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ASYMPTOTIC BEHAVIOR IN PURE YANG-MILLS THEORY WITH NON-COVARIANT GAUGES

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The β-function of pure Yang-Mills theory is gauge independent for linear non-singular renormalizable gauges in lowest order. The vertex functions scale gauge independently in the ultraviolet region for this general class of gauges

1. Introduction

The interest in perturbation expansion for pure Yang-Mills theory results mainly from the fact, that the large momentum behavior of the Green functions in the deep Euclidean region can be computed reliably using renormalization group arguments [1].

In the quantization procedure of a gauge field theory one has to add a gauge fixing term to the gauge-invariant Lagrangian in order to obtain the Feynman rules [2]. The resulting gauge dependence of the renormalization group parameters [3] leads to the question of gauge dependence of the β-function and the anomalous dimensions of formally gauge-invariant composite operators [4].

In this note a general class of gauge conditions, which are linear, non-singular [5] and renormalizable, are considered *. The well-known classes of covariant gauges like the Feynman and Landau gauge and non-covariant gauges like the Coulomb gauge are included.

In sect. 2 the Feynman rules for the Yang-Mills theory in a general gauge are given and the renormalization is discussed.

In sect. 3 the renormalization group parameters in the one-loop approximation are calculated. The gauge independence of the β-function for linear non-singular and renormalizable gauges [5] in μ-normalized pure Yang-Mills theory follows explicitly in this approximation.

* In quantum electrodynamics similar gauge conditions are discussed by Tatur and Bialynicki-Birula [12]. The authors thank Professor B. Schroer for bringing these papers to their attention.

In sect. 4 the ultra violet limit for the effective gauge parameters is analysed. The Coulomb gauge is approached for a two-parameter subclass of gauges. The asymptotic behavior of vertex functions with insertions of composite operators is discussed in sect. 5.

2. Yang-Mills theory in a general class of gauges

The Lagrangian for the interaction of massless vector mesons with massless fermions is given by

$$\mathcal{L} = -\tfrac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2\alpha} (\partial^\mu \hat{B}^a_\mu)^2 + \bar{C}^a M^{ab} C^b$$

$$+ i \bar{\psi}_\alpha (\not\partial \, \delta_{\alpha\beta} - i g \tau^a_{\alpha\beta} \not{B}^a) \psi_\beta + \text{c.t.} ,$$

with

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g f^a{}_{bc} B^b_\mu B^c_\nu . \tag{1}$$

The structure constants f_{abc} of the gauge group obey $f_{acd} f_{bcd} = 2 c_1 \delta_{ab}$ and $\tau^a_{\alpha\beta}$ are representation matrices for the fermions. The field $\hat{B}^a_\mu$ in the gauge fixing term is defined by $\hat{B}^a_\mu = \hat{T}^\nu_\mu B^a_\nu$ with $\hat{T}^\nu_\mu$ a positive symmetric 4×4 matrix such as to desribe linear renormalizable non-singular gauges [5]. For special gauges we write the matrix $\hat{T}_{\mu\nu}$ in the following way: $\hat{T}_{\mu\nu} = d g_{\mu\nu} - a \eta_\mu \eta_\nu$ where η_μ is a fixed vector. The Landau, Feynman, Coulomb, and axial gauge [7] are obtained for the values of the parameters $(\alpha, d, a) = (0, 1, 0), (1, 1, 0), (0, 1, 1)$ resp. $(0, 0, -1)$. In the sequel we do not consider the axial gauge since it is singular. $\hat{T}$ has to be normalized, e.g. one can set the biggest eigenvalue of $\hat{T}^\nu_\mu$ equal to 1. $\bar{C}^a$ and C^a are the Faddeev-Popov ghost fields and

$$M^{ab} = \partial^\mu \hat{D}^{ab}_\mu \quad \text{with}\, \hat{D}^{ab}_\mu C^b = (\hat{T}\partial)_\mu C^a + g f^{abc} (\hat{T} B^b)_\mu C^c .$$

