1^\alpha$ through all the $B$'s of (4.10) and use $M_1^\alpha\, \Omega = 0$ for $\alpha > 1$ (cf. (4.4)). The $A$'s and $D$'s appearing are also commuted through all the $B$'s using the commutation rules (2.13) and (2.14). In each summand exactly one $B$-operator disappears. Therefore the result is of the form of eq. (4.11). Contributions to $G_m$ arise when we commute $M_1^\alpha$ through $B_{\beta_m}(u_m)$ and then push the $A(u_m)$ and $D(u_m)$ through the other $B$'s $(j < m)$, only taking the first terms of (2.13) and (2.14) into account. All other terms would contain a $B(u_m)$ and would therefore contribute to one of the other $G_j$ $(j < m)$. Finally we apply $A(u_m)$ and $D(u_m)$ to $\Omega$

$$A(u_m)\, \Omega = \Omega\ , \quad D_\beta^\alpha(u_m)\, \Omega = \delta_{\alpha\beta} \prod_{i=1}^{n} b(x_i - u_m)\, \Omega$$

and get eq. (4.12).

For $\alpha' > \alpha > 1$ we again use the commutation rule (4.7)

$$[M_\alpha^{\alpha'}, B_\beta(u)] = \delta_{\alpha'\beta}\, B_\alpha(u)$$

and get

$$M_\alpha^{\alpha'}\, B_\beta(u_m) \dots B_\beta(u_1) = B_{\beta_m}(u_m) \dots B_{\beta_1}(u_1)\, M_\alpha^{\alpha'} + B_{\beta'_m}(u_m) \dots B_{\beta'_1}(u_1)\, M^{(1)}{}_{\underline{\beta},\alpha}^{\beta',\alpha'}$$

with $M^{(1)}_{1\dots m,a} = P^{(1)}_{1a} + \dots + P^{(1)}_{ma}$ analogously to (4.3). Because of $M_\alpha^{\alpha'}\,\Omega = 0$ for $\alpha' > 1$ (cf. (4.4)) we get eq. (4.11)

The diagonal elements of $M$ are the weight operators $W_\alpha = M_\alpha^\alpha$, they act on the basis vectors in $V$ as

$$W_\alpha \,|\, \alpha_1, \dots, \alpha_n \,\rangle = \sum_{i=1}^{n} \delta_{\alpha_i \alpha}\ |\, \alpha_1, \dots, \alpha_n \,\rangle \tag{4.14}$$

which follows from $P_{i\,\alpha}^{\alpha'}\,|\,\alpha_i\,\rangle = \delta_{\alpha\alpha_i}\,|\,\alpha'\,\rangle$. In particular we get for the Bethe Ansatz reference state (3.14)

$$W_\alpha\,\Omega = \delta_{\alpha 1}\, n\,\Omega. \tag{4.15}$$

**Lemma 4.2** *Let $F[g] \in V^{1\dots n}$ be as in Lemma 4.1. Then*

$$W_\alpha\, F[g] = \begin{cases} (n-m)\, F[g] & \text{for } \alpha = 1 \\ F[W_\alpha^{(1)}\, g] & \text{for } \alpha > 1, \end{cases} \tag{4.16}$$

*where the $W_\alpha^{(1)}$'s are the $su(N-1)$ weight operators acting in $V^{(1)}$, i.e. the diagonal elements of generator matrix $M^{(1)}{}_\alpha^{\alpha'}$ (analogously to (4.3)).*

**Proof:** By means of the commutation relation (4.7) for $\alpha' = \alpha = \beta' = 1,\ \beta > 1$

$$[W_1, B_\beta] = -B_\beta$$

we commute $W_1$ through all $m$ $B$'s of eq. (4.10) and with (4.15) we get the first equation. For the second equation we again use (4.7) now for $\alpha' = \alpha > 1,\ \beta' = 1,\ \beta > 1$

$$[W_\alpha, B_\beta] = \delta_{\alpha\beta}\, B_\beta.$$

Again commuting $W_\alpha$ through all the $B$'s of eq. (4.10) we get with eq. (4.15)

$$\begin{aligned} W_\alpha\, B_{\beta_m} \dots B_{\beta_1}\, \Omega\ g^{\beta_1 \dots \beta_m} &= B_{\beta_m} \dots B_{\beta_1} \left( W_\alpha + \sum_{i=1}^{m} \delta_{\beta_i \alpha} \right) \Omega\ g^{\beta_1 \dots \beta_m} \\ &= B_{\beta_m} \dots B_{\beta_1}\, \Omega\ \left( W_\alpha^{(1)}\, g \right)^{\beta_1 \dots \beta_m} \end{aligned}$$

which concludes the proof.

