The Form Factors and Quantum Equation of Motion in the sine-Gordon Model*

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Abstract

Using the methods of the "form factor program" exact expressions of all matrix elements are obtained for several operators of the quantum sine-Gordon model alias the massive Thirring model. A general formula is presented which provides form factors in terms of an integral representation. In particular charge-less operators as for example the current of the topological charge, the energy momentum tensor and all higher currents are considered. In the breather sector it is found the quantum sine-Gordon field equation holds with an exact relation between the "bare" mass and the normalized mass. Also a relation for the trace of the energy momentum is obtained. All results are compared with Feynman graph expansion and full agreement is found.

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1 Introduction

This work continues a previous investigation [1] on exact form factors for the sine-Gordon alias the massive Thirring model. Coleman [2] had shown that these two models are equivalent on the quantum level. The corresponding classical models are defined by their Lagrangian's

$$\begin{aligned}\mathcal{L}^{SG} &= \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha}{\beta^2}(\cos\beta\varphi - 1)\\ \mathcal{L}^{MT} &= \overline{\psi}(i\gamma\partial - M)\psi - \tfrac{1}{2}g\,j^\mu j_\mu\,, \quad \left(j^\mu = \overline{\psi}\gamma^\mu\psi\right).\end{aligned}$$

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We do not use these classical Lagrangians and any quantization procedure to construct the quantum models. We have contact with the classical models only, when at the end we compare our exact results with Feynman graph expansions which are based on the Lagrangians. The 'form factor program' is part of the 'Bootstrap program for integrable quantum field theories in 1+1-dimensions'. This program classifies integrable quantum field theoretic models and in addition it provides their explicit exact solutions in term of all Wightman functions. These results are obtained in three steps:

1. The S-matrix is calculated by means of general properties such as unitarity and crossing, the Yang-Baxter equations (which are a consequence of integrability) and the additional assumption of 'maximal analyticity'. This means that the two-particle S-matrix is an analytic function in the physical plane (of the Mandelstam variable $(p_1+p_2)^2$) and possesses only those poles there which are of physical origin.

2. Generalized form factors which are matrix elements of local operators

$$^{out}\langle\, p'_m, \dots , p'_1\, |\mathcal{O}(x)|\, p_1, \dots , p_n\,\rangle^{in}$$

   are calculated by means of the S-matrix. More precisely, the equations $(i)-(v)$ given below on page 6 are used as an input. These equations follow from LSZ-assumptions and again the additional assumption of 'maximal analyticity' (see also [1]).

3. The Wightman functions are obtained by inserting a complete set of intermediate states. In particular the two point function for a hermitian operator $\mathcal{O}(x)$ reads

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int\cdots\int\frac{dp_1\dots dp_n}{(2\pi)^n 2\omega_1\dots 2\omega_n}\left|\langle\, 0\,|\mathcal{O}(0)|\, p_1, \dots , p_n\,\rangle^{in}\right|^2 e^{-ix\sum p_i}.$$

The on-shell program i.e. the exact determination of the scattering matrix was formulated in [3]. Off-shell considerations were carried out in [4] and in [5], where the concept of a generalized form factor was introduced and consistency equations were formulated which are expected to be satisfied by these objects. Thereafter this approach was developed further and studied in the context of several explicit models (see e.g. [6][1]). More recent papers on solitonic matrix elements in the sine-Gordon model are [7, 8]. There is a nice application [9, 10] of form factors in condensed matter physics. The one dimensional Mott insulators can be described in terms of the quantum sine-Gordon model.

In the previous paper [1] an integral representation for general matrix elements of the fundamental fermi-field of the massive Thirring model has been proposed. In [11, 12, 13] we generalize this formula and investigate in particular charge-less local operators. The strategy is as follows:

For a state of n particles of kind α_i with momenta $p_i = m\sinh\theta_i$ and a local operator $\mathcal{O}(x)$ the generalized form factor is defined by

$$\langle\, 0\,|\,\mathcal{O}(x)\,|\,\alpha_1(p_1),\dots,\alpha_n(p_n)\,\rangle^{in} = e^{-ix(p_1+\dots+p_n)}\mathcal{O}_{\underline{\alpha}}(\underline{\theta}) \tag{1}$$

[1]For more references see [1].

for $\theta_1 > \cdots > \theta_n$. The short notation $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\underline{\theta} = (\theta_1, \ldots, \theta_n)$ has been used. We make the Ansatz

$$\mathcal{O}_{\underline{\alpha}}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, h(\underline{\theta}, \underline{z}) \, p^{\mathcal{O}}(\underline{\theta}, \underline{z}) \, \Psi_{\underline{\alpha}}(\underline{\theta}, \underline{z})$$

with the Bethe state $\Psi_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ defined by eq. (5) and the integration contours $\mathcal{C}_{\underline{\theta}}$ of figure 1. The scalar function $h(\underline{\theta}, \underline{z})$ is uniquely determined by the S-matrix whereas the scalar 'p-function' $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ depends on the operator. By means of the Ansatz we transform the properties $(i) - (v)$ of the co-vector valued function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ (see page 6) to properties $(i') - (v')$ of the scalar function $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ which are easily solved. For example we obtain the p-functions for the local operator[2] $\mathcal{N}\left[\overline{\psi}\psi\right](x)$ as

$$p^{\overline{\psi}\psi}(\underline{\theta}, \underline{z}) = N_n^{\overline{\psi}\psi} \left(\sum_{i=1}^{n} e^{-\theta_i} \sum_{i=1}^{m} e^{z_i} - \sum_{i=1}^{n} e^{\theta_i} \sum_{i=1}^{m} e^{-z_i} \right) .$$