The general gauge fixing term in the Lagrangian (1) gives rise to modified Feynman rules. The vector-meson propagator, the Faddeev-Popov propagator and the Faddeev-Popov vertex read

$$D^{ab}_{\mu\nu}(p) = -\frac{i\delta_{ab}}{p^2 + i\epsilon} \left\{ g_{\mu\nu} + \frac{\hat{p}^2 + \alpha p^2}{(p \cdot \hat{p})^2}\, p_\mu p_\nu - \frac{p_\mu \hat{p}_\nu + \hat{p}_\mu p_\nu}{p \cdot \hat{p}} \right\} ,$$

$$G^{ab}(p) = \frac{i\delta_{ab}}{p \cdot \hat{p} + i\epsilon} ,$$

$$\gamma^{abc}_\mu(q) = -i g f_{abc} \hat{q}_\mu , \quad \text{with } \hat{p}_\mu = \hat{T}^\nu_\mu p_\nu . \tag{2}$$

If $\hat{T}_{\mu\nu}$ is $g_{\mu\nu}$, then the vector meson propagator is simply

$$D^{ab}_{\mu\nu}(p) = -\frac{i\delta_{ab}}{p^2 + i\epsilon}\left\{g_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right\}.$$

Fermions have been added to the pure Yang-Mills Lagrangian in order to simplify perturbation calculation. Since the renormalized charges of the Yang-Mills and the fermion fields are equal [6], the fermion-vector three-point vertex can be used to define the coupling constant g. Pure Yang-Mills theory is formally obtained in the case $c_2 = 0$ [8] where $\mathrm{tr}(\tau^a\tau^b) = 2c_2\delta_{ab}$ for the fermion representation matrices.

As usually the renormalized theory is obtained by specifying the subtraction constants by means of normalizing vertex functions at an arbitrary Euclidean point. The divergences are determined in the one-loop approximation by calculating the self-energy and the vertex diagrams (figs. 1 and 2). These divergences can be absorbed by multiplicative counterterms in the Lagrangian (appendix A) which are obtained as usual by introducing wave function, vertex and gauge parameter renormalization constants. The explicit calculation (appendix B) shows that in the case of the general gauge fixing term in the Lagrangian (1) the renormalization constants turn out to be matrices,

$$B^{a\mu}_u = (\sqrt{Z_3}\,B^a)^\mu = \sqrt{Z_3}^\mu{}_\nu\,B^{a\nu} \tag{3}$$

for the vector-meson field,

$$\hat{T}^{\mu\nu}_u = (\hat{X}\,\hat{T})^{\mu\nu}\,, \qquad \alpha_u = Y\alpha\,. \tag{4}$$

for the gauge parameters, and for the fermion-vector three-point vertex:

$$g_u\sqrt{Z_3}_{\mu\nu}Z^M_2 = gZ^M_{1\mu\nu}\,. \tag{5}$$

Eq. (5) is the result of the one-loop approximation given by eqs. (14) and (17′), which shows the multiplicative counterterm structure (see (A.1)). The coupling renormalization can also be determined by the three-meson vertex:

$$g_u\sqrt{Z_3}_{\mu\nu}Z_{3\rho\sigma} = gZ_{1\mu\nu\rho\sigma} \tag{6}$$

or by the ghost vertex:

$$g_u\sqrt{Z_3}_{\mu\nu}\,\widetilde{Z}_3 = g\,\widetilde{Z}_{1\mu\nu}\,. \tag{7}$$

The Ward-Slavnov identities guarantee the equality of the renormalized coupling constants [6]. The wave function renormalization constants for the Fermi field and the ghost field are defined by

$$\psi_u = \sqrt{Z^M_2}\,\psi\,, \tag{8}$$

$$C^a_u = \sqrt{\widetilde{Z}_3}\,C^a\,. \tag{9}$$

The renormalized one-particle irreducible (1PI) Green functions $\Gamma^{(n)}_{\mu_1,\dots\mu_n}(p_1, \dots p_n, \mu, g, T)$ are functions of the ratio $\hat{T}/\sqrt{\alpha} = T$ only, since the gauge fixing term in the Lagrangian only depends on T and a factor $\sqrt{\alpha}$ can be absorbed in the ghost field renormalization constant.

