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$O(N)$-matrix difference equations and a nested Bethe ansatz

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Abstract

A system of $O(N)$-matrix difference equations is solved by means of the off-shell version of the nested algebraic Bethe ansatz. In the nesting process, a new object, the Π-matrix, is introduced to overcome the complexities of the $O(N)$-group structure. The highest weight property of the solutions is proved and some explicit examples are discussed.

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1. Introduction

$O(N)$ Gross–Neveu and $O(N)$ σ-models are asymptotically free quantum field theories which attract high interest, since they share some common features with QCD. Since perturbation theory fails for these models, exact results, such as exact generalized form factors, are desirable and welcome. The concept of a generalized form factor was introduced in [1, 2], where several consistency equations were formulated. Subsequently, this approach was developed further and investigated in different models by Smirnov [3]. Generalized form factors are matrix elements of fields with many particle states. To construct these objects explicitly, one has to solve generalized Watson's equations that are matrix difference equations. To solve these equations, the so-called off-shell Bethe ansatz is applied [4–6]. The conventional Bethe ansatz introduced by Bethe [7] is used to solve eigenvalue problems and its algebraic formulation was developed by Faddeev and co-workers (see e.g. [8]). The off-shell Bethe ansatz has been introduced in [9] to solve the Knizhnik–Zamolodchikov equations that are differential equations. In

[10], a variant of this technique has been formulated to solve matrix difference equations of the form

$$K(u_1,\ldots,u_i+\kappa,\ldots,u_n)=K(u_1,\ldots,u_i,\ldots,u_n)Q(u_1,\ldots,u_n,;i),\qquad(i=1,\ldots,n),$$

where $K(\underline{u})$ is a co-vector-valued function, $Q(\underline{u},i)$ are matrix valued functions and κ is a constant to be specified. We use here a co-vector formulation because this is more convenient for its application to the form factor program. For higher rank internal symmetry groups, the nested version of this Bethe ansatz has to be applied. The nested Bethe ansatz as a method to solve eigenvalue problems was introduced by Yang [11] and further developed by Sutherland [12, 13].

In this paper, we will solve the $O(N)$-difference equations combining the nested Bethe ansatz with the off-shell Bethe ansatz. This procedure is similar to the $SU(N)$ [14] case, where also a nesting procedure is used. However, the algebraic formulation for $O(N)$ is much more intricate because the R-matrix exhibits an extra new term. In addition, for $SU(N)$ we can use the same R-matrix at every level, while for the group $O(N)$ the R-matrix changes after each level. Therefore, in our construction a new object, called Π-matrix, is introduced in order to overcome these difficulties. This provides a systematic formulation of techniques introduced by Tarasov [15] and also used in [16]. In [17], a different procedure was used to solve the $O(N)$ on-shell Bethe ansatz for even N.

The results of this paper will be applied in [18] to calculate exact form factors of the $O(N)$ σ-models and Gross–Neveu models. We should mention that the first computation of form factors for $O(3)$ σ-model is due to [3] (see also [19, 20]). There are also new developments concerning the connection between 2D conformal field theory (CFT) and integrable models with $N=2$ super Yang–Mills (SYM) theories in different higher dimensions. First, there is a surprising relation between 2D conformal blocks and the instanton partition function in $N=2$, 4D SYM theory [21] (Alday, Gaiotto, Tachikawa—AGT relation) and this is a particular version of the AdS/CFT correspondence, which is a more general part of the gauge/string duality. There is also a q-deformation of the AGT relation which connects the $N=2$ 5D SYM theory and the q-deformed conformal blocks [22]. This last relation offers new insights and gives the intriguing hope that the form factor program can be used to obtain a deeper understanding of this connection. The solution of the difference equations is the first step to obtain the exact form factors and therefore important physical relations and correlation functions for integrable models. In fact, difference equations play a significant role in various contexts of mathematical physics (see e.g. [23] and references therein).

The paper is organized as follows. In section 2, we recall some results and fix the notation concerning the $O(N)$ R-matrix, the monodromy matrix and some commutation rules. We also introduce a new object, which we call the Π-matrix and which is a central element in our construction of the nested off-shell Bethe vector. In section 3, we introduce the nested generalized Bethe ansatz to solve a system of $O(N)$-difference equations and present the solutions in terms of 'Jackson-type integrals'. We introduce a new type of monodromy matrix fulfilling a new type of Yang–Baxter relation and which is adapted to the difference problem. In particular, this yields a relatively simple proof of our main result, which is theorem 3.4. In section 4, we prove the highest weight property of the solutions and calculate the weights. Section 5 contains some examples of solutions of the $O(N)$-difference equations. The appendices provide the more complicated proofs of additional results we have obtained. In particular, in appendix B we prove that all seven types of 'unwanted terms' in the Bethe ansatz cancel.

2. General setting and notion of the Π-matrix

2.1. The $O(N)$ R-matrix

Let $V^{1,\dots,n}$ be the tensor product space,

$$V^{1,\dots,n} = V_1 \otimes \cdots \otimes V_n, \tag{1}$$

where the vector spaces $V_i \cong \mathbf{C}^N$, $(i = 1,\dots,n)$ are copies of the fundamental vector representation space of $O(N)$ with the (real) basis vectors

$$|\alpha\rangle_r \in V, (\alpha = 1,\dots,N).$$

It is straightforward to generalize the results of this paper to the case where the V_i are vector spaces for other representations. We denote the canonical basis vectors of $V^{1,\dots,n}$ by

$$|\,\alpha_1,\dots,\alpha_n\,\rangle \in V^{1,\dots,n}, \qquad (\alpha_i = 1,\dots,N)\,. \tag{2}$$

A vector $v^{1,\dots,n} \in V^{1,\dots,n}$ is given in terms of its components by

$$v^{1,\dots,n} = \sum_{\underline{\alpha}} |\,\alpha_1,\dots,\alpha_n\,\rangle_r\, v^{\alpha_1,\dots,\alpha_n}. \tag{3}$$

A matrix acting on $V^{1,\dots,n}$ is denoted by

$$A_{1,\dots,n} : V^{1,\dots,n} \to V^{1,\dots,n}. \tag{4}$$

We will also use the dual space $V_{1,\dots,n} = (V^{1,\dots,n})^\dagger$.

The $O(N)$ spectral parameter-dependent R-matrix was found by Zamolodchikov–Zamolodchikov [24].[4] It acts on the tensor product of two (fundamental) representation spaces of $O(N)$. It may be written as

$$R_{12}(u_{12}) = (\mathbf{1}_{12} + c(u_{12})\,\mathbf{P}_{12} + d(u_{12})\,\mathbf{K}_{12}) : \, V^{12} \to V^{21}\,, \tag{5}$$

where $\mathbf{P}_{12}$ is the permutation operator, $\mathbf{K}_{12}$ is the annihilation-creation operator and $u_{12} = u_1 - u_2$. Here and in the following, we associate with each space V_i a variable (spectral parameter) $u_i \in \mathbf{C}$. The components of the R-matrix are

$$R_{\alpha\beta}^{\delta\gamma}(u_{12}) = \delta_\alpha^\gamma\,\delta_\beta^\delta + \delta_\alpha^\delta\,\delta_\beta^\gamma\,c(u_{12}) + \delta^{\gamma\delta}\delta_{\alpha\beta}\,d(u_{12}) \tag{6}$$

from which $\mathbf{P}_{12}$ and $\mathbf{K}_{12}$ can be read off. The functions

$$c(u) = \frac{-1}{u}, \qquad d(u) = \frac{1}{u - 1/\nu}, \qquad \nu = \frac{2}{N-2} \tag{7}$$

are obtained as the rational solution of the Yang–Baxter equation

$$R_{12}(u_{12})\,R_{13}(u_{13})\,R_{23}(u_{23}) = R_{23}(u_{23})\,R_{13}(u_{13})\,R_{12}(u_{12})\,, \tag{8}$$

where we have employed the usual notation [11]. We will also use

$$\tilde{R}(u) = R(u)/a(u)$$

with

$$a(u) = 1 + c(u) = \frac{u-1}{u}.$$

The 'unitarity' of the R-matrix reads as

$$\tilde{R}_{21}(u_{21})\tilde{R}_{12}(u_{12}) = 1$$

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[4] We use here the normalization $R = S/\sigma_2$ and the parameterization $u = \theta/\mathrm{i}\pi\nu$, which is more convenient for our purpose.

and the three eigenvalues of the R-matrix are

$$R_\pm(u) = 1 \pm c(u) = \frac{u \mp 1}{u}, \qquad R_0 = 1 + c(u) + Nd(u)\,. \tag{9}$$

The crossing relation may be written as

$$R_{12}(u_{12}) = \mathbf{C}^{2\bar{2}}\, R_{\bar{2}1}(\hat{u}_{12})\, \mathbf{C}_{\bar{2}2} = \mathbf{C}^{1\bar{1}}\, R_{2\bar{1}}(\hat{u}_{12})\, \mathbf{C}^{\bar{1}1}\,, \tag{10}$$

where $\hat{u} = 1/\nu - u$. Here, $\mathbf{C}^{1\bar{1}}$ and $\mathbf{C}_{1\underline{\bar{1}}}$ are the charge conjugation matrices. Their matrix elements are $\mathbf{C}^{\alpha\bar{\beta}} = \mathbf{C}_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$, where $\bar{\beta}$ denotes the anti-particle of β. In the real basis used up to now the particles are chargeless, which means that $\bar{\beta} = \beta$ and $\mathbf{C}$ is diagonal.

In the following, we will use instead of the real basis $|\alpha\rangle_r$, $(\alpha = 1, 2, \ldots, N)$ the complex basis given by

$$\begin{aligned} |\alpha\rangle &= \tfrac{1}{\sqrt{2}}\,(|2\alpha - 1\rangle_r + \mathrm{i}|2\alpha\rangle_r) \\ |\bar{\alpha}\rangle &= \tfrac{1}{\sqrt{2}}\,(|2\alpha - 1\rangle_r - \mathrm{i}|2\alpha\rangle_r) \end{aligned}$$

for $\alpha = 1, 2, \ldots, [N/2]$. If N is odd, there is in addition $|0\rangle = |\bar{0}\rangle = |N\rangle_r$. The weight vector $w = \big(w_1, \ldots, w_{[N/2]}\big)$ and the charges of the one-particle states are given by

$$\begin{aligned} &\text{for } |\alpha\rangle : w_k = \delta_{k\alpha}, \qquad Q = 1 \\ &\text{for } |\bar{\alpha}\rangle : w_k = -\delta_{k\alpha}, \qquad Q = -1 \\ &\text{for } |0\rangle : w_k = 0, \qquad Q = 0. \end{aligned}$$

Remark 2.1. *For even N, this means that we consider $O(N)$ as a subgroup of $U(N/2)$ and the charge Q is its $U(1)$ charge. For $N = 3$, we may identify the particles* $1, \bar{1}, 0$ *with the pions* $\pi_\pm, \pi_0$.

The highest weight eigenvalue of the R-matrix is

$$R_{11}^{11}(u) = R_+(u) = a(u).$$

We order the states as $1, 2, \ldots, (0), \ldots, \bar{2}, \bar{1}$ (0 only for N odd). Then the charge conjugation matrix in the complex basis is of the form

$$\mathbf{C}^{\delta\gamma} = \delta^{\delta\bar{\gamma}}, \quad \mathbf{C}_{\alpha\beta} = \delta_{\alpha\bar{\beta}}$$

$$\mathbf{C} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 1 \\ \vdots & \ddots & \vdots & . & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & . & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 \end{pmatrix}. \tag{11}$$

The annihilation-creation matrix in (5) may be written as

$$\mathbf{K}_{\alpha\beta}^{\delta\gamma} = \mathbf{C}^{\delta\gamma}\mathbf{C}_{\alpha\beta}.$$

2.2. *The monodromy matrix*

We consider a state with n particles and as is usual in the context of the algebraic Bethe ansatz we define [25, 8] the monodromy matrix by

$$T_{1,\ldots,n,0}(\underline{u}, u_0) = R_{10}(u_{10}) \cdots R_{n0}(u_{n0}) \tag{12}$$

with $\underline{u} = u_1, \ldots, u_n$. It is a matrix acting on the tensor product of the 'quantum space' $V^{1,\ldots,n} = V_1 \otimes \cdots \otimes V_n$ and the 'auxiliary space' V_0. All vector spaces V_i are isomorphic to a space V whose basis vectors label all kinds of particles. Here, $V \cong \mathbb{C}^N$ is the space of the vector representation of $O(N)$.