**Theorem 4.3** *Let the vector valued function $f(\underline{x}) \in V$ be given by the BETHE ANSATZ 3.5 fulfilling the assumptions of Theorem 3.7. If in addition $f^{(1)}$ is a highest weight vector and an eigenvector of the weight operators with*

$$W_\alpha^{(1)} f^{(1)} = w_\alpha^{(1)} f^{(1)}, \tag{4.17}$$

*then also $f$ is a highest weight vector*

$$M_\alpha^{\alpha'} f = 0 \ , \quad (\alpha' > \alpha) \tag{4.18}$$

*and an eigenvector of the weight operators*

$$W_\alpha f = w_\alpha f \ , \quad w_\alpha = \begin{cases} n-m & \text{for } \alpha = 1 \\ w_\alpha^{(1)} & \text{for } \alpha > 1 \end{cases} \tag{4.19}$$

*with*

$$w_\alpha \geq w_\beta \ , \quad (1 \leq \alpha < \beta \leq N). \tag{4.20}$$

**Proof:** To prove the highest weight property we apply Lemma 4.1. By assumption $f^{(1)}$ fulfills the difference equation

$$f^{(1)}(u_1, \ldots, u_m + 2) = Q^{(1)}(\underline{u}; m)\, f^{(1)}(u_1, \ldots, u_m).$$

Together with eq. (3.18), (3.19) and (4.12) we obtain after summation $\sum_{u_m} G_m(\underline{x}, \underline{u}) = 0$, if $u_m = \tilde{u}_m - 2l_m$ ($l_m \in \mathbf{Z}$). The same is true for the other $G_i$ in eq. (4.11), since $g$ fulfills the symmetry property of Condition 3.1 (i) and thereby $F[g](\underline{x}, \underline{u})$ of eq. (4.10) is symmetric with respect to the $u_i$. Therefore in eq. (4.12) we have $M_\alpha^{\alpha'} f = 0$ for $\alpha' > \alpha > 1$ and for $\alpha' > \alpha = 1$ by assumption on $f^{(1)}$. The weights of $f$ follow from Lemma 4.2 and also by assumption on $f^{(1)}$. From the commutation rule (4.8) and $M_\alpha^{\beta\dagger} = M_\beta^\alpha$ follows

$$0 \leq M_\alpha^\beta M_\beta^\alpha = M_\beta^\alpha M_\alpha^\beta + W_\alpha - W_\beta$$

which implies (4.20).

Since the states $f^{(k_{max}-1)}$ of eq. (3.25) are highest weight states in $V^{(k_{max}-1)}$ with weight $w_{k_{max}}^{(k_{max}-1)} = n_{k_{max}-1}$ we have the

**Corollary 4.4** *If $f(\underline{x})$ is a solution of the system of $SU(N)$ matrix difference equations (3.2)*

$$f(\ldots, x_i + 2, \ldots) = Q(\underline{x}; i)\, f(\ldots, x_i, \ldots) \ , \quad (i = 1, \ldots, n)$$

*given by the generalized nested Bethe Ansatz of Corollary 3.9, then $f$ is a highest weight vector with weights*

$$w = (w_1, \ldots, w_N) = (n - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1}), \tag{4.21}$$

*where $n_k$ is the number of $B^{(k-1)}$ operators in the Bethe Ansatz of level $k$, $(k = 1, \ldots, N-1)$. Further non-highest weight solutions of (3.2) are given by*

$$f_{\alpha}^{\alpha'} = M_{\alpha}^{\alpha'} f \ , \quad (\alpha' < \alpha). \tag{4.22}$$

The interpretation of eq. (4.21) is that each $B^{(k)}$-operator reduces $w_k$ and lifts a $w_l$ $(l > k)$ by one.