The renormalization group equation for the general class of gauges for pure Yang-Mills theory is

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\gamma_{\mu\nu}\left(T^{\nu\lambda}\frac{\partial}{\partial T^{\lambda}_{\mu}}-n^{\mu\nu}\right)\right\}\Gamma^{(n)}(p_1,\dots p_n;\mu,g,T)=0\,, \qquad (10)$$

where

$$\beta=\mu\frac{\partial g}{\partial\mu}=gZ_1^M\sqrt{Z_3}^{-1}(Z_2^M)^{-1}\mu\frac{\partial}{\partial\mu}(Z_1^M)^{-1}\sqrt{Z_3}\,Z_2^M\,, \qquad (11)$$

$$\gamma_{\mu\nu}=\frac{1}{2}\left(Z_3^{-1}\,\mu\frac{\partial}{\partial\mu}Z_3\right)_{\mu\nu}\,, \qquad (12)$$

$$\gamma_{\mu\nu}(n^{\mu\nu}\Gamma^{(n)})_{\mu_1\dots\mu_n}=\gamma^{\nu}_{\mu_1}\Gamma^{(n)}_{\nu\mu_2\dots\mu_n}+\gamma^{\nu}_{\mu_2}\Gamma^{(n)}_{\mu_1,\nu,\mu_3\dots\mu_n}+\dots\,.$$

The renormalization group eq. (10) is a generalisation of

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-n\gamma+\delta\frac{\partial}{\partial\alpha}\right)\Gamma^{(n)}_{\mu_1\dots\mu_n}(p_1,\dots p_n;\mu,g,\alpha)=0\,,$$

which is valid for covariant gauges. The well-known relation $\delta=-2\alpha\gamma$ is given in the general case by

$$\mu\frac{\partial}{\partial\mu}T_{\mu\nu}=\left(T\sqrt{Z_3}^{-1}\mu\frac{\partial}{\partial\mu}\sqrt{Z_3}\right)_{\mu\nu}=(T\gamma)_{\mu\nu}\,, \qquad (13)$$

as a consequence of (see appendix A)

$$Y^{\frac{1}{2}}(\mathcal{X}^{-1})_{\mu\nu}=\sqrt{Z_3}_{\mu\nu}\,.$$

The solution of the renormalization group equation (10) is expressed in terms of effective parameters for the coupling constant and the gauge parameters in sect. 4.

3. Renormalization group parameters in the one-loop approximation

For computational purposes it is useful to calculate the logarithmically divergent terms in the charge- and wave-function renormalization constants of order g^2 in order to determine the renormalization group parameters using eqs. (11) and (12). Explicite calculation shows the logarithmically divergent part of the second-order self-energy diagrams in the general gauge is equal to

$$\delta_{ab}\{(Z_3-1)_{\mu\nu}q^2-\tfrac{1}{2}(Z_3-1)_{\mu\lambda}q^{\lambda}q_{\nu}-\tfrac{1}{2}(Z_3-1)_{\nu\lambda}q^{\lambda}q_{\mu}\} \qquad (14)$$

with

$$(Z_3 - 1)_{\mu\nu} = -\frac{g^2 C_1}{16\,\pi^2} \ln\frac{\Lambda^2}{\mu^2}\, C_{\mu\nu}$$

(Λ is a ultra violet cut-off). The elements $C_{\mu\nu}$ depend only on the gauge parameters. The result for the general class of gauges and further details are given in appendix B.

If the matrix T is diagonal and has the elements $T_{00} = (1-a)/\sqrt{\alpha}$, $T_{11} = T_{22} = T_{33} = -1/\sqrt{\alpha}$ one gets rotational invariant gauges. The gauge-fixing term in the Lagrangian (1) then reads

$$\mathcal{L}_g = -\frac{1}{2\alpha}\,\{\partial^\mu B^a_\mu - a\,\partial^\nu \eta_\nu(\eta^\mu B^a_\mu)\}^2 \quad \text{with } \eta_\nu = (1, \mathbf{0})\,. \tag{15}$$

For $a = 1$, $\alpha = 0$ this gives the Coulomb gauge and for $a = 0$ the covariant gauges like the Landau gauge and Feynman gauge are obtained. $C_{\mu\nu}$ now has non-vanishing elements.

$$-C_{00} = \tfrac{22}{3} - 3\sqrt{1-a} - \frac{\alpha}{\sqrt{1-a}}\,,$$

$$C_{11} = C_{22} = C_{33} = \frac{1}{3}\left(21 + \frac{a-16}{1+\sqrt{1-a}}\right) - \frac{\alpha}{\sqrt{1-a}}\,. \tag{16}$$

If in addition $a = 0$, the known expression for covariant gauges follows [4].