Suppressing the indices $1, \ldots, n$, we write the monodromy matrix as (following the notation of Tarasov [15])

$$T_\alpha^{\alpha'} = \begin{pmatrix} A_1 & (B_1)_{\mathring{\alpha}} & B_2 \\ (C_1)^{\mathring{\alpha}'} & (A_2)_{\mathring{\alpha}}^{\mathring{\alpha}'} & (B_3)^{\mathring{\alpha}'} \\ C_2 & (C_3)_{\mathring{\alpha}} & A_3 \end{pmatrix}, \tag{13}$$

where α, α' assume the values $1, 2, \ldots, (0), \ldots, \bar{2}, \bar{1}$ corresponding to the basis vectors of the auxiliary space $V \cong \mathbb{C}^N$ and $\mathring{\alpha}, \mathring{\alpha}'$ assume the values $2, \ldots, (0), \ldots, \bar{2}$ corresponding to the basis vectors of $\mathring{V} \cong \mathbb{C}^{N-2}$. We will also use the notation $A = A_1, B = B_1, C = C_1$ and $D = A_2$, which is an $(N-2) \times (N-2)$ matrix in the auxiliary space. The Yang–Baxter algebra relation for the R-matrix (8) yields

$$T_{1,\ldots,n,a}(\underline{u}, u_a)\, T_{1,\ldots,n,b}(\underline{u}, u_b)\, R_{ab}(u_{ab}) = R_{ab}(u_{ab})\, T_{1,\ldots,n,b}(\underline{u}, u_b)\, T_{1,\ldots,n,a}(\underline{u}, u_a)\,. \tag{14}$$

2.3. A lemma

In our approach of the algebraic Bethe ansatz, the following lemma replaces commutation rules of the entries of the monodromy matrix. In the conventional approach one derives them from the Yang–Baxter algebra relations (14) and uses them for the algebraic Bethe ansatz.

Lemma 2.2. *For the monodromy matrix the following identity holds:*

$$\begin{aligned} T_{1,\ldots,n,a}(\underline{u}, v) = \mathbf{1}_1, \ldots, \mathbf{1}_n \mathbf{1}_a &+ \sum_{i=1}^{n} c(u_i - v) R_{1a}(u_{1i}) \cdots \mathbf{P}_{ia} \cdots R_{na}(u_{ni}) \\ &+ \sum_{j=1}^{n} d(u_i - v) R_{1a}(\hat{u}_{i1}) \cdots \mathbf{K}_{ia} \cdots R_{na}(\hat{u}_{in}) \end{aligned} \tag{15}$$

with $\hat{u} = 1/\nu - u\,.$

Proof. The R-matrix $R(u)$ (see (6) and (7)) is meromorphic and has simple poles at $u = 0$ and $u = 1/\nu$ due to the form of $c(u)$ and $d(u)$ such that

$$\begin{aligned} \mathrm{Re\,s}_{v=u_i} T_{1,\ldots,n,a}(\underline{u}, v) &= \mathrm{Re\,s}_{v=u_i} c(u_i - v) R_{1a}(u_{1i}) \cdots \mathbf{P}_{ia} \cdots R_{na}(u_{ni}) \\ \mathrm{Re\,s}_{v=u_i-1/\nu} T_{1,\ldots,n,a}(\underline{u}, v) &= \mathrm{Re\,s}_{v=u_i-1/\nu} d(u_j - v) R_{1a}(\hat{u}_{i1}) \cdots \mathbf{K}_{ia} \cdots R_{ma}(\hat{u}_{in}) \end{aligned}$$

holds. The claim follows by Liouville's theorem because $\lim_{v\to\infty} T_{1,\ldots,n,a}(\underline{u}, v) = \mathbf{1}_1, \ldots, \mathbf{1}_n \mathbf{1}_a$. □

Similarly, we have for the crossed monodromy matrix,

$$T_{a,1,\ldots,n}(v, \underline{u}) = R_{an}(v - u_n) \cdots R_{a1}(v - u_1),$$

the relation

$$\begin{aligned} T_{a,1,\ldots,n}(v, \underline{u}) = \mathbf{1}_a \mathbf{1}_1, \ldots, \mathbf{1}_n &+ \sum_{i=1}^{n} c(v - u_i) R_{an}(u_{in}) \cdots \mathbf{P}_{ai} \cdots R_{a1}(u_{i1}) \\ &+ \sum_{i=1}^{n} d(v - u_i) R_{am}(\hat{u}_{mi}) \cdots \mathbf{K}_{ai} \cdots R_{a1}(\hat{u}_{1i}). \end{aligned} \tag{16}$$

Note that the crossing relation (10) implies

$$T_{a,1,\ldots,n}(v_a, \underline{u}) = \mathbf{C}_{ba} T_{1,\ldots,n,b}(\underline{u}, v_b) \mathbf{C}^{ab} \tag{17}$$

with $v_b = v_a - 1/\nu$.

2.4. The matrix Π

The nested Bethe ansatz relies on the principle that after each level, the rank of the group (or quantum group) is reduced by 1. For $SU(N)$ the rank is $N-1$ and for $O(N)$ it is $[N/2]$. This means that the dimension of the vector representation (where the R-matrix usually acts) is reduced by 1 for the case of $SU(N)$ and by 2 for the case of $O(N)$. A more essential difference is that for $SU(N)$ one can use at every level the same R-matrix, because (with a suitable normalization and parameterization) the $SU(N)$ R-matrix does not depend on N. In contrast for $O(N)$, the R-matrix changes after each level, because it depends on N. Therefore, we need a new object called matrix Π, which maps the $O(N)$ R-matrix to the $O(N-2)$ one. We use the notation

$$\begin{aligned} \mathring{R}(u) &= R(u, N-2) = \mathbf{1} + \mathbf{P}c(u) + \mathbf{K}\mathring{d}(u) \\ \mathring{d}(u) &= \frac{1}{u - 1/\mathring{\nu}} \end{aligned} \tag{18}$$

with $\mathring{\nu} = 2/(N-4)$. The components of the R-matrix $\mathring{R}(u)$ will be denoted by

$$\mathring{R}^{\mathring{\delta}\mathring{\gamma}}_{\mathring{\alpha}\mathring{\beta}}(u), \mathring{\alpha}, \mathring{\beta}, \mathring{\gamma}, \mathring{\delta} = 2, 3, \dots, (0), \dots, \bar{3}, \bar{2}.$$

In addition to $V^{1,\dots,m} = V_1 \otimes \dots \otimes V_m$ (1), we introduce

$$\mathring{V}^{1,\dots,m} = \mathring{V}_1 \otimes \dots \otimes \mathring{V}_m\,, \tag{19}$$

where the vector spaces $\mathring{V}_i \cong \mathbf{C}^{N-2}$, $(i = 1, \dots, m)$ are considered as fundamental (vector) representation spaces of $O(N-2)$. The space V_i is spanned by the complex basis vectors $|1\rangle, |2\rangle, \dots, |\bar{2}\rangle, |\bar{1}\rangle$ and $\mathring{V}_i$ by $|2\rangle, \dots, |\bar{2}\rangle$.

Definition 2.3. *We define the map*

$$\Pi_{1,\dots,m} : V^{1,\dots,m} \to \mathring{V}^{1,\dots,m}$$

recursively by $\Pi_1 = \pi_1$ *and*

$$\Pi_{1,\dots,m}(\underline{u}) = (\pi_1 \Pi_{2,\dots,m})\bar{e}_a T_{1,\dots,m,a}(\underline{u}, u_a)\bar{e}^a \tag{20}$$

with the projector $\pi : V \to \mathring{V} \subset V$*, the monodromy matrix* (12) *and* $u_a = u_1 - 1$*. The vector* $\bar{e}^a \in V_a$ *(acting in the auxiliary space of* $T_{1,\dots,m,a}$*) and the co-vector* $\bar{e}_a \in (V_a)^\dagger$ *correspond to the state* $\bar{1}$ *and have the components* $\bar{e}^\alpha = \delta^\alpha_{\bar{1}}$ and $\bar{e}_\alpha = \delta^{\bar{1}}_\alpha$.

Lemma 2.4. *In particular for* $m = 2$*, the matrix* $\Pi_{12}(u_1, u_2)$ *may be written as*

$$\Pi_{12}(\underline{u}) = \pi_1\pi_2 + f(u_{12})\mathring{\mathbf{C}}^{12}\bar{e}_1 e_2 \tag{21}$$

with $e_2 = \mathbf{C}_{2a}\bar{e}^a$ *and* $f(u) = -d(1-u)$*. It satisfies the fundamental relation*

$$\mathring{R}_{12}(u_{12})\Pi_{12}(u_1, u_2) = \Pi_{21}(u_2, u_1)R_{12}(u_{12})\,, \tag{22}$$

where $\mathring{R}_{12}$ *is the* $O(N-2)$ *R-matrix.*

Proof. Equation (21) can be easily derived

$$\begin{aligned} \Pi_{12}(\underline{u}) &= \pi_1\pi_2\bar{e}_a T_{12,a}(\underline{u}, u_a)\bar{e}^a \\ &= \pi_1\pi_2\bar{e}_a R_{1a}(u_1 - u_a)R_{2a}(u_2 - u_a)\bar{e}^a \\ &= \pi_1\pi_2\bar{e}_a \left(\mathbf{1}_1\mathbf{1}_a + c(1)\mathbf{P}_{1a}\right)\left(\mathbf{1}_2\mathbf{1}_a + d(u_{21}+1)\mathbf{K}_{2a}\right)\bar{e}^a \\ &= \pi_1\pi_2 + c(1)d(u_{21}+1)\mathring{\mathbf{C}}^{12}\bar{e}_1 e_2. \end{aligned}$$

Use has been made of $c(1) = -1$ and $\pi_1\pi_2\bar{e}_a\mathbf{P}_{1a}\mathbf{K}_{2a}\bar{e}^a = \mathring{\mathbf{C}}^{12}\bar{e}_1 e_2$. Equation (22) is derived for all components. Obviously,

$$\mathring{R}^{\mathring{\beta}'\mathring{\alpha}'}_{\mathring{\alpha}\mathring{\beta}}(u)\mathring{\mathbf{C}}^{\mathring{\alpha}\mathring{\beta}} = \mathring{R}_0(u)\mathring{\mathbf{C}}^{\mathring{\beta}'\mathring{\alpha}'}$$

holds, where the scalar R-matrix eigenvalue is (see (9))

$$\mathring{R}_0(u) = a(u) + (N-2)\,\mathring{d}(u).$$

Therefore, the relations

$$(\mathring{R}_{12}(u_{12})\Pi_{12})^{\mathring{\beta}'\mathring{\alpha}'}_{\alpha\beta} = \mathring{R}^{\mathring{\beta}'\mathring{\alpha}'}_{\mathring{\alpha}\mathring{\beta}}(u_{12})\pi^{\mathring{\alpha}}_{\alpha}\pi^{\mathring{\beta}}_{\beta} + f(u_{12})\mathring{R}_0(u_{12})\mathring{\mathbf{C}}^{\mathring{\beta}'\mathring{\alpha}'}\delta^{\bar{1}}_{\alpha}\delta^{1}_{\beta}$$

and

$$(\Pi_{21}R_{12}(u_{12}))^{\mathring{\beta}'\mathring{\alpha}'}_{\alpha\beta} = \pi^{\mathring{\beta}'}_{\beta'}\pi^{\mathring{\alpha}'}_{\alpha'}R^{\beta'\alpha'}_{\alpha\beta}(u_{12}) + f(u_{21})\mathring{\mathbf{C}}^{\mathring{\beta}'\mathring{\alpha}'}R^{\bar{1}1}_{\alpha\beta}(u_{12})$$

are valid. The claim of the lemma is then equivalent to

$$\begin{array}{lll} (i): \mathring{d}(u) = d(u) + f(-u)d(u) & \text{for} & \alpha \text{ or } \beta \neq 1, \bar{1} \\ (ii): 0 = d(u) + f(-u)\,(1 + d(u)) & \text{for} & \alpha = 1, \beta = \bar{1} \\ (iii): f(u)\mathring{R}_0(u) = d(u) + f(-u)\,(c(u) + d(u)) & \text{for} & \alpha = \bar{1}, \beta = 1. \end{array}$$

These equations may be easily checked with the amplitudes (7). □

These results can be extended to general m, as presented below.