## 4.2 The $U(N)$ - case

The results of this subsection and also the techniques used are very similar to the corresponding ones of the previous subsection. Therefore we only point out the main differences. By an asymptotic expansion of the $R$-matrix and the monodromy matrix $T$ (cf. eqs.(2.5), (2.18), (2.20) and (2.23)) we get for $u \to \infty$

$$R_{ab}(u) = \mathbf{1}_{ab} - \frac{2}{Nu} P_{ab} + O(u^{-2}) \tag{4.23}$$

$$R_{\bar{a}\bar{b}}(u) = \mathbf{1}_{ab} + \frac{2}{Nu} K_{\bar{a}\bar{b}} + O(u^{-2}) \tag{4.24}$$

$$T_{1\ldots n,a}(\underline{x}, u) = \mathbf{1}_{1\ldots n,a} + \frac{2}{Nu} M_{1\ldots n,a} + O(u^{-2}). \tag{4.25}$$

Explicitly we get from eq. (2.23)

$$M_{1\ldots n,a} = \sum_{i \in I_+} P_{ia} - \sum_{i \in I_-} K_{ia}\,, \tag{4.26}$$

where $I_{\pm}$ denote the particles and anti-particles, respectively.

In the following we will suppress the indices like $1\ldots n$. In terms of matrix elements in the auxiliary space $V_a$ the generators act on the basis states as

$$M_{\alpha}^{\alpha'} \,|\, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n \,\rangle = \left( \sum_{i \in I_+} \delta_{\alpha'\alpha_i} - \sum_{i \in I_-} \delta_{\alpha'\alpha_i} \right) \,|\, \alpha_1, \ldots, \alpha, \ldots, \alpha_n \,\rangle. \tag{4.27}$$

The commutation relations of the $M$ and $T$ which follow from the Yang-Baxter relations are the same as in Subsection 4.1. Also Lemma 4.1 holds here for the $U(N)$ case. However, one has to replace $\Omega\, g$ in eq. (4.10) by $g$ as in BETHE ANSATZ 3.11, and the Q-matrix here is given by the trace of (3.39).

The slight difference to the $SU(N)$ case arises from the action of the Cartan subalgebra; i.e. from the diagonal elements of $M$ which are the weight operators $W_\alpha = M_\alpha^\alpha$. They act on the basis vectors in $V$ as

$$W_{\alpha} \,|\, \alpha_1, \ldots, \alpha_n \,\rangle = \left( \sum_{i \in I_+} \delta_{\alpha\alpha_i} - \sum_{i \in I_-} \delta_{\alpha\alpha_i} \right) \,|\, \alpha_1, \ldots, \alpha_n \,\rangle \tag{4.28}$$

which follows from eq. (4.27). In particular we have for the Bethe Ansatz reference state $g$

$$W_1\, g = n_+\, g. \tag{4.29}$$

This means that also Lemma 4.2 holds here for the $U(N)$ case where, however, in eq. (4.16) the number $n$ is replaced by $n_+$.

**Theorem 4.5** *Let the vector valued function $f(\underline{x}) \in V$ be given by the BETHE ANSATZ 3.11 fulfilling the conditions of Theorem 3.13, i.e. eqs. (3.38) and (3.19). If in addition $f^{(1)}$ is a highest weight vector and an eigenvector of the weight operators with*

$$W_\alpha^{(1)}\, f^{(1)} = w_\alpha^{(1)}\, f^{(1)}, \tag{4.30}$$

*then also $f$ is a highest weight vector*

$$M_\alpha^{\alpha'}\, f = 0 \ , \quad (\alpha' > \alpha) \tag{4.31}$$

*and an eigenvector of the weight operators*

$$W_\alpha\, f = w_\alpha\, f \ , \quad w_\alpha = \begin{cases} n_+ - m & \text{for } \alpha = 1 \\ w_\alpha^{(1)} & \text{for } \alpha > 1 \end{cases} \tag{4.32}$$

*with*

$$w_\alpha \geq w_\beta \ , \quad (1 \leq \alpha < \beta \leq N), \tag{4.33}$$

The proof of this theorem is again parallel to the corresponding one Subsection 4.1.

The states $f^{(k_{max}-1)}$ of Lemma 3.14 which define the start of the iteration of the nested Bethe Ansatz are obviously highest weight states in $V^{(k_{max}-1)}$ with weight $w_{k_{max}}^{(k_{max}-1)} = n_{k_{max}-1} - n_-$ by eq. (4.28).

**Corollary 4.6** *If $f(\underline{x})$ is a solution of the system of $U(N)$ matrix difference equations (3.3)*

$$f(\ldots, x_i + 2, \ldots) = Q(\underline{x}; i)\, f(\ldots, x_i, \ldots) \ , \quad (i = 1, \ldots, n)$$

*given by the generalized nested Bethe Ansatz of Corollary 3.15, then $f$ is a highest weight vector with weights*

$$w = (w_1, \ldots, w_N) = (n_+ - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1} - n_-), \tag{4.34}$$

*where $n_k$ is the number of $B^{(k)}$ operators in the Bethe Ansatz of level $k$. Further non-highest weight solutions of (3.2) are given by*

$$f_\alpha^{\alpha'} = M_\alpha^{\alpha'}\, f \ , \quad (\alpha' < \alpha). \tag{4.35}$$

Note that in contrast to the $SU(N)$ case, here the weights may also be negative.