The charge renormalization constant is calculated in second order by means of the logarithmic divergent part of the fermion-vector three-point vertex

$$g\,\tau^a\gamma^\nu\Lambda_{\mu\nu}(\Lambda/\mu) = g\,\tau^a\gamma^\nu(Z_1^M - 1)_{\mu\nu}\,, \tag{17}$$

which is calculated in appendix B. The result is

$$(Z_1^M - 1)_{\mu\nu} = \frac{g^2 C_1}{16\,\pi^2} \ln\frac{\Lambda^2}{\mu^2}\; \tfrac{1}{2}(C_{\mu\nu} + \tfrac{22}{3}\, g_{\mu\nu})\,, \tag{17'}$$

where only the pure Yang-Mills contribution proportional to C_1 is written down.

The renormalization group parameters follow from eqs. (11) and (12)

$$\beta = -g\,\frac{g^2 C_1}{16\,\pi^2}\,\frac{22}{3} = -b_0 g^3\,, \tag{18}$$

$$\gamma_{\mu\nu} = \frac{g^2 C_1}{16\,\pi^2}\, C_{\mu\nu}\,. \tag{19}$$

This shows the gauge independence of the β-function for the general class of renormalizable gauges in lowest order of perturbation theory in a μ-normalized theory.

4. Ultraviolet behavior of vertex functions in the general gauge

The gauge independence of the β-function eq. (18) confirms pure Yang-Mills theory to be "asymptotically free" in all non-singular renormalizable gauges. The solution of the renormalization group equation (10) is expressed in terms of effective coupling constant $\bar{g}(\lambda, g)$ and effective gauge parameters $\bar{T}(\lambda, g, T)$ using the method of characteristics with the scaling parameter λ. This determines the ultraviolet behaviour of the vertex functions.

For covariant gauges it is known that the effective gauge parameter $\bar{\alpha}$ tends to the finite value $\frac{13}{3}$ as $\lambda \to \infty$ [3], while for a class of rotational invariant gauges with $\alpha = 0$ the Coulomb gauge is approached [9].

The differential equations for the effective parameters are in the general case

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}\ln\lambda} = \beta(\bar{g}) , \qquad \frac{\mathrm{d}\bar{T}_{\mu\nu}}{\mathrm{d}\ln\lambda} = \frac{\bar{g}^2 C_1}{16\pi^2} (\bar{T}\, C(\bar{T}))_{\mu\nu} , \tag{20}$$

with boundary conditions $\bar{g}(1) = g$; $\bar{T}(1) = T$. The solution for the effective coupling constant is always given by

$$\bar{g}^2(\lambda) = g^2(1 + 2b_0 g^2 \ln\lambda)^{-1} ,$$

while the effective gauge parameters in general obey a system of coupled differential equations.

As a first step we state the solution for the two-parameter family of gauges where the gauge fixing term is given by eq. (15). The functions which appear in the two coupled differential equations are then given by eqs. (16) and their behaviour show that the point $(a^*, \alpha^*) = (1, 0)$ is approached for positive values $a > 0$, $\alpha > 0$ in the limit $\lambda \to \infty$, i.e. the Coulomb gauge is the stagnant point for this subclass of gauges.

By expanding the coefficients C_{00} and C_{11} of eq. (16) around $a = 1$, the differential equations may be solved analytically:

$$\bar{T}_{00}(x) = \mathrm{e}^{-\frac{12}{3}x + 3x_1} \sqrt{\tfrac{3}{10} + \exp(-\tfrac{20}{3}x + x_2)} = \frac{1 - \bar{a}}{\sqrt{\bar{\alpha}}} ,$$

$$-\bar{T}_{11}(x) = \mathrm{e}^{\frac{4}{3}x - x_1} \sqrt{\tfrac{3}{10} + \exp(-\tfrac{20}{3}x + x_2)} = \frac{1}{\sqrt{\bar{\alpha}}} , \tag{21}$$

with

$$x = \tfrac{3}{44} \ln\left(\ln\lambda + \frac{1}{2b_0 g^2}\right)$$

and x_1, x_2 being constants fixed by the boundary conditions. In the limit $\lambda \to \infty$ the vertex functions therefore behave like:

$$\Gamma^{(n_\eta, n_T)}(\lambda p_1, \ldots \lambda p_n; \mu, g, T) \overset{\lambda\to\infty}{\sim} \lambda^{4-n_\eta-n_T}$$

$$\times (\ln\lambda)^{(n_\eta C_{00}^\infty - n_T C_{11}^\infty)/2b}\, \Gamma^{(n_\eta, n_T)}(p_1, \ldots p_n; \mu, \bar{g}, \bar{T}_{(\infty)}) \tag{22}$$

with

$$\lim_{\lambda\to\infty} C_{00}(\bar{T}(\lambda)) = C_{00}^\infty = -4\,; \quad \lim_{\lambda\to\infty} C_{11}(\bar{T}(\lambda)) = C_{11}^\infty = -\tfrac{4}{3}, \quad b = \tfrac{22}{3}\,.$$

In addition the Coulomb gauge ($\alpha = 0, a = 1$) has the property that there is no contribution to the "longitudinal part" of the charge renormalization constant as follows from eqs. (16) and (17′).