In section 4 we propose in addition the p-functions for $\mathcal{N}\left[\overline{\psi}\gamma^5\psi\right](x)$, the current $j^\mu(x)$, the energy momentum tensor $T^{\mu\nu}(x)$ and the infinitely many higher conserved currents $J_L^\mu(x)$. The identification with the operators is made by comparing the exact results with Feynman graph expansions. Properties as charge, behavior under Lorentz transformations etc. will also become obvious.

2 Recall of formulae

In this section we recall some formulae which we shall need in the following sections to present our results. All this material can be found in [1] including the original references.

2.1 The S-matrix

The sine-Gordon model alias massive Thirring model describes the interaction of several types of particles: solitons, anti-solitons alias fermions and anti-fermions and a finite number of charge-less breathers, which may be considered as bound states of solitons and anti-solitons. Integrability of the model implies that the n-particle S-matrix factorizes into two particle S-matrices. In particular scattering conserves the number of particles and even their momenta. The two particle soliton-soliton amplitude $a(\theta)$, the soliton anti-solitons forward and backward amplitudes $b(\theta)$ and $c(\theta)$ are given by [14][3]

$$\begin{aligned} b(\theta) &= \frac{\sinh \theta/\nu}{\sinh(i\pi - \theta)/\nu} \, a(\theta) \, , \qquad c(\theta) = \frac{\sinh i\pi/\nu}{\sinh(i\pi - \theta)/\nu} \, a(\theta) \, , \\ a(\theta) &= \exp \int_0^\infty \frac{dt}{t} \, \frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \, \cosh \frac{1}{2} t} \, \sinh t \frac{\theta}{i\pi} \, . \end{aligned} \tag{2}$$

[2]The symbol $\mathcal{N}$ refers to normal products of local quantum fields.

The parameter θ is the absolute value of the rapidity difference $\theta = |\theta_1 - \theta_2|$ where θ_i are the rapidities of the particles given by the momenta $p_i = M \sinh \theta_i$. The parameter ν is related to the sine-Gordon and massive Thirring model coupling constant by

$$\nu = \frac{\beta^2}{8\pi - \beta^2} = \frac{\pi}{\pi + 2g}$$

where the second equality is due to Coleman [2].

We list some general properties of the two-particle S-matrix. As usual in this context we use in the notation

$$v^{1\ldots n} \in V^{1\ldots n} = V_1 \otimes \cdots \otimes V_n$$

for a vector in a tensor product space. The vector components are denoted by $v^{\alpha_1 \ldots \alpha_n}$. A linear operator connecting two such spaces with matrix elements $A_{\alpha_1 \ldots \alpha_n}^{\alpha'_1 \ldots \alpha'_{n'}}$ is denoted by

$$A_{1\ldots n}^{1'\ldots n'} \,:\, V^{1\ldots n} \to V^{1'\ldots n'}$$

where we omit the upper indices if they are obvious. All vector spaces V_i are isomorphic to a space V whose basis vectors are label all kinds of particles (here solitons and anti-solitons, i.e. $V \cong \mathbb{C}^2$). An S-matrix as S_{ij} acts nontrivial only on the factors $V_i \otimes V_j$.

The physical S-matrix in the formulas above is given for positive values of the rapidity parameter θ. For later convenience we will also consider an auxiliary matrix $\dot{S}(\theta_1, \theta_2)$ regarded as a function depending on the individual rapidities of both particles θ_1, θ_2 or some times also on the difference $\theta_1 - \theta_2$

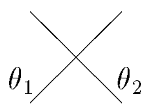
$$\dot{S}_{12}(\theta_1, \theta_2) = \dot{S}_{12}(\theta_1 - \theta_2) = \begin{cases} (\sigma S)_{12}(|\theta_1 - \theta_2|) & \text{for} \quad \theta_1 > \theta_2 \\ (S\sigma)_{21}^{-1}(|\theta_1 - \theta_2|) & \text{for} \quad \theta_1 < \theta_2 \end{cases}$$

with σ taking into account the statistics of the particles. It is a diagonal matrix σ_{12} with entries -1 if both particles are fermions and $+1$ otherwise. The matrix $\dot{S}(\theta_1, \theta_2)$ is an analytic function in terms of both variables θ_1 and θ_2. The auxiliary matrix $\dot{S}_{12}$ acts nontrivial on the factors $V_1 \otimes V_2$ and in addition exchanges these factors, e.g.