Lemma 2.5. *The matrix* $\Pi_{1,\dots,m}(\underline{u})$ *satisfies*

(a) in addition to (20) the recursion relation

$$\Pi_{1,\dots,m}(\underline{u}) = (\Pi_{1,\dots,m-1}\pi_m)\bar{e}_b T_{1,\dots,m,b}(\underline{u}, u_b)\bar{e}^b \tag{23}$$

with $u_b = u_m - 1/\nu + 1$, *and*

(b) the fundamental relation

$$\mathring{R}_{ij}(u_{ij})\Pi_{\dots ij\dots}(\underline{u}) = \Pi_{\dots ji\dots}(\underline{u})R_{ij}(u_{ij})\,. \tag{24}$$

(c) The matrix $\bar{e}_0 T_{1,\dots,m,0}(\underline{u}, u_0)\bar{e}^0$ *acts on* $\Pi_{1,\dots,m}(\underline{u})$ *as the unit matrix for arbitrary* u_0

$$\Pi_{1,\dots,m}(\underline{u})\bar{e}_0 T_{1,\dots,m,0}(\underline{u}, u_0)\bar{e}^0 = \Pi_{1,\dots,m}(\underline{u})\,. \tag{25}$$

(d) Special components of Π *satisfy*

$$\Pi^{\mathring{\alpha}_1,\dots,\mathring{\alpha}_m}_{1\alpha_2,\dots,\alpha_m}(u_1,\dots,u_m) = 0 \tag{26}$$

$$\Pi^{\mathring{\alpha}_1,\dots,\mathring{\alpha}_m}_{\mathring{\alpha}\alpha_2,\dots,\alpha_m}(u_1,\dots,u_m) = \delta^{\mathring{\alpha}_1}_{\mathring{\alpha}}\Pi^{\mathring{\alpha}_2,\dots,\mathring{\alpha}_m}_{\alpha_2,\dots,\alpha_m}(u_2,\dots,u_m) \tag{27}$$

$$\Pi^{\mathring{\alpha}_1,\dots,\mathring{\alpha}_m}_{\alpha_1,\dots,\alpha_{m-1}\bar{1}}(u_1,\dots,u_m) = 0 \tag{28}$$

$$\Pi^{\mathring{\alpha}_1,\dots,\mathring{\alpha}_m}_{\alpha_1,\dots,\alpha_{m-1}\mathring{\alpha}}(u_1,\dots,u_m) = \Pi^{\mathring{\alpha}_1,\dots,\mathring{\alpha}_{m-1}}_{\alpha_1,\dots,\alpha_{m-1}}(u_1,\dots,u_{m-1})\delta^{\mathring{\alpha}_m}_{\mathring{\alpha}}. \tag{29}$$

with $\mathring{\alpha} \neq 1, \bar{1}$.

The proof of this lemma is presented in appendix A.

Definitions (20) and (23) can be rewritten as (see also lemma 2.4 for $m = 2$)

$$\Pi_{1,\dots,m}(\underline{u}) = \pi_1\Pi_{2,\dots,m} + \sum_{j=2}^{m} f(u_{1j})\mathring{R}_{jj-1}\cdots\mathring{R}_{j2}\mathring{\mathbf{C}}^{1j}\Pi_{2,\dots,\hat{\jmath},\dots,m}\bar{e}_1 e_j R_{jm}\cdots R_{jj+1} \tag{30}$$

$$= \Pi_{1,\dots,m-1}\pi_m + \sum_{j=1}^{m-1} f(u_{jm})\mathring{R}_{j+1j}\cdots\mathring{R}_{m-1j}\mathring{\mathbf{C}}^{jm}\Pi_{1,\dots,\hat{\jmath},\dots,m-1}\bar{e}_j R_{1j}\cdots R_{j-1j}e_m. \tag{31}$$

3. The $O(N)$-difference equation

Let $K_{1,\ldots,n}(\underline{u}) \in V_{1,\ldots,n}$ be a co-vector-valued function of $\underline{u} = u_1, \ldots, u_n$ with values in $V_{1,\ldots,n}$. The components of this vector are denoted by

$$K_{\alpha_1,\ldots,\alpha_n}(\underline{u}), \qquad (\alpha_i = 1, 2, \ldots, (0), \ldots, \bar{2}, \bar{1}).$$

The following symmetry and periodicity properties of this function are supposed to be valid.

Conditions 3.1.

(i) The symmetry property under the exchange of two neighboring spaces V_i and V_j and the variables u_i and u_j, at the same time, is given by

$$K_{\ldots ij\ldots}(\ldots, u_i, u_j, \ldots) = K_{\ldots ji\ldots}(\ldots, u_j, u_i, \ldots)\tilde{R}_{ij}(u_{ij}), \tag{32}$$

where $\tilde{R}(u) = R(u)/a(u)$ and $R(u)$ is the $O(N)$ R-matrix.

(ii) The *system of matrix difference equations* holds

$$\boxed{K_{1,\ldots,n}(\ldots, u_i', \ldots) = K_{1,\ldots,n}(\ldots, u_i, \ldots)Q_{1,\ldots,n}(\underline{u}; i), \; (i = 1, \ldots, n)} \tag{33}$$

with $u_i' = u_i + 2/\nu$. The matrix $Q_{1,\ldots,n}(\underline{u}; i) \in End(V^{1,\ldots,n})$ is defined as the trace

$$Q_{1,\ldots,n}(\underline{u}; i) = \mathrm{tr}_0 \tilde{T}_{Q,1,\ldots,n,0}(\underline{u}, i) \tag{34}$$

of a modified monodromy matrix

$$\tilde{T}_{Q,1,\ldots,n,0}(\underline{u}, i) = \tilde{R}_{10}(u_1 - u_i') \cdots \mathbf{P}_{i0} \cdots \tilde{R}_{n0}(u_n - u_i).$$

The Yang–Baxter equations for the R-matrix guarantee that these properties are compatible. The shift of $2/\nu$ in equation (33) could be replaced by an arbitrary κ. For the application to the form factor problem, however, it is fixed to be equal to $2/\nu$ in order to be compatible with crossing symmetry. Instead of the Yang–Baxter relation (14), the modified monodromy matrix $\tilde{T}_Q$ satisfies the Zapletal rules [14, 4]. We have for $i = 1, \ldots, n$,

$$\tilde{T}_Q(\underline{u}; i)\, T_0(\underline{u}', v)\, R_{i0}(u_i - v) = R_{i0}\big(u_i' - v\big)\, T_0(\underline{u}, v)\, \tilde{T}_Q(\underline{u}; i) \tag{35}$$

with $\underline{u}' = u_1, \ldots, u_i', \ldots, u_n$ and $u_i' = u_i + 2/\nu$. The $Q_{1,\ldots,n}(\underline{u}; i)$ satisfy the commutation rules

$$\begin{aligned} &Q_{1,\ldots,n}(\ldots u_i \ldots u_j \ldots; i)Q_{1,\ldots,n}\big(\ldots u_i' \ldots u_j \ldots; j\big) \\ &\quad = Q_{1,\ldots,n}(\ldots u_i \ldots u_j \ldots; j)Q_{1,\ldots,n}\big(\ldots u_i \ldots u_j' \ldots; i\big). \end{aligned} \tag{36}$$

The following proposition is obvious.

Proposition 3.2. *Let the vector valued function $K_{1,\ldots,n}(\underline{u}) \in V_{1,\ldots,n}$ satisfy (i). Then for all $i = 1, \ldots, n$, the relations (3.2) are equivalent to each other and also equivalent to the following periodicity property under cyclic permutation of the spaces and the variables*

$$K_{\alpha_1\alpha_2,\ldots,\alpha_n}(u_1', u_2, \ldots, u_n) = K_{\alpha_2,\ldots,\alpha_n\alpha_1}(u_2, \ldots, u_n, u_1). \tag{37}$$

Remark 3.3. *Equations (32) and (37) imply Watson's equations for the form factors [26].*

Because of proposition 3.2, we mainly consider $Q_{1,\ldots,n}(\underline{u}, i)$ for $i = 1$:

$$Q_{1,\ldots,n}(\underline{u}) = \mathrm{tr}_0 \tilde{T}_{Q,1,\ldots,n,0}(\underline{u}) = \prod_{k=2}^{n} \frac{1}{a(v_{ki})} \mathrm{tr}_0 T_{Q,1,\ldots,n,0}(\underline{u}) \tag{38}$$

with $T_{Q,1,\ldots,n,0}(\underline{u}) = T_{Q,1,\ldots,n,0}(\underline{u}, 1)$. In analogy to equation (13), we introduce (suppressing the indices $1, \ldots, n$)

$$T_Q(\underline{u}) \equiv \begin{pmatrix} A_Q(\underline{u}) & B_Q(\underline{u}) & B_{Q,2}(\underline{u}) \\ C_Q(\underline{u}) & D_Q(\underline{u}) & B_{Q,3}(\underline{u}) \\ C_{Q,2}(\underline{u}) & C_{Q,3}(\underline{u}) & A_{Q,3}(\underline{u}) \end{pmatrix}. \tag{39}$$

3.1. The off-shell Bethe ansatz

We will express the co-vector-valued function $K_{\underline{\alpha}}(\underline{u})$ in terms of the co-vectors

$$\Psi_{\underline{\alpha}}(\underline{u},\underline{v}) = L_{\underline{\mathring{\beta}}}(\underline{v})\,\Phi_{\underline{\alpha}}^{\underline{\mathring{\beta}}}(\underline{u},\underline{v}) = \big(L(\underline{v})\Phi(\underline{u},\underline{v})\big)_{\underline{\alpha}}\,, \tag{40}$$

where summation over $\mathring{\beta}_1,\dots,\mathring{\beta}_m$, $\mathring{\beta}_i = 2,\dots,0,\dots,\bar{2}$, is assumed and $L_{\underline{\mathring{\beta}}}(\underline{v})$ is a co-vector-valued function with values in $\mathring{V}_{1,\dots,m} \simeq \mathbb{C}^{N-2}\otimes\cdots\otimes\mathbb{C}^{N-2}$. We assume that for $L_{\mathring{\beta}}(\underline{v})$, the higher level conditions of 3.1 hold with R and Q replaced by $\mathring{R}$ and $\mathring{Q}$ (which means N is replaced by $N-2$):

$$(\mathrm{i})^{(1)}: L_{\dots ij\dots}(\dots,v_i,v_j,\dots) = L_{\dots ji\dots}(\dots,v_j,v_i,\dots)\tilde{\mathring{R}}_{ij}(v_{ij}) \tag{41}$$

$$(\mathrm{ii})^{(1)}: L_{1,\dots,m}(\dots,v_i',\dots) = L_{1,\dots,n}(\dots,v_i,\dots)\mathring{Q}_{1,\dots,m}(\underline{v},i). \tag{42}$$

The Bethe ansatz states are[5]

$$\Phi_{\underline{\alpha}}^{\underline{\mathring{\beta}}}(\underline{u},\underline{v}) = \Pi_{\underline{\beta}}^{\underline{\mathring{\beta}}}(\underline{v})\left(\Omega T_1^{\beta_m}(\underline{u},v_m)\cdots T_1^{\beta_1}(\underline{u},v_1)\right)_{\underline{\alpha}}\,. \tag{43}$$

The reference state Ω (pseudo-vacuum) is the highest weight co-vector (with weights $w=(n,0,\dots,0)$)

$$\Omega_{\underline{\alpha}} = \delta_{\alpha_1}^{1}\cdots\delta_{\alpha_n}^{1}\,. \tag{44}$$

It satisfies

$$\Omega T(\underline{u},v) = \Omega\begin{pmatrix} a_1(\underline{u},v) & 0 & 0\\ * & a_2(\underline{u},v) & 0\\ * & * & a_3(\underline{u},v)\end{pmatrix}, \tag{45}$$

$$a_1(\underline{u},v) = \prod_{k=1}^{n} a(u_k-v),\; a_2(\underline{u},v) = 1,\; a_3(\underline{u},v) = \prod_{k=1}^{n}(1+d(u_k-v))\,.$$

We also have for $T_Q(\underline{u}) = T_Q(\underline{u},1)$,

$$\Omega T_Q(\underline{u}) = \Omega\prod_{k=2}^{n} a(u_{k1})\begin{pmatrix}1&0&0\\ *&0&0\\ *&0&0\end{pmatrix}. \tag{46}$$

The system of difference equations (33) can be solved by means of a nested 'off-shell' Bethe ansatz. The first level is given by the following Bethe ansatz.