# 5 Examples

## 5.1 The $SU(N)$ - case

From a solution of the matrix difference equations (3.2) one gets a new solution by multiplication of a scalar function which is symmetric with respect to all variables $x_i$ and periodic with period 2. Therefore the solutions of the following examples may be multiplied by such functions.

**Example 5.1** The simplest example is obtained for $k_{max} = 1$ which means the trivial solution of the difference equations

$$f^{1\ldots n} = \Omega^{1\ldots n}$$

The weights of $f^{1\ldots n}$ are $w = (n, 0, \ldots, 0)$.

In the language of spin chains this case corresponds to the ferro-magnetic ground state.

**Example 5.2** For the case $k_{max} = 1$ and $n^{(1)} = 1$ the solution reads

$$f^{1\ldots n}(\underline{x}) = \sum_u B_{1\ldots n,\beta}(\underline{x}, u)\, \Omega^{1\ldots n}\ g^{\beta}(\underline{x}, u).$$

with $u = \tilde{u} - 2l$ ($l \in \mathbf{Z}$, $\tilde{u}$ an arbitrary constant) and

$$g^{\beta}(\underline{x}, u) = \delta_{\beta 2} \prod_{i=1}^{n} \psi(x_i - u).$$

The weights of this vector $f^{1\ldots n}$ are $w = (n-1, 1, 0, \ldots, 0)$. The action of the creation operator $B_{1\ldots n\beta}(x, y; u)$ on the reference state is easily calculated with help of eqs. (2.6) (2.9) and (2.11).

As a particular case of this example we determine explicitly the solution for the following

**Example 5.3** The action of the $B$-operator on the reference state for the case of $n = 2$ of Example 5.2 yields

$$B_{12,\beta}(x, y; u)\,|\,11\,\rangle = c(x-u)\,b(y-u)\ |\,\beta 1\,\rangle + c(y-u)\ |\,1\beta\,\rangle.$$

Therefore we obtain

$$f^{12}(x, y) = \sum_u \psi(x-u)\,\psi(y-u)\{c(x-u)\,b(y-u)\ |\,21\,\rangle + c(y-u)\ |\,12\,\rangle\}$$

with $u = \tilde{u} - 2l,\ (l \in \mathbf{Z})$. Using the expressions for the functions $b,\ c,\ \psi$ given by eqs. (2.7) and (3.20) we get (by Dougall's formula) up to a constant

$$f^{12}(x,y) = \Big(\sin\pi\left(\tfrac{x-\tilde{u}}{2}-\tfrac{1}{N}\right)\sin\pi\left(\tfrac{y-\tilde{u}}{2}-\tfrac{1}{N}\right)\Gamma\left(\tfrac{y-x}{2}+\tfrac{1}{N}\right)\Gamma\left(1+\tfrac{x-y}{2}+\tfrac{1}{N}\right)\Big)^{-1}(\,|\,21\,\rangle - |\,12\,\rangle).$$

This solution could also be obtained by means of the method used in [3], namely by diagonalization of the R-matrix. One obtains the difference equations

$$f_-(x,y) = R_-(x-y)\,f_-(y,x)\ , \quad f_-(x,y) = f_-(y,x+2)$$

with the eigenvalue $R_-(x) = (x+2/N)/(x-2/N)$ of the antisymmetric tensor representation.

**Example 5.4** Next we consider for $N > 2$ the case of the quantum space $V_{123} = V_1 \otimes V_2 \otimes V_3$ and the case that the nested Bethe Ansatz has only two levels with two creation operators in the first level and one in the second level. This means $k_{max} = 3,\ n = 3,\ n^{(1)} = 2,\ n^{(2)} = 1$ and the weights $w = (1,1,1,0,\ldots,0)$. The first level Bethe Ansatz is given by

$$f^{123}(x,y,z) = \sum_{u,v}\ B_{123,\beta}(x,y,z;v)\,B_{123,\alpha}(x,y,z;u)\,\Omega^{123}\ g^{\alpha\beta}(x,y,z;u,v).$$

where the summation is specified by $u = \tilde{u} - 2k,\ v = \tilde{u} - 2l,\ (k,l \in \mathbf{Z})$. By eq. (3.18) $g^{12}$ is related to the next level function ${f^{(1)}}^{12}$ by

$$g^{12}(x,y,z;u,v) = \prod_{x_i = x,y,z}\ \prod_{u_j = u,v}\ \psi(x_i - u_j)\,\tau(u-v)\,{f^{(1)}}^{12}(u,v).$$