$$\dot{S}_{12}(\theta) \,:\, V_1 \otimes V_2 \to V_2 \otimes V_1\,.$$

It may be depicted as

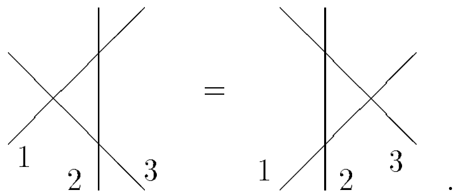
$$\dot{S}_{12}(\theta_1, \theta_2) \quad = $$



Here and in the following we associate a rapidity variable $\theta_i \in \mathbb{C}$ to each space V_i which is graphically represented by a line labeled by θ_i or simply by i. In terms of the auxiliary S-matrix the Yang-Baxter equation has the general form

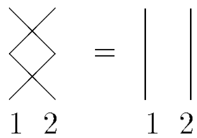
$$\dot{S}_{12}(\theta_{12})\, \dot{S}_{13}(\theta_{13})\, \dot{S}_{23}(\theta_{23}) = \dot{S}_{23}(\theta_{23})\, \dot{S}_{13}(\theta_{13})\, \dot{S}_{12}(\theta_{12})$$

which graphically simply reads

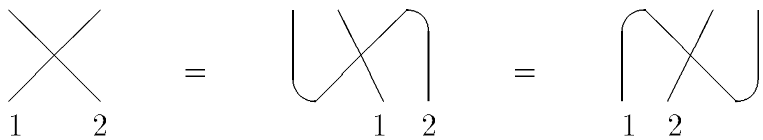


Unitarity and crossing may be written and depicted as

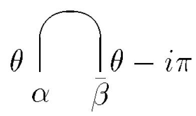
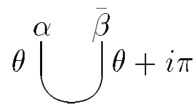
$$\dot{S}_{21}(\theta_{21})\dot{S}_{12}(\theta_{12}) = 1 \quad : \tag{3}$$



$$\dot{S}_{12}(\theta_1 - \theta_2) = \mathbf{C}^{2\bar{2}}\,\dot{S}_{\bar{2}1}(\theta_2 + i\pi - \theta_1)\,\mathbf{C}_{\bar{2}2} = \mathbf{C}^{1\bar{1}}\,\dot{S}_{2\bar{1}}(\theta_2 - (\theta_1 - i\pi)\,\mathbf{C}^{\bar{1}1} \tag{4}$$



where $\mathbf{C}^{1\bar{1}}$ and $\mathbf{C}_{1\bar{1}}$ are charge conjugation matrices. For the sine-Gordon model the matrix elements are $\mathbf{C}^{\alpha\bar{\beta}} = \mathbf{C}_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$ where $\bar{\beta}$ denotes the anti-particle of β. We have introduced the graphical rule, that a line changing the "time direction" also interchanges particles and anti-particles and changes the rapidity as $\theta \to \theta \pm i\pi$, as follows

$\mathbf{C}_{\alpha\bar{\beta}} =$  , $\mathbf{C}^{\alpha\bar{\beta}} =$  .

Similar crossing relations will be used below to investigate the properties of form factors.

Finally we denote a property of the two-particle S-matrix

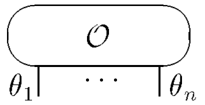
$$\dot{S}^{\delta\gamma}_{\alpha\beta}(0) = -\delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta}$$

which turns out to be true for all examples. This means that $\dot{S}$ for zero momentum difference is equal to minus the permutation operator.

2.2 Form factors

For a state of n particles of kind α_i with momenta p_i and a local operator $\mathcal{O}(x)$ we define the form factor functions $\mathcal{O}_{\alpha_1,\dots,\alpha_n}(\theta_1,\dots,\theta_n)$ by eq. (1) on page 2 for the specific order of the rapidities $\theta_1 > \cdots > \theta_n$. For all other arrangements of the rapidities the functions $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ are given by analytic continuation. Note that in general this analytic continuation does <u>not</u> provide the physical values of the form factor. These are given for ordered rapidities as indicated above and for other orders by the statistics of the particles, of course. The $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ are considered as the components of a co-vector valued function $\mathcal{O}_{1\dots n}(\underline{\theta}) \in V_{1\dots n} = (V^{1\dots n})^\dagger$which may be depicted as

$$\mathcal{O}_{1\dots n}(\underline{\theta}) =$$



Now we formulate the main properties of form factors in terms of the functions $\mathcal{O}_{1\dots n}(\underline{\theta})$ which follow from general LSZ-assumptions and "maximal analyticity". The later condition means that $\mathcal{O}_{1\dots n}(\underline{\theta})$ is a meromorphic function with respect to all θ's and all poles in the 'physical' strips $0 < \operatorname{Im}\theta_{ij} < \pi$ ($\theta_{ij} = \theta_i - \theta_j \; i < j$) are of physical origin, as for example bound state poles.