Next we consider the general case when all the eigenvalues of T are different. The differential equation for the effective gauge parameters eq. (10) is now given by a system of four coupled differential equations. This case is much more involved, but it can be shown that there exists a stagnant point T^* (see appendix B). The coefficients $C_{\mu\nu}$ of the anomalous dimension matrix (19) vanish:

$$\lim_{\lambda\to\infty} C_{\mu\nu}(\bar{T}(\lambda)) = 0 \tag{23}$$

and therefore in the limit $\lambda\to\infty$ the two-point function has a power behavior. The logarithmic factors only occur for subclasses of gauges, namely when the Landau gauge is chosen initially or in the subclass considered above (see eq. (22)).

5. Example of a composite operator

We now determine the asymptotic behavior of the vertex functions with insertions of composite operators.

As an example the composite operators $(G^a_{\mu\nu})^2$ and $g(\delta\mathcal{L}/\delta g)$ are considered and the anomalous dimension matrix is calculated for linear non-singular renormalizable gauges.

The insertion of the operators $O_1 = -\frac{1}{4}\int d^4x (G^a_{\mu\nu}(x))^2$ and $O_2 = \int d^4x g\, \delta\mathcal{L}(x)/\delta g$ can be defined by using the expression for the tree approximation and imposing suitable renormalization conditions [10].

For the pure Yang-Mills Lagrangian and Green functions with n external vector meson legs we define

$$\Gamma^{(n)}_{O_2} = g\frac{\partial}{\partial g}\,\Gamma^{(n)}\,, \tag{24}$$

$$\Gamma^{(n)}_{O_2' \mu\nu} = -\left(T\frac{\partial}{\partial T} - n\right)_{\mu\nu} \Gamma^{(n)}\,, \tag{25}$$

with

$$\left(T\frac{\partial}{\partial T}-n\right)_{\mu\nu}=T_{\mu\lambda}\frac{\partial}{\partial T_{\lambda}{}^{\nu}}-n_{\mu\nu}\,.$$

Then the counting identity [11] for the vector field leads to

$$2\Gamma^{(n)}_{O_1\mu\nu}+g_{\mu\nu}\Gamma^{(n)}_{O_2}-\Gamma^{(n)}_{O_2'\mu\nu}=0\,. \tag{26}$$

Let D denote the operator

$$D=\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\gamma_{\mu\nu}\left\{T^{\nu\lambda}\frac{\partial}{\partial T_{\mu}{}^{\lambda}}-n^{\mu\nu}\right\}.$$

Calculating the commutators $[g\,\partial/\partial g, D]$ and $[(T\,\partial/\partial T-n)_{\mu\nu}, D]$ gives

$$D\Gamma^{(n)}_{O_2}=-g\frac{\partial}{\partial g}\left(\frac{\beta}{g}\right)\Gamma^{(n)}_{O_2}+g\frac{\partial}{\partial g}(\gamma^{\mu\nu})\,\Gamma^{(n)}_{O_2'\mu\nu}\,, \tag{27a}$$

$$D\Gamma^{(n)}_{O_2'\rho\sigma}=\left(T\frac{\partial}{\partial T}\right)_{\rho\sigma}\left(\frac{\beta}{g}\right)\Gamma^{(n)}_{O_2}-\left(T\frac{\partial}{\partial T}\right)_{\rho\sigma}(\gamma^{\mu\nu})\,\Gamma^{(n)}_{O_2'\mu\nu}\,. \tag{27b}$$