The second level Bethe Ansatz reads

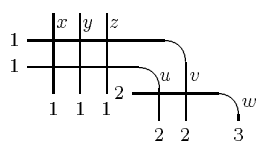
$${f^{(1)}}^{12}(u,v) = \sum_{w}\ B^{(1)}_{12\gamma}(u,v;w)\,{\Omega^{(1)}}^{12}\ {g^{(1)}}^{\gamma}(u,v;w)$$

where $w = \tilde{w} - 2m,\ (m \in \mathbf{Z})$. The second level reference state is ${\Omega^{(1)}}^{12} = |\,22\,\rangle^{(1)} \in {V^{(1)}}^{12}$. Again according to eq. (3.18)

$${g^{(1)}}^{\gamma}(u,v;w) = \psi(u-w)\,\psi(v-w)\,{f^{(2)}}^{\gamma},$$

with ${f^{(2)}}^{\gamma} = \delta_{\gamma 3}$. Similar as in Example 5.3 the action of the operators $B$ and $B^{(1)}$ on their reference states may be calculated.

For this example the two level nested Bethe Ansatz may be depicted as



## 5.2 The $U(N)$ - case

**Example 5.5** Let us consider the trivial case that there is no $B$-operator in each level of the nested Bethe Ansatz, which means that $k_{max}$ of Section 3 is equal to 1. In the language of the conventional Bethe Ansatz for quantum chains this correspond to the "ferro-magnetic vacuum". By Section 4 this means that $f^{1\dots n}(\underline{x})$ has the weights

$$w = (n_+, 0, \dots, 0, -n_-).$$

For fixed positions of the particles $I_+$ and anti-particles $I_-$ by Lemma 3.14 the vector $f^{1\dots n} \in V^{I_-I_+}$ is given by

$$f^{\alpha_1\dots\alpha_n}(\underline{x}) = \prod_{i\in I_-} \delta_{\alpha_i \bar{N}} \prod_{i\in I_+} \delta_{\alpha_i 1}.$$

or if the anti-particles are sitting at the first places

$$f^{1\dots n} = |\, \bar{N} \dots \bar{N} 1 \dots 1 \,\rangle.$$

This $f^{1\dots n}$ is a highest weight vector in $V_{ref}$.

**Example 5.6** For $N > 2$ and $n_+ > 2$ let us take the case where there is one $B$-operator in the first level of the nested Bethe Ansatz and no $B$'s in higher levels, which means that the $k_{max}$ of Section 3 is equal to 2. By Section 4 this means that the weights are

$$w = (n_+ - 1, 1, 0, \dots, 0, -n_-).$$

The first level Ansatz reads

$$f^{1\dots n}(\underline{x}) = \sum_u B_{1\dots n}\beta(\underline{x}; u)\; g^{1\dots n\beta}(\underline{x}; u). \tag{5.1}$$

For fixed positions of the particles $I_+$ and anti-particles $I_-$, if the anti-particles are sitting on the left of all particles, the function $g^{1\dots n\beta}$ is given by

$$g^{I_-I_+\beta}(\underline{x}, u) = \prod_{i\in I_+} \psi(x_i - u)\, {f^{(1)}}^{I_-\beta} \otimes \Omega^{I_+} \tag{5.2}$$

with $\psi(x) = \frac{\Gamma(1-\frac{1}{N}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}$ (see eq. (3.19)) and by Lemma 3.14

$${f^{(1)}}^{I_-I_1} = |\, \bar{N} \dots \bar{N} 2 \,\rangle\ , \quad \Omega^{I_+} = |\, 1 \dots 1 \,\rangle. \tag{5.3}$$

i.e. $f^{(1)}$ is the highest weight vector in $V^{(1)}_{ref}$. Note that the function $\psi$ appears only with respect to the parameter $x_i$ which correspond to the particles. The action of the B-operators in eq. (5.1) can easily be obtained from the definition of the R-matrices (2.17) and (2.18). In particular we consider

**Example 5.7** As a simple case of Example 5.6 we take $n_- = 1$ and $n_+ = 2$, which means $w = (1, 1, 0, \ldots, 0, -1)$

$$f^{\bar{1}23}(x,y,z) = \sum_u \psi(y-u)\,\psi(z-u)\Big\{c(y-u)\,b(z-u)\,|\,\bar{N}21\,\rangle + c(z-u)\,|\,\bar{N}12\,\rangle\Big\}.$$

The sum over $u$ can be performed and gives the same result as in Example 5.3.