Properties: The co-vector valued function $\mathcal{O}_{1\dots n}(\underline{\theta})$ is meromorphic with respect to all variables $\theta_1, \dots, \theta_n$ and

(i) it satisfies the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces i, j at the same time

$$\mathcal{O}_{\dots ij\dots}(\dots, \theta_i, \theta_j, \dots) = \mathcal{O}_{\dots ji\dots}(\dots, \theta_j, \theta_i, \dots)\, \dot{S}_{ij}(\theta_i - \theta_j)$$

for all possible arrangements of the θ's,

(ii) it satisfies the periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\mathcal{O}_{1\dots n}(\theta_1, \theta_2, \dots, \theta_n,) = \mathcal{O}_{2\dots n1}(\theta_2, \dots, \theta_n, \theta_1 - 2\pi i)\sigma_{\mathcal{O}1}$$

(iii) and it has poles determined by one-particle states in each sub-channel. In particular the function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} \mathcal{O}_{1\dots n}(\theta_1, \dots, \theta_n) = 2i\, \mathbf{C}_{12}\, \mathcal{O}_{3\dots n}(\theta_3, \dots, \theta_n)\, (\mathbf{1} - S_{2n} \dots S_{23})$$

where $\mathbf{C}_{12}$ is the charge conjugation matrix.

(iv) If the model also possesses bound states, the function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = iu_{12}^{(12)}$ $(0 < u_{12}^{(12)} < \pi)$ such that

$$\operatorname*{Res}_{\theta_{12}=iu_{12}^{(12)}} \mathcal{O}_{12\dots n}(\theta_1, \theta_2, \dots, \theta_n) = \mathcal{O}_{(12)\dots n}(\theta_{(12)}, \dots, \theta_n)\, \sqrt{2}\Gamma_{12}^{(12)}$$

where the bound state intertwiner $\Gamma_{12}^{(12)}$ and the relations of the rapidities $\theta_1, \theta_2, \theta_{(12)}$ and the fusion angle $u_{12}^{(12)}$ [11].

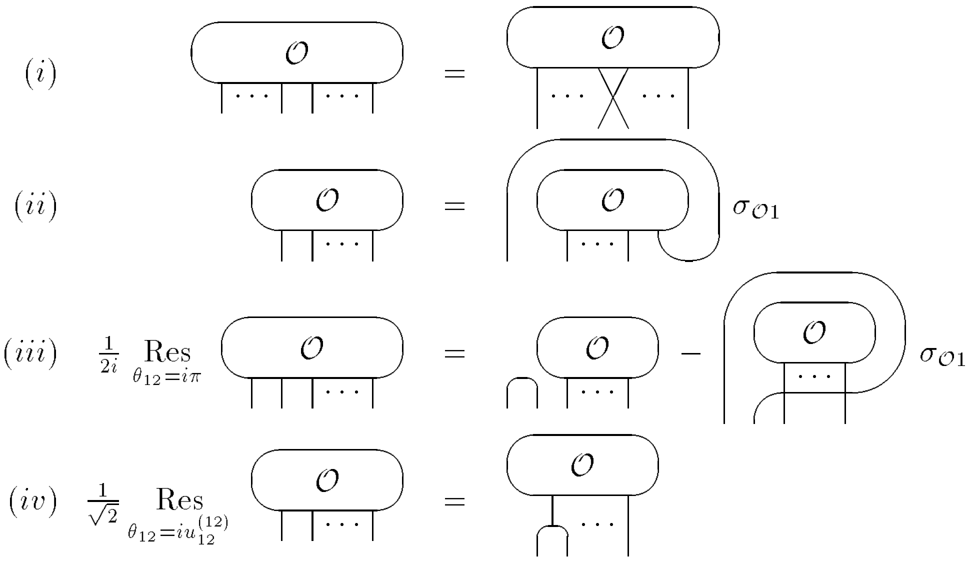
(v) Naturally, since we are dealing with relativistic quantum field theories we finally have Lorentz covariance

$$\mathcal{O}_{1\ldots n}(\theta_1+\mu,\ldots,\theta_n+\mu)=e^{s\mu}\,\mathcal{O}_{1\ldots n}(\theta_1,\ldots,\theta_n)$$

if the local operator transforms as $\mathcal{O}\to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

In the formulae (i) the statistics of the particles is taken into account by $\dot{S}$ which means that $\dot{S}_{12}=-S_{12}$ if both particles are fermions and $\dot{S}_{12}=S_{12}$ otherwise. In (ii) the statistics of the operator $\mathcal{O}$ is taken into account by $\sigma_{\mathcal{O}1}=-1$ if both the operator $\mathcal{O}$ and particle 1 are fermionic and $\sigma_{\mathcal{O}1}=1$ otherwise.

The property $(i)-(iv)$ may be depicted as

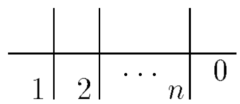


We will now provide a constructive and systematic way of how to solve the properties $(i)-(v)$ for the co-vector valued function f once the scattering matrix is given. These solutions are candidates of form factors. To capture the vectorial structure of the form factors we will employ the techniques of the algebraic Bethe Ansatz which we now explain briefly.