Bethe ansatz 1.

$$\boxed{K_{\underline{\alpha}}(\underline{u}) = \sum_{\underline{v}} g(\underline{u},\underline{v})\,\Psi_{\underline{\alpha}}(\underline{u},\underline{v})\,,} \tag{47}$$

where Ψ is given by (40) and

$$g(\underline{u},\underline{v}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\psi(u_i-v_j)\prod_{1\leqslant i<j\leqslant m}\tau(v_i-v_j)\,. \tag{48}$$

The functions $\psi(u)$ and $\tau(v)$ satisfy the functional equations

$$\psi(u') = a(u)\psi(u), \qquad \tau(v')a(v') = a(-v)\tau(v), \tag{49}$$

---

[5] The $\Phi_{\underline{\alpha}}^{\underline{\mathring{\beta}}}$ are generalizations of the states introduced by Tarasov in [15].

with $u' = u + 2/\nu$. The summation over $\underline{v}$ is specified by

$$\underline{v} = (v_1, \ldots, v_m) = (\tilde{v}_1 - 2l_1/\nu, \ldots, \tilde{v}_m - 2l_m/\nu), \qquad l_i \in \mathbf{Z}\,, \tag{50}$$

where $\tilde{v}_i$ are arbitrary constants.

The sums (47) are also called the 'Jackson-type integrals' (see e.g. [10] and references therein). The solutions of (49) are

$$\psi(u) = \frac{\Gamma\left(-\frac{1}{2}\nu + \frac{u}{2}\nu\right)}{\Gamma\left(\frac{u}{2}\nu\right)} \tag{51}$$

$$\tau(v) = v\frac{\Gamma\left(\frac{1}{2}\nu + \frac{v}{2}\nu\right)}{\Gamma\left(1 - \frac{1}{2}\nu + \frac{v}{2}\nu\right)}. \tag{52}$$

We are now in a position to formulate the main result of this paper.

Theorem 3.4. *Let the co-vector-valued function* $K_{1,\ldots,n}(\underline{u}) \in V_{1,\ldots,n}$ *be given by the Bethe ansatz 1 and let* $g(\underline{x}, \underline{u})$ *be of the form (48). If in addition the co-vector-valued function* $L_{1,\ldots,m}(\underline{v}\,) \in \mathring{V}_{1,\ldots,m}$ *satisfies the properties* $(i)^{(1)}$ *and* $(ii)^{(1)}$*, i.e. equations (32) and (33) for* $O(N-2)$*, then* $K_{1,\ldots,n}(\underline{u})$ *satisfies equations (32) and (33) for* $O(N)$*, i.e.* $K_{1,\ldots,n}(\underline{u})$ *is a solution of the set of difference equations.*

The proof of this theorem can be found in appendix B.

Iterating (47), (40) and theorem 3.4, we obtain the nested off-shell Bethe ansatz with levels $k = 1, \ldots, [(N-1)/2]$. The ansatz for level k reads

$$K^{(k-1)}_{1,\ldots,n_{k-1}}(\underline{u}^{(k-1)}) = \sum_{\underline{u}^{(k)}} g^{(k-1)}\left(\underline{u}^{(k-1)}, \underline{u}^{(k)}\right)\Psi^{(k-1)}_{1,\ldots,n_{k-1}}\left(\underline{u}^{(k-1)}, \underline{u}^{(k)}\right) \tag{53}$$

$$\Psi^{(k-1)}_{1,\ldots,n_{k-1}}(\underline{u}^{(k-1)}, \underline{u}^{(k)}) = (K^{(k)}(\underline{u}^{(k)})\Phi^{(k-1)}(\underline{u}^{(k-1)}, \underline{u}^{(k)}))_{1,\ldots,n_{k-1}}, \tag{54}$$

where $\Phi^{(k)}$ is the Bethe ansatz state (43) and $g^{(k)}$ is the function (48) for $O(N-2k)$. The start of the iteration is given by a $1 \leqslant k_{\max} < [(N+1)/2]$ with

$$\Psi^{(k_{\max}-1)}_{\alpha_1\ldots\alpha_{n_{k-1}}} = \delta^{k_{\max}}_{\alpha_1}\ldots\delta^{k_{\max}}_{\alpha_{n_{k-1}}}, \qquad \text{and} \quad n_k = 0 \text{ for } k \geqslant k_{\max} \tag{55}$$

which is the reference state of level $k_{\max} - 1$ and trivially fulfils conditions 3.1. The start of the iteration for $k_{\max} = [(N+1)/2]$ is given by the solution for $O(3)$ for N odd and $O(4)$ for N even. These two cases are investigated below.

Corollary 3.5. *The system of* $O(N)$*-matrix difference equations (33) is solved by the nested Bethe ansatz (53) with (55) and* $K_{1,\ldots,n}(\underline{u}) = K^{(0)}{}_{1,\ldots,n}(\underline{u})$.

3.1.1. The off-shell Bethe ansatz for $O(3)$. The R-matrix is given by (5) and (7) for $\nu = 2$:

$$R(u) = \mathbf{1} + c(u)P + d(u)K, \qquad c(u) = \frac{-1}{u}, \qquad d(u) = \frac{1}{u - 1/2}.$$

The solution of the difference equations is given by the off-shell Bethe ansatz (47) with

$$\psi(u) = \frac{1}{u-1}, \qquad \tau(v) = v^2.$$

The Bethe vector Ψ is expressed in terms of the co-vectors (43)

$$\Psi_{\underline{\alpha}}(\underline{u}, \underline{v}) = L(\underline{v})\Phi_{\underline{\alpha}}(\underline{u}, \underline{v}),$$

where the scalar function $L(\underline{v})$ has to satisfy

$$L(\dots, v_i, v_j, \dots) = L(\dots, v_j, v_i, \dots)\tilde{\mathring{R}}(v_{ij})$$
$$L(v_1', v_2, \dots, v_m) = L(v_2, \dots, v_m, v_1)$$

with $\tilde{\mathring{R}}(v) = \frac{(v+1)(v-1/2)}{(v-1)(v+1/2)}$ and $v' = v + 1$. The minimal solution of these equations is

$$L(\underline{v}) = \prod_{1 \leqslant i < j \leqslant m} L(v_{ij})$$
$$L(v) = \frac{\pi}{4} \frac{(v-1/2)}{v\,(v-1)} \tan \pi v .$$

The $O(3)$ weight of the state is

$$w = n - m . \tag{56}$$

3.1.2. The off-shell Bethe ansatz for $O(4)$. The R-matrix is given by (5) and (7) for $\nu = 1$:

$$R(u) = \mathbf{1} + c(u)P + d(u)K, \qquad c(u) = \frac{-1}{u}, \qquad d(u) = \frac{1}{u-1}.$$

We could apply theorem 3.4 and write the off-shell Bethe ansatz for $O(4)$ in terms of an $O(2)$ problem. However, the latter cannot be solved by the Bethe ansatz because the R-matrix is diagonal (note that $R_{11}^{1\bar{1}} = 0$). But there is another way to solve the $O(4)$ problem. The group isomorphism $O(4) \simeq SU(2) \otimes SU(2)$ reflects itself in terms of the corresponding R-matrices. Indeed, the $O(4)$ R-matrix can be written as a tensor product of two $SU(2)$ R-matrices, or more precisely

$$(\tilde{R}^{O(4)})_{\alpha\beta}^{\delta\gamma} \Gamma_{AB}^{\alpha} \Gamma_{CD}^{\beta} = \Gamma_{C'D'}^{\delta} \Gamma_{A'B'}^{\gamma} \big(\tilde{R}_{+}^{SU(2)}\big)_{AC}^{C'A'} (\tilde{R}_{-}^{SU(2)})_{BD}^{D'B'} .$$

The $SU(2)$ R-matrices $\tilde{R}_{\pm}^{SU(2)}(u) = R_{\pm}^{SU(2)}(u)/a(u)$ correspond to the spinor representations of $O(4)$ with positive (negative) chirality

$$R_{\pm}^{SU(2)} = \mathbf{1} + c(u)\mathbf{P},$$

where the amplitude $c(u)$ is given by (7). The relative R-matrix for states of different chirality is trivial $\tilde{R} = \mathbf{1}$. The intertwiners are

$$\Gamma_{AB}^{\alpha} = (\gamma_{+}\gamma^{\alpha}C)_{AB}$$

with the $O(4)$ gamma matrices γ^{α}, $\gamma_{+} = \frac{1}{2}(1 + \gamma^{5})$ and the charge conjugation matrix C. For more details, see [27, 28]. In the complex basis of the $O(4)$ and the fundamental representations of the $SU(2)$, the states have the $O(4)$ weights

vector states	$O(4)$ weights
1	$(1, 0)$
2	$(0, 1)$
$\bar{2}$	$(0, -1)$
$\bar{1}$	$(-1, 0)$

spinor states	$O(4)$ weights
$\uparrow_{+}$	$(\frac{1}{2}, \frac{1}{2})$
$\downarrow_{+}$	$(-\frac{1}{2}, -\frac{1}{2})$
$\uparrow_{-}$	$(\frac{1}{2}, -\frac{1}{2})$
$\downarrow_{-}$	$(-\frac{1}{2}, \frac{1}{2})$.

(57)

Because of weight conservation the intertwiner matrix is diagonal in this basis and is calculated to be

$$\Gamma_{AB}^{\alpha} = \begin{pmatrix} \Gamma_{\uparrow_{+}\uparrow_{-}}^{1} & 0 & 0 & 0 \\ 0 & \Gamma_{\uparrow_{+}\downarrow_{-}}^{2} & 0 & 0 \\ 0 & 0 & \Gamma_{\downarrow_{+}\uparrow_{-}}^{\bar{2}} & 0 \\ 0 & 0 & 0 & \Gamma_{\downarrow_{+}\downarrow_{-}}^{\bar{1}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{58}$$

We also use the dual intertwiner Γ_α^{AB} with

$$\sum_{A,B}\Gamma_{AB}^{\alpha'}\Gamma_\alpha^{AB}=\delta_\alpha^{\alpha'},\quad \sum_\alpha \Gamma_\alpha^{A'B'}\Gamma_{AB}^\alpha=\delta_A^{A'}\delta_B^{B'}. \tag{59}$$

We write the co-vector-valued function $K_{\underline{\alpha}}(\underline{u})$ as

$$K_{\underline{\alpha}}(\underline{u})=K_{\underline{A}}^{(+)}(\underline{u})K_{\underline{B}}^{(-)}(\underline{u})\Gamma_{\underline{\alpha}}^{\underline{A}\,\underline{B}}, \tag{60}$$

where $\Gamma_{\underline{\alpha}}^{\underline{A}\,\underline{B}}=\prod_{i=1}^n\Gamma_{\alpha_i}^{A_iB_i}$. The transfer matrix $\mathrm{tr}\tilde{T}^{O(4)}(\underline{u},v)$ can also be decomposed such that

$$\begin{aligned}K_{\underline{\alpha}}(\underline{u})(\tilde{T}^{O(4)})_\gamma^\gamma(\underline{u},v)&=K_{\underline{\alpha}}(\underline{u})\Gamma_{A'B'}^\gamma\big(\tilde{T}_+^{SU(2)}\big)_A^{A'}(\underline{u},v)\big(\tilde{T}_-^{SU(2)}\big)_B^{B'}(\underline{u},v)\Gamma_\gamma^{AB}\\&=\big(K_{\underline{A}}^{(+)}(\underline{u})\big(\tilde{T}_+^{SU(2)}\big)_A^A(\underline{u},v)\big)\big(K_{\underline{B}}^{(-)}(\underline{u})\big(\tilde{T}_-^{SU(2)}\big)_B^B(\underline{u},v)\big)\Gamma_{\underline{\alpha}}^{\underline{AB}},\end{aligned}$$

where (59) has been used. Therefore, $K_{\underline{\alpha}}(\underline{u})$ satisfies the $O(4)$ symmetry relation (32) and the difference equation (33) if the $K_{\underline{A}}^{(\pm)}(\underline{u})$ satisfy the corresponding $SU(2)$ relations.