The anomalous dimension matrix $\hat{\gamma}_{ij}$ in the renormalization group equation

$$\{D\delta_{ij}-\hat{\gamma}_{ij}\}\Gamma^{(n)}_{O_j}=0$$

reads explicitly in the basis O_1 and O_2

$$\hat{\gamma}_{ij}=\begin{pmatrix}-\left\{\left(T\frac{\partial}{\partial T}\right)_{\rho\sigma}+g_{\rho\sigma}g\frac{\partial}{\partial g}\right\}\gamma^{\mu\nu} & \frac{1}{2}\left\{\left(T\frac{\partial}{\partial T}\right)_{\rho\sigma}+g_{\rho\sigma}g\frac{\partial}{\partial g}\right\}\left(\frac{\beta}{g}-\gamma^{\lambda}{}_{\lambda}\right)\\ 2g\frac{\partial}{\partial g}\gamma^{\mu\nu} & g\frac{\partial}{\partial g}\left(\gamma^{\lambda}{}_{\lambda}-\frac{\beta}{g}\right)\end{pmatrix} \tag{28}$$

with $i=1,2$ and $j=1,2$. The matrix $\hat{\gamma}_{ij}$ is not diagonal, showing that the operators O_1 and O_2 defined by eqs. (24)–(26) are not multiplicatively renormalizable.

The composite operator for the gauge fixing term, eq. (A.4) is multiplicatively renormalizable and there is no mixing. The mixing with Faddeev-Popov ghosts has to be considered for the renormalization of the operators O_1 and O_2 [10]. We only take into account vertex functions with O_1 resp. O_2 insertions defined by eqs. (24)–(26), without external ghost legs. Therefore insertions of ghost operators do not appear [10]. Furthermore we obtain from eqs. (27a) and (27b) or alternatively from the commutators $[\mu\,\partial/\partial\mu, D]=0$ and $[c^{\mu\nu}(T\,\partial/\partial T-n)_{\mu\nu}, D]=0$ two linear combinations which are diagonal under renormalization with $\hat{\gamma}=0$:

$$D\left\{\frac{\beta}{g}\Gamma^{(n)}_{O_2}-\gamma^{\mu\nu}\Gamma^{(n)}_{O_2'\mu\nu}\right\}=0\,, \tag{29a}$$

$$D\,\{C^{\mu\nu}\Gamma^{(n)}_{O_2'\mu\nu}\}=0\,, \tag{29b}$$

if we only take the second-order contribution eq. (19). The solutions of eqs. (29a) and (29b) give the asymptotic behavior of the vertex functions with a composite operator inserted: With the help of eq. (26) we have

$$\frac{\beta}{g}\Gamma^{(n)}_{O_2}(\lambda p_i, g_1 T) \overset{\lambda\to\infty}{\sim} \lambda^{4-n}\gamma^{\mu\nu}(g, T)\,\Gamma^{(n)}_{O_1\mu\nu}(p_i, \bar{g}, \bar{T}) \times \exp\left(-n_{\mu\nu}\int_0^{\ln\lambda} \mathrm{d}t\, \gamma^{\mu\nu}(\bar{g}(t), \bar{T}(t))\right),$$

$$\gamma^{\mu\nu}\Gamma^{(n)}_{O_1\mu\nu}(\lambda p_i, g, T) \overset{\lambda\to\infty}{\sim} \lambda^{4-n}\gamma^{\mu\nu}(g, T)\left(1-\gamma^{\lambda}{}_{\lambda}\frac{g}{\beta}\right)\Gamma_{O_1\mu\nu}(p_i, \bar{g}, \bar{T}) \times \exp\left(-n_{\mu\nu}\int_0^{\ln\lambda} \mathrm{d}t\, \gamma^{\mu\nu}(\bar{g}(t), \bar{T}(t))\right).$$

For the various gauges the asymptotic behavior of the coefficients $C_{\mu\nu}(\bar{T})$ have been calculated in sect. 4: constant values for the Landau gauge and the generalized Coulomb gauges eq. (22) and powers of $(\ln\lambda)^y$ with $y = -\frac{13}{22}$ for the covariant gauges ($\alpha \neq 0$) and $y = -\frac{21}{55}$ for the general case.

6. Conclusion

Massless Yang-Mills theory is multiplicatively renormalizable in the one loop approximation for the general class of linear non-singular renormalizable gauges. The β-function is then gauge independent for this class of gauges.

For the two-dimensional subclass of all rotational invariant gauges the effective gauge parameter approaches the Coulomb gauge in the ultraviolet limit. The non-covariance of these gauges remains in the logarithmic factors of the asymptotic expression of the vertex functions. In the general case the vertex functions have a power behavior.