**Example 5.8** For $N = 2$ let $n_- = n_+$ be $= 1$. In addition to the trivial case of Example 5.5 with no B-operator ($w = (1, -1)$) there is only the possibility analogous to Examples 5.6 and 5.7 with one B-operator. By Section 4 $f^{\bar{1}2}(x,y)$ is an $U(2)$-singlet vector with the weights $w = (0, 0)$. For $N = 2$ we must take into account the annihilation-creation contribution, for the action of the B-operator on the reference state as well as for $f^{(1)}$ due to Lemma 3.14:

$$f^{\bar{1}2}(x,y) = \sum_u \bar{\psi}_1(x-u)\,\psi(y-u)\Big\{d(x-u)\,b(y-u)\,|\,\bar{1}1\,\rangle + c(y-u)\,|\,\bar{2}2\,\rangle\Big\}$$

with $\psi(x) = \bar{\psi}_1(x) = \frac{\Gamma(\frac{1}{2}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}$. Similar as in Example 5.7 the sum over $u = \tilde{u} - 2l,\ (l \in \mathbf{Z})$ can be performed and we get for any $\tilde{u}$ the solution

$$f^{\bar{1}2}(x,y) = \frac{\cos\frac{\pi}{2}(x-y)}{\cos\frac{\pi}{2}(x-\tilde{u})\cos\frac{\pi}{2}(y-\tilde{u})}\,\frac{1}{x-y+1}\Big\{\,|\,\bar{1}1\,\rangle + \,|\,\bar{2}2\,\rangle\Big\}.$$

As a generalization of this formula for arbitrary $N$ we consider

**Example 5.9** Let us take for $n_- = n_+ = 1$ the case where there is exactly one $B$-operator in each level of the nested Bethe Ansatz, which means that the $k_{max}$ of Section 3 is equal to $N - 1$. By Section 4 this means the weights are $w = (0, \ldots, 0)$, i.e. $f^{12}(x,y)$ is an $U(N)$-singlet. The first level Ansatz reads

$$f^{\bar{1}2}(x,y) = \sum_u B_{\bar{1}2,\beta}(x,y;u)\; g^{\bar{1}2\beta}(x,y;u) \tag{5.4}$$

where $u = \tilde{u} + 2l$, $l \in \mathbf{Z}$ and

$$g^{\bar{1}20}(x,y,u) = \psi(y-u)\,f^{(1)\,\bar{1}0}(x,u)\,\Omega^2. \tag{5.5}$$

The higher level Ansätze are of the same form. A particular solution is

$$f^{\bar{\alpha}\beta}(x,y) = \frac{1}{x-y+1}\,\delta_{\alpha\beta}. \tag{5.6}$$

**Proof:** The solution (5.6) is a up to a constant the particular case $k = 0$ of a general formula valid for all levels. This general formula (again up to unimportant constants)

$$f^{(k)^{\bar{\alpha}\beta}}(x,y) = \bar{\psi}_k(x-y)\,\delta_{\alpha\beta}\ , \quad (k < \alpha,\beta \leq N) \tag{5.7}$$

where as an extension of eq. (3.46)

$$\bar{\psi}_k(x) = \frac{\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x}{2}\right)} \tag{5.8}$$

will be proved inductively. For $k = N-1$ as the start of the iteration formula (5.7) follows from Lemma 3.14. For the other values of $k$ it follows recursively from eqs. (3.41) and (3.42):

$$f^{(k-1)^{\bar{1}2}}(x,y) = \sum_u B^{(k-1)}_{\bar{1}2\beta}(x,y;u)\, g^{(k-1)^{\bar{1}2\beta}}(x,y;u) \tag{5.9}$$

$$\sum_u B^{(k-1)}_{\bar{1}2\beta}(x,y;u)\,|\,\bar{\alpha}k\,\rangle = d(x-u)\,b(y-u)\,|\,\bar{k}k\,\rangle + c(y-u)\,|\,\bar{\alpha}\beta\,\rangle \tag{5.10}$$

$$g^{(k-1)^{\bar{\alpha}\beta\gamma}}(x,y;u) = \psi(y-u)\,f^{(k)^{\bar{\alpha}\gamma}}(x,u)\,\delta_{\beta k}. \tag{5.11}$$