The $SU(2)$ on-shell Bethe ansatz is well known and the off-shell case has been solved in [14, 6, 29]:

$$K_{\underline{A}}(\underline{u})=\sum_{\underline{v}}g(\underline{u},\underline{v})\Psi_{\underline{A}}(\underline{u},\underline{v}) \tag{61}$$

$$\Psi(\underline{u},\underline{v})=\Omega C(\underline{u},v_m)\cdots C(\underline{u},v_1), \tag{62}$$

where $g(\underline{u},\underline{v})$ is given by (48) and

$$\psi(u)=\frac{\Gamma\left(-\frac{1}{2}+\frac{u}{2}\right)}{\Gamma\left(\frac{u}{2}\right)},\qquad \tau(v)=v.$$

The summation over $\underline{v}$ is specified by

$$\underline{v}=(v_1,\ldots,v_m)=(\tilde{v}_1-2l_1,\ldots,\tilde{v}_m-2l_m),\qquad l_i\in\mathbf{Z}, \tag{63}$$

where the $\tilde{v}_i$ are arbitrary constants. The $SU(2)$ weights of the state (62) are $w=(n-m,m)$ and due to (57) the $O(4)$ weights are

$$w=\begin{cases}(n-m,-m) & \text{for positive chirality spinors}\\(n-m,m) & \text{for negative chirality spinors.}\end{cases}$$

Therefore, the $O(4)$ weights of (60) are

$$w=(n-n_--n_+,n_--n_+), \tag{64}$$

where $n_\pm$ are the numbers of positive (negative) chirality C-operators.

4. Weights of off-shell $O(N)$ Bethe vectors

In this section, we analyze some group theoretical properties of off-shell Bethe states. We show that they are highest weight states and we calculate the weights. The first result is not only true for the conventional Bethe ansatz, which solves an eigenvalue problem and which is well known, but it is also true, as we will show, for the off-shell one which solves a difference equation (or a differential equation).

By the asymptotic expansion of the R-matrix (5) and the monodromy matrix (12), we obtain for $u\to\infty$:

$$R_{ab}(u)=\mathbf{1}_{ab}-\frac{1}{u}M_{ab}+O(u^{-2}) \tag{65}$$

$$M_{ab}=\mathbf{P}_{ab}-\mathbf{K}_{ab}. \tag{66}$$

More explicitly, equation (12) gives

$$T_{1,\dots,n,a}(\underline{u},u)=\mathbf{1}_{1,\dots,n,a}+\frac{1}{u}M_{1,\dots,n,a}+O(u^{-2}) \tag{67}$$

$$M_{1,\dots,n,a}=(\mathbf{P}_{1a}-\mathbf{K}_{1a})+\cdots+(\mathbf{P}_{na}-\mathbf{K}_{na})\,. \tag{68}$$

The matrix elements of $M_{1,\dots,n,a}$, as a matrix in the auxiliary space, are the $O(N)$ Lie algebra generators. In the following, we will consider only operators acting in the fixed tensor product space $V=V^{1,\dots,n}$ of (1). Therefore, we will omit the indices $1,\dots,n$. In terms of the matrix elements in the auxiliary space V_a, the generators act on the basis states as

$$\langle\,\alpha_1,\dots,\alpha_i,\dots,\alpha_n\,|M_\alpha^{\alpha'}=\sum_{i=1}^{n}(\delta_{\alpha\alpha_i}\,\langle\,\alpha_1,\dots,\alpha',\dots,\alpha_n\,|-\delta_{\alpha'\bar\alpha_i}\,\langle\,\alpha_1,\dots,\bar\alpha,\dots,\alpha_n\,|). \tag{69}$$

The diagonal elements of $M_\alpha^{\alpha'}$ are the weight operators W_α with

$$\begin{aligned}&\langle\alpha_1,\dots,\alpha_i,\dots,\alpha_n|W_\alpha=w_\alpha\langle\alpha_1,\dots,\alpha_i,\dots,\alpha_n|\\&w_\alpha=n_\alpha-n_{\bar\alpha},\end{aligned}$$

where n_α is the number of particles α in the state. It is sufficient to consider only the weights

$$w=(w_1,\dots,w_{[N/2]}) \tag{70}$$

because of $W_\alpha=-W_{\bar\alpha}$ and $\langle\,\underline{\alpha}\,|W_0=0$ for N odd.

The Yang–Baxter relations (14) yield for $u_a\to\infty$,

$$[M_a+M_{ab},T_b(u_b)]=0 \tag{71}$$

and if additionally $u_b\to\infty$, we obtain

$$[M_a+M_{ab},M_b]=0, \tag{72}$$

or for the matrix elements (in the real basis)

$$\left[M_\alpha^{\alpha'},T_\beta^{\beta'}(u)\right]=-\delta_\alpha^{\beta'}T_\beta^{\alpha'}(u)+\delta^{\alpha'\beta'}T_\beta^{\alpha}(u)+T_\alpha^{\beta'}(u)\delta_\beta^{\alpha'}-T_{\alpha'}^{\beta'}(u)\delta_{\alpha\beta} \tag{73}$$

$$\left[M_\alpha^{\alpha'},M_\beta^{\beta'}\right]=-\delta_\alpha^{\beta'}M_\beta^{\alpha'}+\delta^{\alpha'\beta'}M_\beta^{\alpha}+M_\alpha^{\beta'}\delta_\beta^{\alpha'}-M_{\alpha'}^{\beta'}\delta_{\alpha\beta}. \tag{74}$$

Equation (74) represents the structure relations of the $O(N)$ Lie algebra and (73) the $O(N)$-covariance of T. In particular, the transfer matrix is invariant

$$[M_\alpha^{\alpha'},\mathrm{tr}T(u)]=0. \tag{75}$$

Theorem 4.1.

1. *If the co-vector-valued function*

$$K_{\underline{\alpha}}(\underline{u})=\sum_{\underline{v}}g(\underline{u},\underline{v})L_{\underline{\mathring{\beta}}}(\underline{v})\Phi_{\underline{\alpha}}^{\underline{\mathring{\beta}}}(\underline{u},\underline{v})$$

is given by the nested off-shell Bethe ansatz (53), then the weights (70) are w

$$=(w_1,\dots,w_{[N/2]})=\begin{cases}\left(n-n_1,\dots,n_{[N/2]-1}-n_{[N/2]}\right) & \text{for } N \text{ odd}\\ \left(n-n_1,\dots,n_{[N/2]-2}-n_--n_+,n_--n_+\right) & \text{for } N \text{ even}\,.\end{cases}$$

2. *If* $K_{\underline{\alpha}}(\underline{u})$ *satisfies the conditions of theorem 3.4 and if* $L_{\underline{\mathring{\beta}}}(\underline{v})$ *is a highest weight state, then* $K_{\underline{\alpha}}(\underline{u})$ *is a highest weight state:*

$$K(\underline{u})M_{\alpha}^{\alpha'} = 0 \qquad \text{for} \qquad \alpha' < \alpha.$$

(Recall that $\alpha' < \alpha$ *is to be understood corresponding to the ordering* $1, 2, \ldots, [N/2], (0), \overline{[N/2]}, \ldots, \bar{2}, \bar{1}$ *.)*

3. *The weights satisfy the highest weight condition*

$$\begin{cases} w_1 \geqslant w_2 \geqslant \ldots \geqslant w_{[N/2]} \geqslant 0 & \text{for } N \text{ odd} \\ w_1 \geqslant w_2 \geqslant \ldots \geqslant |w_{[N/2]}| & \text{for } N \text{ even.} \end{cases}$$

The proof of this theorem can be found in appendix C. We mention that for N even, the highest weight property was already discussed in appendix B of [17].

5. Examples

In this section, we present some explicit examples of the solutions of the $O(N)$-difference equations.

Example 5.1. The simplest example is obtained for $k_{\max} = 1$, which means the trivial solution of the difference equations

$$K_{\underline{\alpha}} = \Omega_{\underline{\alpha}} = \delta_{\alpha_1}^1, \ldots, \delta_{\alpha_n}^1.$$

The weights of $K_{\underline{\alpha}}$ are $w = (n, 0, \ldots, 0)$. In the language of spin chains this case corresponds to the ferromagnetic ground state.

Example 5.2. For $N \neq 4$, $k_{\max} = 2$ and $n^{(1)} = 1$, the solution reads

$$K_{\underline{\alpha}}(\underline{u}) = \sum_v g(\underline{u}, v) L_{\mathring{\beta}}(v) (\Omega C^{\mathring{\beta}}(\underline{u}, v))_{\underline{\alpha}},$$

with $v = \tilde{v} - 2l/\nu (l \in \mathbf{Z}, \tilde{v}$ an arbitrary constant) and

$$g(\underline{u}, v) = \prod_{i=1}^n \psi(u_i - v), \qquad \psi(u) = \frac{\Gamma\left(-\frac{1}{2}\nu + \frac{u}{2}\nu\right)}{\Gamma\left(\frac{u}{2}\nu\right)}$$

$$L_{\mathring{\beta}}(v) = \delta_{\mathring{\beta}}^2 \quad \text{for} \quad N > 4 \quad \text{and} \quad = \delta_{\mathring{\beta}}^0 \quad \text{for} \quad N = 3.$$

The weights of this co-vector $K_{\underline{\alpha}}(\underline{u})$ are $w = (n - 1, 1, 0, \ldots, 0)$. The action of the operator $C^{\mathring{\beta}}(\underline{u}, v)$ on the reference state is easily calculated with the help of equations (6), (12) and (13).

As a particular case of this example, we determine explicitly the solution for the following.

Example 5.3. The action of the C-operator on the reference state for the case of $n = 2$ and $N > 4$ of example 5.2 yields

$$\left(\Omega C^2(\underline{u}, v)\right)_{\underline{\alpha}} = \delta_{\alpha_1}^2 \delta_{\alpha_2}^1 c(u_1 - v) a(u_2 - v) + \delta_{\alpha_1}^1 \delta_{\alpha_2}^2 c(u_2 - v).$$

Therefore, we obtain (with the help of Dougall's formula)

$$\begin{aligned} K_{\underline{\alpha}}(\underline{u}) &= \sum_v \psi(u_1 - v)\psi(u_2 - v) \left(\delta_{\alpha_1}^2 \delta_{\alpha_2}^1 c(u_1 - v) a(u_2 - v) + \delta_{\alpha_1}^1 \delta_{\alpha_2}^2 c(u_2 - v)\right) \\ &= \frac{\pi^2 \Gamma(\nu) / \left(\Gamma\left(\frac{1}{2}\nu\right)\right)^2}{\sin \frac{1}{2}\pi\nu(1 - u_1 + \tilde{v}) \sin \frac{1}{2}\pi\nu(1 - u_2 + \tilde{v})} \frac{\delta_{\alpha_1}^{\mathring{\beta}} \delta_{\alpha_2}^1 - \delta_{\alpha_1}^1 \delta_{\alpha_2}^{\mathring{\beta}}}{\Gamma\left(1 + \frac{1}{2}\nu(u_{21} + 1)\right) \Gamma\left(-\frac{1}{2}\nu(u_{21} - 1)\right)}. \end{aligned}$$

The weights are $w = (1, 1, 0, \ldots, 0)$.

Example 5.4. The special case $N = 3$ gives

$$K_{\underline{\alpha}}(\underline{u}) = \frac{-\pi}{\sin\pi\,(-u_1+\tilde{v})\sin\pi\,(-u_2+\tilde{v})}\frac{\sin\pi u_{12}}{u_{12}\,(u_{12}-1)}\left(\delta^0_{\alpha_1}\delta^1_{\alpha_2}-\delta^1_{\alpha_1}\delta^0_{\alpha_2}\right)$$

and weight $w = (1)$.