The vertex function with insertion of $\frac{1}{4}\int \mathrm{d}^4x\,(G^a_{\mu\nu}(x))^2$ has asymptotic behavior with logarithmic deviation from scaling in all gauges coming either from the Yang-Mills fields or from the insertion. The composite operator $(G^a_{\mu\nu})^2$ is not multiplicatively renormalizable.

Appendix A

Renormalization of the Yang-Mills theory with non-covariant gauge conditions

Introducing the various renormalized quantities defined by eqs. (3)–(9) in the Lagrangian (1) leads to the counter terms

$$\text{c.t.} = -\tfrac{1}{2}((Z_3-1)B^a)^\mu \Box B^a_\mu + \tfrac{1}{2}\{(\partial^\mu(\sqrt{Z_3}\,B^a)_\mu)^2 - (\partial^\mu B^a_\mu)^2\}$$
$$-\tfrac{1}{2} g f_{abc}\{((Z_1-1)\partial B^a)_{\mu\nu} - ((Z_1-1)\partial B^a)_{\mu\nu}\} B^{b\mu} B^{c\nu}$$
$$-\tfrac{1}{4} g^2 f_{abc} f_{ade}((Z_1^2 Z_3^{-1}-1)B^b B^c)_{\mu\nu} B^{d\mu} B^{e\nu}$$
$$-\frac{1}{2\alpha}(\partial^\mu(Y^{-\frac{1}{2}}\hat{X}\sqrt{Z_3}-1)\hat{B}^a_\mu)^2 + \bar{C}^a \partial^\mu((\widetilde{Z}_3\hat{X}-1)\hat{\partial})_\mu C^a$$
$$+ g f_{abc}\bar{C}^a \partial_\mu((\hat{X}\widetilde{Z}_1-1)\hat{B}^b)^\mu C^c \tag{A.1}$$
$$+ i\bar{\psi}\,\partial\!\!\!/\,\psi(Z_2^M-1) + g\bar{\psi}\gamma_\mu((Z_1^M-1)B^a)^\mu \tau^a \psi$$

with

$$(Z_1\partial B^a)_{\mu\nu} = Z_{1\,\mu\nu}^{\;\rho\sigma}\,\partial_\rho B^a_\sigma\,, \qquad (Z_1^2 Z_3^{-1} B^b B^c)_{\mu\nu} = Z_{1\,\mu\nu}^{\;\rho\sigma}\, Z_{1\,\rho\sigma}^{\;\kappa\lambda}(Z_3^{-1})^\delta_\lambda B^b_\kappa B^c_\delta\,.$$

The generalized Ward identity [13] for the vector-meson two-point function reads

$$\frac{1}{\alpha}\langle T\hat{\partial}^\mu B^a_\mu(x)\hat{\partial}^\nu B^b_\nu(y)\rangle = -\delta_{ab}\delta(x-y)\,. \tag{A.2}$$

As a consequence the renormalization constants of the gauge fixing term in eq. (A.1) are given by

$$(Y^{-\frac{1}{2}}\hat{X}\sqrt{Z_3})_{\mu\nu} = g_{\mu\nu}\,. \tag{A.3}$$

The gauge fixing term for non-covariant gauges is not renormalized

$$\frac{1}{2\alpha_u}(\partial_\mu \hat{B}^{a\,\mu}_u)^2 = \frac{1}{2\alpha}(\partial_\mu \hat{B}^{a\mu})^2\,. \tag{A.4}$$

In momentum space the first and second term in eq. (A.1) give $(Z_3-1)_{\mu\nu}q^2$ and $((\sqrt{Z_3 q})_\mu(\sqrt{Z_3 q})_\nu - q_\mu q_\nu)$ and formula (14) for the two-point function follows. The last term of eq. (A.1) corresponds to the fermion-vector three-point vertex $g(Z_1^M-1)_{\mu\nu}\gamma^\mu q^\nu$ given in eq. (17). The other terms contribute to the renormalization constants of eqs. (5)–(9).

Appendix B

Details of the one-loop approximation

The vector-meson propagator eq. (2), which can be written as

$$D^{ab}_{\mu\nu}(p) = -\frac{i\delta_{ab}}{p^2+i\epsilon}\left\{g_{\mu\nu} + \frac{\bar{p}^2+p^2}{(p\cdot\bar{p})^2}\,p_\mu p_\nu - \frac{\bar{p}_\mu p_\nu + p_\mu\bar{p}_\nu}{(p\cdot\bar{p})}\right\}\,, \tag{B.1}$$

with $\bar{p}_\mu = (Tp)_\mu$, $T = \hat{T}/\sqrt{\alpha}$, produces additional terms in the one-loop calculation of the renormalization group parameters.