2.3 The 'off-shell' Bethe Ansatz co-vectors

As usual in the context of algebraic Bethe Ansatz we define the monodromy matrix as

$$T_{1\ldots n,0}(\underline{\theta},\theta_0)=\dot{S}_{10}(\theta_1-\theta_0)\,\dot{S}_{20}(\theta_2-\theta_0)\cdots\dot{S}_{n0}(\theta_n-\theta_0)=$$

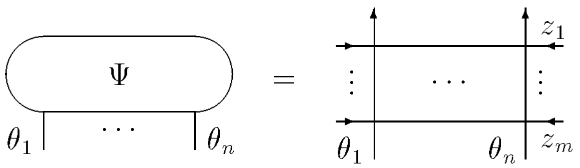


It is a matrix acting in the tensor product of the "quantum space" $V^{1\ldots n}=V_1\otimes\cdots\otimes V_n$ and the "auxiliary space" V_0 (all $V_i\cong\mathbb{C}^2=$ soliton-anti-soliton space). The sub-matrices A,B,C,D with respect to the auxiliary space are defined by

$$T_{1\ldots n,0}(\underline{\theta},z)\equiv\begin{pmatrix}A_{1\ldots n}(\underline{\theta},z) & B_{1\ldots n}(\underline{\theta},z)\\ C_{1\ldots n}(\underline{\theta},z) & D_{1\ldots n}(\underline{\theta},z)\end{pmatrix}.$$

A Bethe Ansatz co-vector in $V_{1\dots n}$ is given by

$$\Psi_{1\dots n}(\underline{\theta},\underline{z}) \quad = \quad \Omega_{1\dots n} C_{1\dots n}(\underline{\theta},z_1)\cdots C_{1\dots n}(\underline{\theta},z_m)$$



(5)

where $\underline{z}=(z_1,\dots,z_m)$. Usually one has the restriction $2m\le n$ and the charge of the state is $q=n-2m=$ number of solitons minus number of anti-solitons. The solitons are depicted by $\uparrow$ or $\leftarrow$ and anti-solitons by $\downarrow$ or $\rightarrow$. The co-vector $\Omega_{1\dots n}$ is the "pseudo-vacuum" consisting only of solitons (highest weight states)

$$\Omega_{1\dots n}=\uparrow\otimes\cdots\otimes\uparrow\ .$$

It satisfies

$$\begin{aligned}
\Omega_{1\dots n}\,B_{1\dots n}(\underline{\theta},z) &= 0\\
\Omega_{1\dots n}\,A_{1\dots n}(\underline{\theta},z) &= \prod_{i=1}^n \dot{a}(\theta_i-z)\Omega_{1\dots n}\\
\Omega_{1\dots n}\,D_{1\dots n}(\underline{\theta},z) &= \prod_{i=1}^n \dot{b}(\theta_i-z)\Omega_{1\dots n}\,.
\end{aligned}$$

The eigenvalues of the matrices A and D, i.e. the functions $\dot{a}=-a$ and $\dot{b}=-b$ are given by the amplitudes of the scattering matrix (2). In the following we use the co-vector $\Psi_{1\dots n}(\underline{\theta},\underline{z})$ in its 'off-shell' version which means that we do not fix the parameters $\underline{z}$ by means of Bethe Ansatz equations but we integrate over the z's.

3 The general form factor formula

In this section we present our main result. We derive a general formula in terms of an integral representation which allows to construct form factors i.e. matrix elements of local fields as given by eq. (1). More precisely, we construct co-vector valued functions which satisfy the properties $(i)-(v)$ on page 6.

As a candidate of a generalized form factor of a local operator $\mathcal{O}(0)$ we make the following Ansatz for the co-vector valued function

$$\mathcal{O}_{1\dots n}(\underline{\theta})=\int_{\mathcal{C}_{\underline{\theta}}}dz_1\cdots\int_{\mathcal{C}_{\underline{\theta}}}dz_m\,h(\underline{\theta},\underline{z})\,p^{\mathcal{O}}(\underline{\theta},\underline{z})\,\Psi_{1\dots n}(\underline{\theta},\underline{z}) \tag{6}$$

with the Bethe Ansatz state $\Psi_{1\dots n}(\underline{\theta},\underline{z})$ defined by eq. (5). For all integration variables z_j $(j=1,\dots,m)$ the integration contours $\mathcal{C}_{\underline{\theta}}$ consists of several pieces (see figure 1):

a) A line from $-\infty$ to ∞ avoiding all poles such that $\operatorname{Im}\theta_i-\pi-\epsilon<\operatorname{Im}z_j<\operatorname{Im}\theta_i-\pi$.

b) Clock wise oriented circles around the poles (of the $\phi(\theta_i-z_j)$) at $z_j=\theta_i$ $(i=1,\dots,n)$.