Example 5.5. To obtain the results in examples 5.2 and 5.3 for $N = 4$, we have to use the corresponding $SU(2)$ formulas with $\psi(u) = \Gamma(-\frac{1}{2}+\frac{u}{2})/\Gamma(\frac{u}{2})$:

$$K^{(\pm)}_{\underline{A}}(\underline{u}) = \sum_v g(\underline{u}, v)\left(\Omega C(\underline{u}, v)\right)_{\underline{A}}.$$

For $n = 2$,

$$\begin{aligned}K^{(\pm)}_{\underline{A}}(\underline{u}) &= \sum_v \psi(u_1-v)\psi(u_2-v)\big(\delta^{\downarrow}_{A_1}\delta^{\uparrow}_{A_2}c(u_1-v)a(u_2-v)+\delta^{\uparrow}_{A_1}\delta^{\downarrow}_{A_2}c(u_2-v)\big)\\ &= K^{(\pm)}(\underline{u})\big(\delta^{\downarrow}_{A_1}\delta^{\uparrow}_{A_2}-\delta^{\uparrow}_{A_1}\delta^{\downarrow}_{A_2}\big),\end{aligned}$$

$$\begin{aligned}K^{(\pm)}(\underline{u}) = \pi &\frac{1}{\sin\frac{1}{2}\pi(1-u_1+\tilde{v}^{(\pm)})\sin\frac{1}{2}\pi(1-u_2+\tilde{v}^{(\pm)})}\\ &\times\frac{1}{\Gamma\left(1+\frac{1}{2}\,(u_{21}+1)\right)\Gamma\left(-\frac{1}{2}\,(u_{21}-1)\right)}.\end{aligned}$$

There are three possibilities: using the intertwiners (58), we find

$$\begin{aligned}K_{\underline{\alpha}}(\underline{u}) &= K^{(+)}_{\underline{A}}(\underline{u})\Omega^{(-)}_{\underline{B}}\Gamma^{\underline{AB}}_{\underline{\alpha}} = K^{(+)}(\underline{u})\big(\delta^{\downarrow}_{A_1}\delta^{\uparrow}_{A_2}-\delta^{\uparrow}_{A_1}\delta^{\downarrow}_{A_2}\big)\big(\delta^{\uparrow}_{B_1}\delta^{\uparrow}_{B_2}\big)\Gamma^{\underline{AB}}_{\underline{\alpha}}\\ &= K^{(+)}(\underline{u})\big(-\delta^{\bar{2}}_{\alpha_1}\delta^1_{\alpha_2}+\delta^1_{\alpha_1}\delta^{\bar{2}}_{\alpha_1}\big)\end{aligned}$$

with weights $w = (1, -1)$, and

$$\begin{aligned}K_{\underline{\alpha}}(\underline{u}) &= \Omega^{(+)}_{\underline{A}}(\underline{u})K^{(-)}_{\underline{B}}(\underline{u})\Gamma^{\underline{AB}}_{\underline{\alpha}} = \big(\delta^{\uparrow}_{A_1}\delta^{\uparrow}_{A_2}\big)\big(\delta^{\downarrow}_{B_1}\delta^{\uparrow}_{B_2}-\delta^{\uparrow}_{B_1}\delta^{\downarrow}_{B_2}\big)K^{(-)}(\underline{u})\Gamma^{\underline{AB}}_{\underline{\alpha}}\\ &= K^{(-)}(\underline{u})\big(-\delta^2_{\alpha_1}\delta^1_{\alpha_2}+\delta^1_{\alpha_1}\delta^2_{\alpha_2}\big)\end{aligned}$$

with weights $w = (1, 1)$ and

$$\begin{aligned}K_{\underline{\alpha}}(\underline{u}) &= K^{(+)}_{\underline{A}}(\underline{u})K^{(-)}_{\underline{B}}\Gamma^{\underline{AB}}_{\underline{\alpha}}\\ &= K^{(+)}(\underline{u})K^{(-)}(\underline{u})\big(\delta^{\downarrow}_{A_1}\delta^{\uparrow}_{A_2}-\delta^{\uparrow}_{A_1}\delta^{\downarrow}_{A_2}\big)\big(\delta^{\downarrow}_{B_1}\delta^{\uparrow}_{B_2}-\delta^{\uparrow}_{B_1}\delta^{\downarrow}_{B_2}\big)\Gamma^{\underline{AB}}_{\underline{\alpha}}\\ &= K^{(+)}(\underline{u})K^{(-)}(\underline{u})\big(-\delta^{\bar{1}}_{\alpha_1}\delta^1_{\alpha_2}-\delta^2_{\alpha_1}\delta^{\bar{2}}_{\alpha_2}-\delta^{\bar{2}}_{\alpha_1}\delta^2_{\alpha_2}-\delta^1_{\alpha_1}\delta^{\bar{1}}_{\alpha_2}\big)\end{aligned}$$

with weights $w = (0, 0)$.

6. Conclusion

In this paper, we solved the $O(N)$-matrix difference equations by means of the off-shell algebraic nested Bethe ansatz. We introduced a new object called Π-matrix to overcome the difficulties connected to the special peculiarities of the $O(N)$ symmetric R-matrix structure. The highest weight properties of the solutions were analyzed. We believe that our construction can also be applied to the cases with similar group theoretical complexities, such as B_n, C_n, D_n Lie algebras and superalgebra $Osp(n|2m)$ (see [16]).

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Appendix A. Proof of lemma 2.5

Proof (a) We prove (23) by induction: it is true for $m = 2$, because similar to (21)

$$\pi_1\pi_2\bar{e}_b T_{12,b}(\underline{u}, u_b)\bar{e}^b = \pi_1\pi_2 + f(u_{12})\mathring{\mathbf{C}}^{12}\bar{e}_1 e_2.$$

We assume (23) for $m - 1$ and replace in definition (20) $\Pi_{2,\dots,m}$ as given by (23)

$$\begin{aligned}\Pi_{1,\dots,m} &= \pi_1\Pi_{2,\dots,m}\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\\ &= \pi_1(\Pi_{2,\dots,m-1}\pi_m\bar{e}_b T_{2\dots m,b}\bar{e}^b)\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\\ &= (\pi_1\Pi_{2,\dots,m-1}\pi_m)\bar{e}_b T_{1,\dots,m,b}\bar{e}^b\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\\ &= (\pi_1\Pi_{2,\dots,m-1}\pi_m)\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\bar{e}_b T_{1,\dots,m,b}\bar{e}^b\\ &= (\pi_1\Pi_{2,\dots,m-1}\pi_m)\bar{e}_a T_{1,\dots,m-1,a}\bar{e}^a\bar{e}_b T_{1,\dots,m,b}\bar{e}^b\\ &= (\Pi_{1,\dots,m-1}\pi_m)\bar{e}_b T_{1,\dots,m,b}\bar{e}^b.\end{aligned}$$

In equality 3, we have $T_{2,\dots,m,b}$ replaced by $T_{1,\dots,m,b}$, because

$$\bar{e}_b R_{1b}(u_{1b})\bar{e}_a R_{1a}(u_{1a}) = \bar{e}_b\bar{e}_a R_{1a}(1) + c(u_{1b})\bar{e}_1\bar{e}_a R_{1a}(1) = \bar{e}_b\bar{e}_a R_{1a}(1)$$

holds, where $\bar{e}_1\bar{e}_a R_{1a}(1) = \bar{e}_1\bar{e}_a a(1) = 0$ has been used. Similarly, equality 5 holds. Equality 4 holds because the Yang–Baxter equation for R implies that $\bar{e}_a T_{1,\dots,m,a}\bar{e}^a$ and $\bar{e}_b T_{1,\dots,m,b}\bar{e}^b$ commute.

(b) Again we prove (24) by induction. For $m = 2$, the claim was proved in section 2.4, for $m > 2$ it follows for $1 < i < j$ from (20) and for $i < j < m$ from (23).

(c) The proof of equation (25) is similar to that of (a). We commute $T(u_a)$ and $T(u_0)$, use $\bar{e}_0\pi_1 R_{10}(u_{10})R_{1a}(1) = \bar{e}_0\pi_1$ and apply induction:

$$\begin{aligned}\Pi_{1,\dots,m}\bar{e}_0 T_{1,\dots,m,0}(u_0)\bar{e}^0 &= (\pi_1\Pi_{2,\dots,m})\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\bar{e}_0 T_{1,\dots,m,0}(u_0)\bar{e}^0\\ &= (\pi_1\Pi_{2,\dots,m})\bar{e}_0 T_{1,\dots,m,0}(u_0)\bar{e}^0\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\\ &= \pi_1(\Pi_{2,\dots,m})\bar{e}_0 T_{2,\dots,m,0}(u_0)\bar{e}^0\bar{e}_a T_{1,\dots,m,a}\bar{e}^a\\ &= (\pi_1\Pi_{2,\dots,m})\bar{e}_a T_{1,\dots,m,a}\bar{e}^a = \Pi_{1,\dots,m}.\end{aligned}$$

(d) Equations (26) and (27) follow from (20) and $R^{\bar{1}\mathring{\alpha}_1}_{1\alpha}(1) = 0, R^{\bar{1}\mathring{\alpha}_1}_{\mathring{\alpha}\alpha}(1) = \delta^{\mathring{\alpha}_1}_{\mathring{\alpha}}\delta^{\bar{1}}_{\alpha}$ and analogously (28) and (29).

Appendix B. Proof of the main theorem 3.4

In the following, we use the convention that α etc take the values $1, 2, \dots, (0), \dots, \bar{2}, \bar{1}$ and $\mathring{\alpha}$ etc take the values $2, \dots, (0), \dots, \bar{2}$:

$$K_{\underline{\alpha}}(\underline{u}) = \sum_{\underline{v}} g(\underline{u}, \underline{v})L_{\underline{\mathring{\beta}}}(\underline{v})\Pi^{\underline{\mathring{\beta}}}_{\underline{\beta}}(\underline{v})\big(\Omega T^{\beta_m}_1(\underline{u}, v_m)\cdots T^{\beta_1}_1(\underline{u}, v_1)\big)_{\underline{\alpha}}.$$

(i) Proof. Property (i) in the form of (32) follows directly from the Yang–Baxter equations and the action of the R-matrix on the pseudo-ground state Ω :

$$\begin{aligned}&(T_{\dots ji\dots})^{\beta}_1(\dots u_j, u_i\dots)R_{ij}(u_{ij}) = R_{ij}(u_{ij})(T_{\dots ij\dots})^{\beta}_1(\dots u_i, u_j\dots)\\ &\Omega_{\dots ij\dots}R_{ij}(u_{ij}) = a(u_{ij})\Omega_{\dots ij\dots}.\end{aligned}$$

□

(ii) Proof. We prove

$$K_{1,\dots,n}(\underline{u}') = K_{1,\dots,n}(\underline{u})\,Q_{1,\dots,n}(\underline{u}), \tag{B.1}$$

where $\underline{u}' = (u_1 + 2/\nu, u_2, \dots, u_n)$. The matrix $Q(\underline{u}) = Q(\underline{u}, 1)$ is given by (38). Note that we assign to the auxiliary space of $\tilde{T}_Q(\underline{u})$ the spectral parameter u_1 on the right-hand side and $u_1' = u_1 + 2/\nu$ on the left-hand side. In the following, we will suppress the indices $1, \dots, n$. We are now going to prove (B.1) in the form

$$K(\underline{u})\big(A_Q(\underline{u}) + D_{Q}{}^{\mathring{\beta}}_{\mathring{\beta}}(\underline{u}) + A_{3,Q}(\underline{u})\big) = \prod_{k=2}^{n} a(u_{k1}) K(\underline{u}')\,, \tag{B.2}$$

where $K(\underline{u})$ is a co-vector-valued function as given by equation (47) and the Bethe ansatz state (40). To analyze the left-hand side of equation (B.2), we proceed as usual in the algebraic Bethe ansatz and push $A_Q(\underline{u})$, $D_{Q}{}^{\mathring{\beta}}_{\mathring{\beta}}(\underline{u})$ and $A_{3,Q}(\underline{u})$ through all the $T_1^{\beta_i}$-operators. As usual, we obtain wanted and unwanted terms. We first find that the wanted contribution from $A_Q(\underline{u})$ already gives the result we are looking for. Secondly, the wanted contributions from $D_{Q}{}^{\mathring{\beta}}_{\mathring{\beta}}(\underline{u})$ and $A_{3,Q}(\underline{u})$ applied to Ω give zero because of (46). Thirdly, the unwanted contributions from $A_Q(\underline{u})$, $D_{Q}{}^{\mathring{\beta}}_{\mathring{\beta}}(\underline{u})$ and $A_{3,Q}(\underline{u})$ can be written as differences which cancel after summation over the v_j. All these three facts can be seen as follows.