We calculate the right hand side of eq. (5.9) inserting eqs. (5.10) and (5.11) with $f^{(k)}$ given by (5.7) and with $u = x + 2l$

$$\begin{aligned}
&\sum_l \psi(x-u)\,\bar{\psi}_k(y-x)\left\{d(x-u)\,b(y-u)\,(N-k)\,|\,\bar{k}k\,\rangle + c(y-u)\sum_{\alpha=k+1}^{N}|\,\bar{\alpha}\alpha\,\rangle\right\} \\
&= \frac{1}{N}\sum_l\left\{\frac{\Gamma\left(-\frac{1}{N}+l\right)}{\Gamma(l)}\frac{\Gamma\left(-\frac{1}{2}+\frac{x-y}{2}+l\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x-y}{2}+l\right)}(N-k)\,|\,\bar{k}k\,\rangle\right. \\
&\qquad \left. + \frac{\Gamma\left(-\frac{1}{N}+l\right)}{\Gamma(1+l)}\frac{\Gamma\left(\frac{1}{2}+\frac{x-y}{2}+l\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x-y}{2}+l\right)}\sum_{\alpha=k+1}^{N}|\,\bar{\alpha}\alpha\,\rangle\right\} \qquad (5.12) \\
&= const.\,\bar{\psi}_{k-1}(x-y)\sum_{\alpha=k}^{N}|\,\bar{\alpha}\alpha\,\rangle = const.\,f^{(k-1)^{\bar{1}2}}(x,y)
\end{aligned}$$

The sums over $l$ have been performed using the Gauss formula

$$\sum_l \frac{\Gamma(a+l)\Gamma(b+l)}{l!\,\Gamma(c+l)} = \frac{\Gamma(c-a-b)\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)}$$

This concludes the proof of formula 5.7.

**Example 5.10** The last example will be used in [17] to calculate the exact three particle form factor of the fundamental field in the chiral Gross-Neveu model. We take $n_- =$

1, $n_+ = 2$ and again exactly one $B$-operator in each level of the nested Bethe Ansatz. By Section 4 this means that $f^{\bar{1}23}(x,y,z)$ is an $U(N)$-vector with weights

$$w = (1,0,\ldots,0).$$

The first level Ansatz reads

$$f^{\bar{1}23}(x,y,z) = \sum_u B_{\bar{1}23,\beta}(x,y,z;u)\, g^{\bar{1}23\beta}(x,y,z;u) \tag{5.13}$$

The higher level ($k > 1$) Bethe Ansatz coincides with that of Example 5.9. A solution of the difference equations for this case is given by

$$\begin{aligned} f^{\bar{1}23}(x,y,z) \;=\; & \sum_u \left\{ \frac{\Gamma\left(-\frac{1}{N}+\frac{x-u}{2}\right)}{\Gamma\left(\frac{x-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{N}+\frac{y-u}{2}\right)}{\Gamma\left(\frac{y-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{2}+\frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+\frac{z-u}{2}\right)} (N-1)\,|\,\bar{1}11\,\rangle \right. \\ & - \frac{\Gamma\left(-\frac{1}{N}+\frac{x-u}{2}\right)}{\Gamma\left(\frac{x-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{N}+\frac{y-u}{2}\right)}{\Gamma\left(1+\frac{y-u}{2}\right)} \frac{\Gamma\left(\frac{1}{2}+\frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+\frac{z-u}{2}\right)} \sum_{\alpha=2}^{N} |\,\bar{\alpha}\alpha 1\,\rangle \\ & \left. - \frac{\Gamma\left(-\frac{1}{N}+\frac{x-u}{2}\right)}{\Gamma\left(1+\frac{x-u}{2}\right)} \frac{\Gamma\left(1-\frac{1}{N}+\frac{y-u}{2}\right)}{\Gamma\left(1+\frac{y-u}{2}\right)} \frac{\Gamma\left(\frac{1}{2}+\frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+\frac{z-u}{2}\right)} \sum_{\alpha=2}^{N} |\,\bar{\alpha}1\alpha\,\rangle \right\} \end{aligned} \tag{5.14}$$

where $u = \tilde{u} + 2l$, $l \in \mathbf{Z}$.