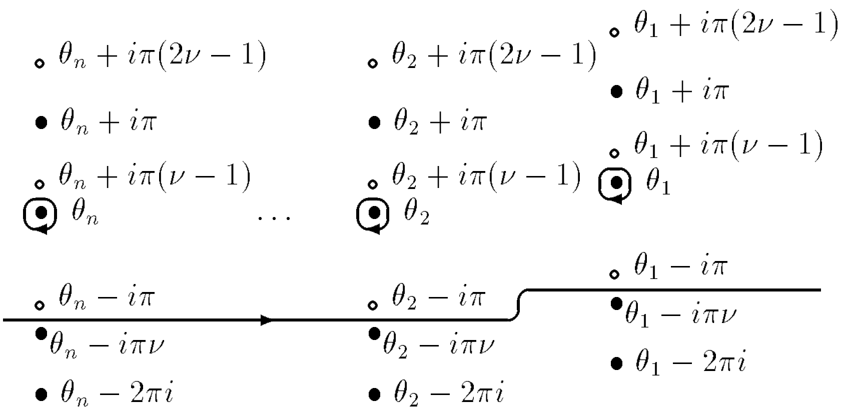


Figure 1: *The integration contour $\mathcal{C}_{\underline{\theta}}$ (for the repulsive case $\nu > 1$). The bullets belong to poles of the integrand resulting from $u(\theta_i - u_j)\,\phi(\theta_i - u_j)$ and the small open circles belong to poles originating from $t(\theta_i - u_j)$ and $r(\theta_i - u_j)$.*

Let the scalar function (c.f. [1])

$$h(\underline{\theta}, \underline{z}) = \prod_{1 \le i < j \le n} F(\theta_{ij}) \prod_{i=1}^{n} \prod_{j=1}^{m} \phi(\theta_i - z_j) \prod_{1 \le i < j \le m} \tau(z_i - z_j)\,, \tag{7}$$

be given by

$$\tau(z) = \frac{1}{\phi(z)\,\phi(-z)}\,, \qquad \phi(z) = \frac{1}{F(z)\,F(z + i\pi)} \tag{8}$$

and

$$\begin{aligned} F(\theta) &= \sin \frac{1}{2i}\theta \; f_{ss}^{(0)}(\theta) \\ f_{ss}^{(0)}(\theta) &= \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \, \cosh \frac{1}{2}t} \frac{1 - \cosh t(1 - \theta/(i\pi))}{2 \sinh t}. \end{aligned} \tag{9}$$

The function $F(\theta)$ is the soliton-soliton form factor fulfilling Watson's equations

$$F(\theta) = F(-\theta)\,\dot{a}(\theta) = F(2\pi i - \theta) \tag{10}$$

with $\dot{a}(\theta) = -a(\theta)$ where $a(\theta)$ is the soliton-soliton scattering amplitude.

Remarks:

- Using Watson's equations (10) for $F(z)$, crossing (4) and unitarity (3) for the sine-Gordon amplitudes one derives the following identities for the scalar functions $\phi(z)$ and $\tau(z)$ from the definitions (8)

$$\phi(z) = \phi(i\pi - z) = \frac{1}{\dot{b}(z)}\,\phi(z - i\pi) = \frac{\dot{a}(z - 2\pi i)}{\dot{b}(z)}\,\phi(z - 2\pi i)\,,$$

$$\tau(z)=\tau(-z)=\frac{b(z)}{a(z)}\frac{a(2\pi i-z)}{b(2\pi i-z)}\tau(z-2\pi i)$$

where $b(z)$ is the soliton-anti-soliton scattering amplitude related to $a(z)$ by crossing $b(z)=a(i\pi-z)$.

- The functions $\phi(z)$ and $\tau(z)$ are of the form

$$\phi(z)=\frac{const}{\sinh z}\exp\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{1}{2}(1-\nu)t\,\left(\cosh t(\frac{1}{2}-z/(i\pi))-1\right)}{\sinh\frac{1}{2}\nu t\sinh t}$$

$$\tau(z)=const.\ \sinh z\sinh z/\nu$$

- The function $h(\underline{\theta},\underline{z})$ and the state $\Psi_{1\ldots n}(\underline{\theta},\underline{z})$ are completely determined by the S-matrix.

In contrast to the functions $F(z),\phi(z)$ and $\tau(z)$ the 'p-function' $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ in the integral representation (6) depends on the local operator $\mathcal{O}(x)$,in particular on the spin, the charge and the statistics. The number of the particles n and the number of integrations m are related by $q=n-2m$ where q is the charge of the operator $\mathcal{O}(x)$. The p-functions is an entire function in the z_j $(j=1,\ldots,m)$ and in order that the form factor satisfies the properties $(i)-(v)$ it has to satisfy the following

Conditions: The p-function $p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$ (where n is the number of particles and the number of variables θ) satisfies

(i') $p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$ is symmetric with respect to the θ's and the z's.