The 'wanted terms' from A_Q are obtained if one writes the Zapletal commutations rule (35) as

$$T_1^{\beta_k}(\underline{u}, v_k) A_Q(\underline{u}) = A_Q(\underline{u}) T_1^{\beta_i}(\underline{u}', v_k) a(u_1 - v_k) + uw.$$

Therefore, we obtain

$$\big(L(\underline{v})\Pi(\underline{u}, \underline{v})\big)_{\underline{\beta}}\, \Omega T_1^{\beta_m}(\underline{u}, v_m) \cdots T_1^{\beta_1}(\underline{u}, v_1) A_Q(\underline{u}) = w^A + uw^A,$$

with

$$\begin{aligned} w^A(\underline{u}, \underline{v}) &= \big(L(\underline{v})\Pi(\underline{u}, \underline{v})\big)_{\underline{\beta}}\, \Omega A_Q(\underline{u}) T_1^{\beta_m}(\underline{u}', v_m) \cdots T_1^{\beta_1}(\underline{u}', v_1) \prod_{k=1}^{m} a(u_1 - v_k) \\ &= \prod_{k=2}^{n} a(u_{k1}) \prod_{k=1}^{m} a(u_1 - v_k) L_{\mathring{\beta}}(\underline{v}) \phi^{\underline{\mathring{\beta}}}(\underline{u}', \underline{v}) \end{aligned} \tag{B.3}$$

and similarly $w^D = w^{A_3} = 0$ because of (46). Inserted into (47), this yields

$$\begin{aligned} \big(K(\underline{u})Q\big)^{\text{wanted}}(\underline{u}) &= \sum_{\underline{u}} g(\underline{u}, \underline{v}) \prod_{k=1}^{m} a(u_1 - v_k) L_{\mathring{\beta}}(\underline{v}) \phi^{\underline{\mathring{\beta}}}(\underline{u}', \underline{v}) \\ &= \sum_{\underline{u}} g(\underline{u}', \underline{v}) L_{\mathring{\beta}}(\underline{v}) \phi^{\underline{\mathring{\beta}}}(\underline{u}', \underline{v}) = K(\underline{u}') \end{aligned}$$

because

$$g(\underline{u}, \underline{v}) \prod_{k=1}^{m} a(u_1 - v_k) = g(\underline{u}', \underline{v}),$$

where (49) has been used. Therefore, it remains to prove that the unwanted terms cancel. This will follow from the lemma below. □

We apply $Q_{1,\dots,n}(\underline{u})$ to the state $\Psi_{1,\dots,n}(\underline{u}, \underline{v})$ (suppressing the quantum space indices)

$$\Psi(\underline{u}, \underline{v})\big(A_Q(\underline{u}) + D_{Q}{}^{\mathring{\beta}}_{\mathring{\beta}}(\underline{u}) + A_{3,Q}(\underline{u})\big) = \text{wanted} + \text{unwanted}.$$

The wanted contribution has been calculated above and the unwanted terms may be written as (see appendix B.1)

$$\begin{aligned}
\text{unwanted} &= uw^{A} + uw^{D} + uw^{A_3}\\
uw^{A} &= \sum_{i=1}^{m} \big(uw_{C}^{A,i}\big)_{\mathring{\gamma}} C_{Q}^{\mathring{\gamma}} + \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} uw_{C_2}^{A,ij} C_{2,Q}\\
uw^{D} &= \sum_{i=1}^{m} \big(uw_{C}^{D,i}\big)_{\mathring{\gamma}} C_{Q}^{\mathring{\gamma}} + \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} uw_{C_2}^{D,ij} C_{2,Q} + \sum_{i=1}^{m} \big(uw_{C_3}^{D,i}\big)_{\mathring{\gamma}} \big(C_{3,Q}\big)_{\mathring{\gamma}'} \mathring{\mathbf{C}}^{\mathring{\gamma}'\mathring{\gamma}}\\
uw^{A_3} &= \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} uw_{C_2}^{A_3,ij} C_{2,Q} + \sum_{i=1}^{m} \big(uw_{C_3}^{A_3,i}\big)_{\mathring{\gamma}} \big(C_{3,Q}\big)_{\mathring{\gamma}'} \mathring{\mathbf{C}}^{\mathring{\gamma}'\mathring{\gamma}}.
\end{aligned}$$

Lemma B.1. *The unwanted terms satisfy the relations*

$$\big(uw_{C}^{D,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}) g(\underline{u}, \underline{v}) = -\big(uw_{C}^{A,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}^{(i)}) g(\underline{u}, \underline{v}^{(i)}) \tag{B.4}$$

$$\big(uw_{C_3}^{A_3,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}) g(\underline{u}, \underline{v}) = -\big(uw_{C_3}^{D,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}^{(i)}) g(\underline{u}, \underline{v}^{(i)}) \tag{B.5}$$

$$uw_{C_2}^{D,ij} (\underline{u}, \underline{v}^{(j)}) g(\underline{u}, \underline{v}^{(j)}) = -uw_{C_2}^{A,ij} (\underline{u}, \underline{v}^{(ij)}) g(\underline{u}, \underline{v}^{(ij)}) - uw_{C_2}^{A_3,ij} (\underline{u}, \underline{v}) g(\underline{u}, \underline{v}) \tag{B.6}$$

with the notation

$$\begin{aligned}
\underline{v} &= v_1, \ldots, v_m\\
\underline{v}^{(i)} &= v_1, \ldots, v_i', \ldots, v_m\\
\underline{v}^{(ij)} &= v_1, \ldots, v_i', \ldots, v_j', \ldots, v_m\\
v' &= v + 2/\nu.
\end{aligned}$$

Equations (B.4)–(B.6) imply that all unwanted terms cancel after summation over the $\underline{v}$ *in the Jackson-type integral (47).*

Proof. We can calculate the following unwanted contributions explicitly (see appendix B.1):

$$\begin{aligned}
\big(uw_{C}^{A,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}) &= -c(u_1' - v_i) X_{\mathring{\gamma}}^{(i)} (\underline{u}, \underline{v})\\
\big(uw_{C}^{D,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}) &= -c(v_i - u_1) X_{\mathring{\gamma}}^{(i)} (\underline{u}, \underline{v}^{(i)}) \chi_i(\underline{u}, \underline{v})\\
\big(uw_{C_3}^{D,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}) &= -f(u_1' - v_i) X_{\mathring{\gamma}}^{(i)} (\underline{u}, \underline{v})\\
\big(uw_{C_3}^{A_3,i}\big)_{\mathring{\gamma}} (\underline{u}, \underline{v}) &= f(u_1 - v_i) X_{\mathring{\gamma}}^{(i)} (\underline{u}, \underline{v}^{(i)}) \chi_i(\underline{u}, \underline{v})
\end{aligned} \tag{B.7}$$

with

$$X_{\mathring{\gamma}}^{(i)} (\underline{u}, \underline{v}) = L(v_i, \underline{v}_i)_{\mathring{\gamma}\underline{\mathring{\beta}}_i} \Phi^{\underline{\mathring{\beta}}_i} (\underline{u}, \underline{v}_i) \prod_{\substack{k=1\\ k\neq i}}^{m} a(v_{ik}) a_1(\underline{u}, v_i) \tag{B.8}$$

and

$$\chi_i(\underline{u}, \underline{v}) = \frac{a_2(\underline{u}, v_i)}{a_1(\underline{u}, v_i')} \prod_{\substack{k=1\\ k\neq i}}^{m} \frac{a(v_{ki})}{a(v_{ik}')}, \tag{B.9}$$

where a_1 and a_2 are defined in (45). The remaining unwanted terms are

$$\begin{aligned} uw_{C_2}^{A,ij}(\underline{u},\underline{v}) &= -c(u_1'-v_i)X^{(ij)}(\underline{v}) \\ uw_{C_2}^{D,ij}(\underline{u},\underline{v}) &= \big(c(u_1-v_i)+f(u_1'-v_j)\big)X^{(ij)}(\underline{v}^{(i)})\chi_i(\underline{u},\underline{v}) \\ uw_{C_2}^{A_3,ij}(\underline{u},\underline{v}) &= -f(u_1-v_j)X^{(ij)}(v_i',v_j',\underline{v}_{ij})\chi_i(\underline{u},\underline{v}^{(j)})\chi_j(\underline{u},\underline{v}) \end{aligned} \tag{B.10}$$

with

$$X^{(ij)}(\underline{u},\underline{v}) = f(v_{ij})a(v_{ij})L(v_i,v_j,\underline{v}_{ij})\mathring{\mathbf{C}}^{ij}\Phi(\underline{u},\underline{v}_{ij})\prod_{k=1,k\neq i,j}^{m}(a(v_{ik})a(v_{jk}))a_1(v_i)a_1(v_j). \tag{B.11}$$

The claims (B.4)–(B.6) follow from the shift property of the function $g(\underline{u},\underline{v})$ defined in (48):

$$\chi_i(\underline{u},\underline{v})g(\underline{u},\underline{v}) = g(\underline{u},\underline{v}^{(i)})$$

which is due to the shift properties (49) of the functions $\psi(v)$ and $\tau(v)$. ☐

Note that for the on-shell Bethe ansatz, the equations for the unwanted terms are quite similar (apart from the fact that there are no shifts) and their cancellation is equivalent to the Bethe ansatz equations

$$\chi_i(\underline{u},\underline{v}) = \frac{a_2(\underline{u},v_i)}{a_1(\underline{u},v_i)}\prod_{\substack{k=1\\k\neq i}}^{m}\frac{a(v_{ki})}{a(v_{ik})} = 1.$$

Appendix B.1. The unwanted terms

The unwanted terms are derived using lemma 2.2. Here we only sketch the derivation of uw^A as an example (more details will be published elsewhere [30]). We start with (B.3), use Yang–Baxter relations

$$\begin{aligned} w^A &= (L\Pi)_{\underline{\beta}}\,\Omega A_Q T_1^{\beta_m}(\underline{u}',v_m)\cdots T_1^{\beta_1}(\underline{u}',v_1)a(u_1-v_m)\cdots a(u_1-v_1) \\ &= (L\Pi)_{\underline{\beta}}\,\Omega\left(R_{0m}(u_1'-v_m)\cdots R_{01}(u_1'-v_1)\right)_{\gamma,\underline{\beta}'}^{\underline{\beta},1} T_1^{\beta_m'}(\underline{u},v_m)\cdots T_1^{\beta_1'}(\underline{u},v_1)(T_Q)_1^{\gamma} \end{aligned}$$

and lemma 2.2 in the form of (16)

$$\begin{aligned} w^A &= (L\Pi)_{\underline{\beta}}\,\Omega T_1^{\beta_m}(z_m)\cdots T_1^{\beta_1}(z_1)A_Q + \sum_{i=1}^{m}c(u_1'-v_i) \\ &\quad\times (L\Pi)_{\underline{\beta}}\,\Omega\left(R_{0m}(v_{im})\cdots\mathbf{P}_{0i}\cdots R_{01}(v_{i1})\right)_{\gamma,\underline{\beta}'}^{\underline{\beta},1} T_1^{\beta_m'}(\underline{u},v_m)\cdots T_1^{\beta_1'}(\underline{u},v_1)(T_Q)_1^{\gamma} \\ &= (L\Pi)_{\underline{\beta}}\,\Omega T_1^{\beta_m}(v_m)\cdots T_1^{\beta_1}(v_1)A_Q - uw^A. \end{aligned}$$

Note that the $d(u_1'-v_i)$ contributions vanish because they produce terms like $\Pi^{\cdots}_{\cdots\bar{1}} = 0$ (see (28)). We commute the $R_{ij}(v_{ij})$ (for $j < i$) with Π using (24) and apply the $\mathring{R}_{ij}(v_{ij})$ to L using (41). Using further Yang–Baxter relations to the $R_{ij}(v_{ij})$ (for $j > i$) the unwanted terms write as

$$\begin{aligned} uw^A = &-\sum_{i=1}^{m}c(u_1'-v_i)\left(L(v_i,\underline{v}_i)\Pi(v_i,\underline{v}_i)\right)_{\gamma\underline{\beta}_i} \\ &\times\prod_{k=1,k\neq i}^{m}a(v_{ik})a_1(\underline{u},v_i)\Omega\left[T_1^{\beta_m}(v_m)\cdots T_1^{\beta_1}(v_1)\right]_i(T_Q)_1^{\gamma}, \end{aligned}$$

where we use the short notations $\underline{v}_i$, $\underline{\beta}_i$ and $\left[T_1^{\beta_m}(v_m)\cdots T_1^{\beta_1}(v_1)\right]_i$ which means that v_i, β_i and $T_1^{\beta_i}(v_i)$ are missing, respectively. For $\gamma = 1$, this vanishes because of $\Pi^{\cdots}_{1\dots} = 0$ (see (26)), for $\gamma = \mathring{\gamma} \neq 1, \bar{1}$ this gives uw_C^A and for $\gamma = \bar{1}$ this gives $uw_{C_2}^A$ in the form of (B.7) and (B.10) where (27) and (30) are used.