The integrals occurring are typically of the form

$$\int_p \frac{\text{Poly}\,(q,p)}{(p^2)^n (p\cdot\bar{p})^m} \qquad \text{with} \int_p f(p) = \lim_{\Lambda\to\infty} \frac{(4\pi)^2}{i} \Lambda^2 \frac{\partial}{\partial\Lambda^2} \int \frac{d^4p}{(2\pi)^4} f_\Lambda(p)\,, \tag{B.2}$$

which gives the logarithmically divergent part.

All contributions can be reduced to the three integrals

$$I_1(T) = \int_p \frac{1}{p^2(p\cdot\bar{p})} = \int_0^1 d\beta \frac{1}{\sqrt{\Pi_{s=0}^3(\beta + (1-\beta)T_s^s)}}\,,$$

$$I_2(T) = \int_p \frac{1}{(p\cdot\bar{p})^2} = \frac{1}{\sqrt{\det T}}\,, \tag{B.3}$$

$$J_r^r(T) = \int_p \frac{p_r p^r}{p^2(p\cdot\bar{p})^2} = \frac{1}{2}\int_0^1 d\beta \frac{1-\beta}{\sqrt{\Pi_{s=0}^3(\beta + (1-\beta)T_s^s}(\beta + (1-\beta)T_r^r)}\,,$$

with

$$T_\mu{}^\nu = \begin{pmatrix} T_0{}^0 & & & 0 \\ & T_1{}^1 & & \\ & & T_2{}^2 & \\ 0 & & & T_3{}^3 \end{pmatrix}, \qquad \text{r} = 0, 1, 2, 3 \text{ (no summation over r)}.$$

The second order self-energy graphs (fig. 1)

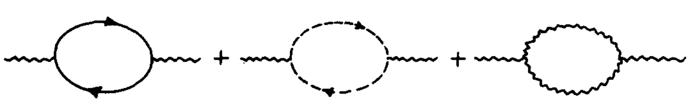

Fig. 1.

give eq. (14) and the vertex correction diagrams (fig. 2)

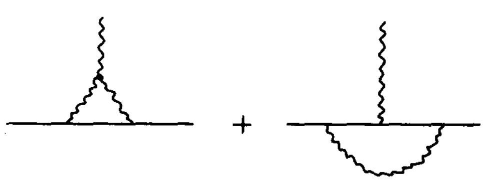

Fig. 2.

lead to eq. (17) with

$$-C_{\mu\nu} = (\tfrac{22}{3} - I_2)g_{\mu\nu} + 4(JT^2)_{\mu\nu} - (JT)_{\mu\nu}\,\mathrm{tr}\,T - 3I_1 T_{\mu\nu}\,. \tag{B.4}$$

For special values of T the integration of eq. (B.3) may be carried out, e.g. if T is diagonal and $T_1{}^1 = T_2{}^2 = T_3{}^3$ the result is eq. (16).

The system of coupled differential equations (20) is in general

$$\frac{\mathrm{d}}{\mathrm{d}x}\,(\ln \overline{T}_r{}^r(x)) = C_r{}^r(\overline{T}) \tag{B.5}$$

with x given in eq. (21). From the integrals (B.3) it is obvious that the functions $C_r{}^r(T)$ are not single valued. Let the matrix $\hat{T} = \sqrt{\alpha}\,T$ be normalized

$$\hat{T} = \begin{pmatrix} \hat{T}_0{}^0 & & & 0 \\ & \hat{T}_1{}^1 & & \\ & & \hat{T}_2{}^2 & \\ 0 & & & 1 \end{pmatrix}$$

with $\hat{T}_r{}^r < 1$. If now all $\hat{T}_r{}^r$ are different it can be shown that in the second sheet for $\lambda \to \infty$ $\hat{T}(\lambda)$ and $\bar{\alpha}(\lambda)$ approach the values

$$\hat{T}^* = \begin{pmatrix} \frac{1}{4} & & & 0 \\ & \frac{1}{4} & & \\ & & \frac{1}{4} & \\ 0 & & & 1 \end{pmatrix}$$

and $\alpha^* = -\frac{5}{3}$ respectively. The $C_r{}^r(\hat{T}(\lambda))$ all vanish at this point.

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