(ii') $p_n^{\mathcal{O}}(\underline{\theta},\underline{z})=\sigma_{\mathcal{O}i}p_n^{\mathcal{O}}(\ldots,\theta_i-2\pi i,\ldots,\underline{z})$ and it is a polynomial in $e^{\pm z_j}$ $(j=1,\ldots,m)$. The statistics factor $\sigma_{\mathcal{O}i}$ is -1 if the operator $\mathcal{O}(x)$ and the particle i are both fermionic and $+1$ otherwise.

(iii')
$$\begin{cases} p_n^{\mathcal{O}}(\theta_1=\theta_n+i\pi,\underline{\tilde\theta},\theta_n;\underline{\tilde z},z_m=\theta_n)=\dfrac{\varkappa}{m}\,p_{n-2}^{\mathcal{O}}(\underline{\tilde\theta},\underline{\tilde z})+\tilde p^{(1)}(\underline{\theta}) \\ p_n^{\mathcal{O}}(\theta_1=\theta_n+i\pi,\underline{\tilde\theta},\theta_n;\underline{\tilde z},z_m=\theta_1)=\sigma_{\mathcal{O}1}\dfrac{\varkappa}{m}\,p_{n-2}^{\mathcal{O}}(\underline{\tilde\theta},\underline{\tilde z})+\tilde p^{(2)}(\underline{\theta})\end{cases}$$

where $\underline{\tilde\theta}=(\theta_2,\ldots,\theta_{n-1})$, $\underline{\tilde z}=(z_1,\ldots z_{m-1})$ and where $\tilde p^{(1,2)}(\underline{\theta})$ are independent of the z's and non-vanishing only for charge less operators $\mathcal{O}(x)$. The constant $\varkappa$ depends on the coupling and is given by (see formula (B.7) in [1])

$$\varkappa=\frac{\left(f_{ss}^{(0)}(0)\right)^2}{4\pi}.$$

(iv') the bound state p-functions are investigated in section 5

(v') $p_n^{\mathcal{O}}(\underline{\theta}+\mu,\underline{z}+\mu)=e^{s\mu}p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$ where s is the 'spin' of the operator $\mathcal{O}(x)$.

As an extension of theorem 4.1 in [1] we prove the following theorem which allow to construct generalized form factors.

Theorem 1 *The co-vector valued function $\mathcal{O}_{1\ldots n}(\underline{\theta})$ defined by (6) fulfills the properties (i), (ii) and (iii) on page 6 if the functions $F(\theta), \phi(z)$ and $\tau(z)$ are given by definition (7) – (9) and if the p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ satisfies the conditions $(i') - (iii')$.*

This theorem is proven in [11].

Remarks:

- The number of C-operators m depends on the charge $q = n - 2m$ of the operator $\mathcal{O}$, e.g. $m = (n-1)/2$ for the soliton field $\psi(x)$ with charge $q = 1$ and $m = n/2$ for charge-less operators like $\overline{\psi}\psi$ or the energy momentum tensor $T^{\mu\nu}$.
- Note that other sine-Gordon form factors can be calculated from the general formula (6) using the bound state formula (iv).
- The general representation of form factors by formula (6) is not specific to the sine-Gordon model. It may be applied to all integrable quantum field theoretic model. The main difficulty is to solve the corresponding Bethe Ansatz.

4 Examples of "p-functions"

In this section we propose the p-functions for various local operators. Since the charge of the operators which we consider is zero the number of integrations m and the number of particles n are related by $m = n/2$ and form factors are non-vanishing only for even number of particles $n = 2, 4, \ldots$. We consider p-functions of the form

$$p_n^{\mathcal{O}}(\underline{\theta}, \underline{z}) = N_n^{\mathcal{O}} \left(p_+^{\mathcal{O}}(P^\mu) \sum_{j=1}^{m} e^{Lz_j} + p_-^{\mathcal{O}}(P^\mu) \sum_{j=1}^{m} e^{-Lz_j} \right) \tag{11}$$

where P^μ is the total energy momentum vector of all particles. The integrals in (6) converge for $L < (1/\nu + 1)(n/2 - m + 1) + 1/\nu$. For large values of L the form factors are defined in general as the analytic continuations of the integral representation from sufficiently small values of ν to other values. Obviously the p-functions (11) satisfy the conditions $(i') - (iii')$ on page 10. From the property (iii') we obtain the recursion relation for the normalization constants

$$N_n^{\mathcal{O}} = N_{n-2}^{\mathcal{O}} \frac{\varkappa}{m} \quad \Rightarrow \quad N_n^{\mathcal{O}} = N_2^{\mathcal{O}} \frac{1}{m!} \varkappa^{m-1}$$

where $N_2^{\mathcal{O}}$ follows from the two-particle form factors. For the local operators $\mathcal{N}\left[\overline{\psi}\psi\right](x)$, $\mathcal{N}\left[\overline{\psi}\gamma^5\psi\right](x)$, the current $j^\mu(x) = \mathcal{N}\left[\overline{\psi}\gamma^\mu\psi\right](x)$, the energy momentum tensor $T^{\mu\nu}(x) =$