Appendix C. Proof of theorem 4.1

Proof. 1. The weights (70) of the reference state Ω (44) are

$$w = (n = n_0, 0, \dots, 0)\,.$$

In level $k = 1, \dots, [(N-3)/2]$ of the Bethe ansatz the weights are changed as

$$w_k \to w_k - n_k, \qquad w_{k+1} \to w_{k+1} + n_k.$$

This means the states $\Phi_{\underline{\alpha}}^{\mathring{\beta}}(\underline{u}, \underline{v})$ of (43) are eigenvectors of the weights. Using in addition (56) for $O(3)$ and (64) for $O(4)$, we obtain w

$$= (w_1, \dots, w_{[N/2]}) = \begin{cases} \left(n - n_1, \dots, n_{[N/2]-1} - n_{[N/2]}, n_{[N/2]}\right) & \text{for} \quad N \text{ odd} \\ \left(n - n_1, \dots, n_{[N/2]-2} - n_- - n_+, n_- - n_+\right) & \text{for} \quad N \text{ even.} \end{cases}$$

2. The proof of the highest weight property

$$\Psi(\underline{v})M^1_{\mathring{\gamma}} = \Psi(\underline{v})M^{\mathring{\gamma}}_{\bar{1}} = \Psi(\underline{v})M^1_{\bar{1}} = 0$$

uses similar techniques as the derivation of the unwanted terms.

(i) We consider

$$\begin{aligned} 0 &= \big(L(\underline{v})\Pi(\underline{v})\big)_{\underline{\beta}}\,\Omega B_{\mathring{\gamma}}(v) T_1^{\beta'_m}(v_m)\cdots T_1^{\beta'_1}(v_1) \\ &= \big(L(\underline{v})\Pi(\underline{v})\big)_{\underline{\beta}}\,\Omega\,(R_{0m}(v-v_m)\cdots R_{01}(v-v_1))^{\underline{\beta},1}_{\gamma,\underline{\beta}'}\,T_1^{\beta''_m}(v_m)\cdots T_1^{\beta''_1}(v_1)T^{\gamma}_{\gamma'}(v) \\ &\quad\times (R_{01}(v_1-v)\cdots R_{0m}(v_m-v))^{\gamma\underline{\beta}''}_{1\dots1,\mathring{\gamma}} + O(v^{-2}). \end{aligned}$$

For $v \to \infty$, applying lemma 2.2, equations (65) and (67), we obtain

$$0 = \Psi(\underline{v})M^1_{\mathring{\gamma}} - \sum_{i=1}^{m} X^{(i)}_{\mathring{\gamma}}(\underline{u}, \underline{v}) + \sum_{i=1}^{m} X^{(i)}_{\mathring{\gamma}}(\underline{u}, \underline{v}^{(i)})\chi_i(\underline{u}, \underline{v}),$$

with $X^{(i)}_{\mathring{\gamma}}$ and χ_i defined in (B.8) and (B.9). After multiplication with $g(\underline{u}, \underline{v})$ and summation over the $\underline{v}$ the terms cancel each other because of $\chi_i(\underline{u}, \underline{v})g(\underline{u}, \underline{v}) = g(\underline{u}, \underline{v}^{(i)})$.

(ii) We consider

$$\begin{aligned} &\big(L(\underline{v})\Pi(\underline{v})\big)_{\underline{\beta}}\,(R_{10}(v_1-v)\cdots R_{m0}(v_m-v))^{\mathring{\gamma}\underline{\beta}}_{\underline{\beta}',\gamma}\,\Omega T^{\gamma}_{\bar{1}}(v) T_1^{\beta'_m}(v_m)\cdots T_1^{\beta'_1}(v_1) \\ &= \big(L(\underline{v})\Pi(\underline{v})\big)_{\underline{\beta}}\,(R_{10}(v_1-v)\cdots R_{m0}(v_m-v))^{\mathring{\gamma}\underline{\beta}}_{\bar{1},\underline{\beta}'}\,\Omega T_1^{\beta'_m}(v_m)\cdots T_1^{\beta'_1}(v_1) + O(v^{-2}) \\ &= \Psi(\underline{v})T^{\mathring{\gamma}}_{\bar{1}}(v) + O(v^{-2}) \\ &\quad + \big(L(\underline{v})\Pi(\underline{v})\big)_{\underline{\beta}}\,\Omega T^{\beta_m}_{\beta'_m}(v_m)\cdots T^{\beta_1}_{\beta'_1}(v_1)\,(R_{10}(v_1-v)\cdots R_{m0}(v_m-v))^{\mathring{\gamma}\underline{\beta}'}_{1,\dots,1,\bar{1}}\,. \end{aligned}$$

It has been used that only $\gamma = \bar{1}$ contributes because of $\Omega B_2 = \Omega B_3 = 0$.

For $v \to \infty$, applying lemma 2.2, equations (65) and (67), we obtain

$$0 = \Psi(\underline{v})M^{\mathring{\gamma}}_{\bar{1}} - \mathbf{C}^{\mathring{\gamma}\mathring{\gamma}'}\sum_{i=1}^{m} X^{(i)}_{\mathring{\gamma}'}(\underline{u}, \underline{v}) + \mathbf{C}^{\mathring{\gamma}\mathring{\gamma}'}\sum_{i=1}^{m} X^{(i)}_{\mathring{\gamma}'}(\underline{u}, \underline{v}^{(i)})\chi_i(\underline{u}, \underline{v}).$$

Again after multiplication with $g(\underline{u}, \underline{v})$ and summation over the $\underline{v}$ the terms cancel each other because of $\chi_i(\underline{u}, \underline{v})g(\underline{u}, \underline{v}) = g(\underline{u}, \underline{v}^{(i)})$.

(iii) We consider

$$\begin{aligned} 0 &= \Omega M_{\bar{1}}^{1} \cdots \\ &= \left(L(\underline{v})\Pi(\underline{v})\right)_{\underline{\beta}} (R(v - v_m) \cdots R(v - v_1))^{\underline{\beta},1}_{\gamma,\underline{\beta}'} \, \Omega T^{\beta'_m}_{\beta''_m}(v_m) \cdots T^{\beta'_1}_{\beta''_1}(v_1) T^{\gamma}_{\gamma'}(v) \\ &\quad \times (R(v_1 - v) \cdots R(v_m - v))^{\gamma',\underline{\beta}''}_{1,\ldots,1,\bar{1}} . \end{aligned}$$

For $v \to \infty$, we apply lemma 2.2, equations (64) and (66), and obtain

$$0 = \Psi(\underline{v})M_{\bar{1}}^{1} - \sum_{i=1}^{m} X^{(ij)}(\underline{v}) + \sum_{i=1}^{m} X^{(ij)}(v'_i, v'_j, \underline{v}_{ij})\chi_i(\underline{u}, \underline{v})\chi_j(\underline{u}, \underline{v}).$$

Again after multiplication with $g(\underline{u}, \underline{v})$ and summation over the $\underline{v}$ the terms cancel each other because of $\chi_i(\underline{u}, \underline{v})\chi_j(\underline{u}, \underline{v})g(\underline{u}, \underline{v}) = g(\underline{u}, \underline{v}^{(ij)})$.

(iv) Next we prove

$$\Psi(\underline{w})M^{\mathring{\gamma}'}_{\mathring{\gamma}} = 0, \, 1 < \mathring{\gamma}' < \mathring{\gamma} < \bar{1}.$$

We consider

$$\begin{aligned} &L_{\underline{\beta}'}(\underline{w})\Pi^{\mathring{\gamma}'\underline{\beta}'}_{\gamma\underline{\beta}}(v, \underline{w})\Omega T^{\beta_m}_{1}(w_m) \cdots T^{\beta_1}_{1}(w_1)T^{\gamma}_{\mathring{\gamma}}(v) + O(v^{-2}) \\ &\qquad = \left(L(\underline{w})\left(T^{(1)}\right)^{\mathring{\gamma}'}_{\mathring{\gamma}}(v)\right)_{\underline{\beta}'} \Pi^{\underline{\beta}'}_{\underline{\beta}}(\underline{w})\Omega T^{\mathring{\gamma}}_{\mathring{\gamma}}(v)T^{\beta_m}_{1}(w_m) \cdots T^{\beta_1}_{1}(w_1) + O(v^{-2}), \end{aligned}$$

where the Yang–Baxter rules and (24) have been used. We have also used that by (30) and (15),

$$\begin{aligned} &\Pi^{\mathring{\gamma}'\underline{\beta}'}_{\gamma\underline{\beta}}(v, \underline{w}) = \delta^{\mathring{\gamma}'}_{\gamma}\Pi^{\underline{\beta}'}_{\underline{\beta}}(\underline{w}) + O(v^{-1}) \\ &(R(w_1 - v) \cdots R(w_m - v))_{1,\ldots,m,0} = \mathbf{1}_{1,\ldots,m}\mathbf{1}_0 + O(v^{-1}). \end{aligned}$$

For $v \to \infty$, the highest weight condition $L(\underline{w})\left(M^{(1)}\right)^{\mathring{\gamma}'}_{\mathring{\gamma}} = 0$ implies the claim.

3. The highest weight properties of the weights are obtained as follows. In the complex basis, relation (73) reads as

$$\left[M_{\alpha}^{\alpha'}, M_{\beta}^{\beta'}\right] = -\delta_{\alpha}^{\beta'}M_{\beta}^{\alpha'} + \mathbf{C}^{\alpha'\beta'}(\mathbf{C}M)_{\alpha\beta} + M_{\alpha}^{\beta'}\delta_{\beta}^{\alpha'} - (M\mathbf{C})^{\beta'\alpha'}\mathbf{C}_{\alpha\beta}.$$

In particular for $\beta \neq \alpha, \bar{\alpha}$,

$$[M_{\alpha}^{\beta}, M_{\beta}^{\alpha}] = M_{\alpha}^{\alpha} - M_{\beta}^{\beta} = M_{\alpha}^{\alpha} + M_{\bar{\beta}}^{\bar{\beta}}.$$

Because of $\left(M_{\alpha}^{\beta}\right)^{\dagger} = M_{\beta}^{\alpha}$

$$0 \leqslant M_{\alpha}^{\beta}\left(M_{\alpha}^{\beta}\right)^{\dagger} = M_{\alpha}^{\beta}M_{\beta}^{\alpha} = M_{\beta}^{\alpha}M_{\alpha}^{\beta} + M_{\alpha}^{\alpha} - M_{\beta}^{\beta}.$$

For highest weight co-vectors,

$$0 = \Psi M_{\beta}^{\alpha} \text{ for } \alpha < \beta$$

which implies for the weights (70)

$$0 \leqslant w_{\alpha} - w_{\beta} \quad \text{for} \quad \alpha < \beta \leqslant N/2.$$

In addition, if N is even, then

$$\begin{aligned} &0 \leqslant w_{\alpha} + w_{\bar{\beta}} \quad \text{for} \quad \alpha \leqslant N/2 < \beta \neq \bar{\alpha} \\ &\quad \Rightarrow w_1 \geqslant w_2 \geqslant \ldots \geqslant w_{N/2-1} \geqslant |w_{N/2}| \end{aligned}$$

and if N is odd, then

$$\begin{aligned} &0 \leqslant w_{\alpha} \quad \text{for} \quad \alpha \leqslant N/2 \quad \text{because} \quad \Psi M_0^0 = 0 \\ &\quad \Rightarrow w_1 \geqslant w_2 \geqslant \ldots \geqslant w_{N/2} \geqslant 0. \end{aligned}$$

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