**Proof:** Analogously to eqs. (5.10) and (5.11) we have for ($k = 1$)

$$\begin{aligned} B_{\bar{1}23,\beta}(x,y,z;u)\,|\,\bar{\alpha}11\,\rangle \;=\; & d(x-u)\,b(y-u)\,b(z-u)\,\delta_{\alpha\beta}\,|\,\bar{1}11\,\rangle \\ & + c(y-u)\,b(z-u)\,|\,\bar{\alpha}\beta 1\,\rangle \\ & + c(z-u)\,|\,\bar{\alpha}1\beta\,\rangle \end{aligned}$$

$$g^{\bar{\alpha}\beta}(x,y,z;u) = \psi(y-u)\psi(z-u)\, f^{(1)\,\bar{\alpha}\beta}(x,u)$$

where $f^{(1)\,\bar{\alpha}\beta}(x,u)$ is the same function as in Example 5.9 and given by eq. (5.7) for $k = 1$. Inserting this into eq. (5.13) we get in analogy to eq. (5.12) the result (5.14).

**Acknowledgment:** The authors have profited from discussions with A. Fring, R. Schrader, F. Smirnov and A. Belavin.

# References

[1] H.M. Babujian, 'Correlation functions in WZNW model as a Bethe wave function for the Gaudin magnets', in: Proc. XXIV Int. Symp. Ahrenshoop, Zeuthen 1990;
H.M. Babujian, J. Phys. A **26** (1993) 6981-6990;
H.M. Babujian and R. Flume, Modern Phys. Lett. A **9** (1994) 2029-2059.

[2] I. Frenkel and N.Yu. Reshetikhin, Commun. Math. Phys. **146**, (1992), 1-60.

[3] M. Karowski and P.Weisz, Nucl. Phys. **B 139** (1978) 445-476.

[4] K.M. Watson, Phys. Rev. **95** (1954) 228.

[5] F.A. Smirnov, 'Form Factors in Completely Integrable Models of Quantum Field Theory', World Scientific Pub. Co, Singapure (1990).

[6] J.L. Cardy and G. Mussardo, Phys. Lett. **B 225** (1989) 275.
Al.B. Zamolodchikov, Nucl. Phys. **B 348** (1991) 619.
V.P. Yurov and Al. B. Zamolodchikov, Int. J. Mod. Phys. **A 6** (1991) 3419.
A. Fring, G. Mussardo and P. Simonetti, Phys. Lett. **B 307** (1993) 83.
A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. **B 393** (1993) 413.
A. Koubek and G. Mussardo, Phys. Lett. **B 311** (1993) 193.
A. Koubek, Nucl. Phys. **B 428** (1994) 655.
C. Ahn, Nucl. Phys. **B 422** (1994) 449.

[7] H. Bethe, Z. Phys. **71** (1931) 205.

[8] L.A. Takhtadzhan and L.D. Faddeev, Russian Math. Surveys **34** (1979) 186.

[9] N.Yu. Reshetikhin, Lett. Math. Phys. **26** (1992) 153.

[10] C.N. Yang, Phys. Rev. Lett. **19** (1967) 1312.

[11] I.H.L. Essler, H. Fram, A.G. Izergin and V.E. Korepin, Commun. Math. Phys. **174** (1995) 191.
P.B Wiegmann and A.V. Zabrodin, 'Algebraization of Difference Equations related to $U_q(sl(2))$', cond-mat/9501129.
A.V. Zabrodin, 'Discrete Hirota's Equation in Quantum Integrable Models', hep-th/9610039.

[12] B. Sutherland, Phys. Rev. Lett. **20** (1967) 98.

[13] P.P. Kulish and N.Yu. Reshetikhin, J. Phys. A: Math. Gen. **16** (1983) L591-L596.
O. Babelon, H.J. de Vega and C.M. Viallet, Nucl. Phys. **B 200** (1982) 266.
C.L. Schultz, Physica **2A** (1983) 71.
Y.K. Zhou, M.L. Yan and B.Y. Hou, Jour. Phys. **A 21** (1988) L929; Nucl. Phys. **B 324** (1989) 715.

[14] A. Matsuo, Commun. Math. Phys. **151** (1993) 263.

[15] A. Nakayashiki, Int. J. Mod. Phys. **A 9** (1994) 5673-5681.

[16] V. Tarasov and A. Varchenko, 'Asymptotic solution to the quantized Knizhnik-Zamolodchikov equation and Bethe vectors', Am. Math. Soc., Ser. 2, **174** (1996) 235-273.

[17] H.M. Babujian, A. Fring and M. Karowski, 'Form factors of the $SU(N)$ - chiral Gross-Neveu model', in preparation.

[18] B. Berg, M. Karowski, V. Kurak and P. Weisz, Nucl. Phys. **B 134** (1978) 125-132.