$\frac{i}{2}\mathcal{N}\left[\overline{\psi}\gamma^\mu\overleftrightarrow{\partial^\nu}\psi\right](x) - g^{\mu\nu}\mathcal{L}^{MT}$ and the infinitely many higher conserved currents $J_L^\mu(x)$ we propose the following p-functions

$$\begin{aligned}
p^{\overline{\psi}\psi}(\underline{\theta},\underline{z}) &= N_n^{\overline{\psi}\psi}\left(\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^m e^{z_i} - \sum_{i=1}^n e^{\theta_i}\sum_{i=1}^m e^{-z_i}\right)\\
p^{\overline{\psi}\gamma^5\psi}(\underline{\theta},\underline{z}) &= N_n^{\overline{\psi}\gamma^5\psi}\left(\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^m e^{z_i} + \sum_{i=1}^n e^{\theta_i}\sum_{i=1}^m e^{-z_i}\right)\\
p^{j^\pm}(\underline{\theta},\underline{z}) &= \pm N_n^j\left(\sum_{i=1}^n e^{\mp\theta_i}\right)^{-1}\left(\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^m e^{z_i} + \sum_{i=1}^n e^{\theta_i}\sum_{i=1}^m e^{-z_i}\right)\\
p^{T^{\pm\pm}}(\underline{\theta},\underline{z}) &= N_n^T\sum_{i=1}^n e^{\pm\theta_i}\left(\sum_{i=1}^n e^{\mp\theta_i}\right)^{-1}\left(\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^m e^{z_i} - \sum_{i=1}^n e^{\theta_i}\sum_{i=1}^m e^{-z_i}\right)\\
p^{T^{+-}}(\underline{\theta},\underline{z}) &= -N_n^T\left(\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^m e^{z_i} - \sum_{i=1}^n e^{\theta_i}\sum_{i=1}^m e^{-z_i}\right)\\
p^{J_L^\pm}(\underline{\theta},\underline{z}) &= \pm N_n^{J_L}\sum_{i=1}^n e^{\pm\theta_i}\sum_{i=1}^m e^{Lz_i}\,,\quad (L=\pm1,\pm3,\dots)\,.
\end{aligned}$$

The identification with the operators has been made by comparing the exact results with Feynman graph expansions. Properties as charge, behavior under Lorentz transformations etc. also become obvious [11] The fundamental sine-Gordon bose field $\varphi(x)$ which correspond to the lowest breather is related to the current by Coleman's formula [2]

$$\epsilon^{\mu\nu}\partial_\nu\varphi = -\frac{2\pi}{\beta}j^\mu \quad \text{or} \quad \partial^\pm\varphi = \pm\frac{2\pi}{\beta}j^\pm.$$

This implies for the p-function

$$p_n^\varphi(\underline{\theta},\underline{z}) = N_n^j\frac{2\pi i}{\beta M}\left(\sum_{i=1}^n e^{\theta}\sum_{i=1}^n e^{-\theta}\right)^{-1}\left(\sum_{i=1}^n e^{-\theta}\sum_{i=1}^m e^{z} + \sum_{i=1}^n e^{\theta}\sum_{i=1}^m e^{-z}\right).$$

5 Soliton Breather form factors

We calculate breather form factors starting with the general formula (6) for the soliton form factors using the property (vi) on page 6 (for details see [11, 13]). The b_k-breather-$(n-2)$-soliton form factor is obtained from $\mathcal{O}_{123\dots n}(\underline{\theta})$ by means of the fusion procedure (iv) with the fusion angle given by $u_{12}^{(12)} = u^{(k)} = \pi(1-k\nu)$ the bound state rapidity $\xi = \theta_{(12)} = \frac{1}{2}(\theta_1+\theta_2)$ and $\underline{\theta}' = \theta_3,\dots,\theta_n$

$$\operatorname*{Res}_{\theta_{12}=iu^{(k)}}\mathcal{O}_{123\dots n}(\underline{\theta}) = \mathcal{O}_{(12)3\dots n}(\xi,\underline{\theta}')\sqrt{2}\Gamma_{12}^{(12)}(iu^{(k)})$$

where the bound state intertwiner [11].

For $\theta_{12} \to iu^{(k)}$ there will be pinchings of the integration contours in formula (6) at the poles $z_i = z^{(l)} = \theta_2 - i\pi l\nu = \xi - \frac{1}{2}i\pi(1 - k\nu + 2l\nu)$ for $l = 0, \ldots, k$ and $i = 1, \ldots, m$. Using the pinching rule of contour integrals and the symmetry with respect to the m z-integrations we obtain

$$\operatorname*{Res}_{\theta_{12}=iu^{(k)}} \mathcal{O}_{123\ldots n}(\underline{\theta}) = \operatorname*{Res}_{\theta_{12}=iu^{(k)}} (-2\pi i)\, m \sum_{l=0}^{k} \operatorname*{Res}_{z_1=z^{(l)}} \int_{\mathcal{C}_{\underline{\theta}}} dz_2 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \\ \times h(\underline{\theta}, \underline{z}) p^{\mathcal{O}}(\underline{\theta}, \underline{z})\, \Psi_{1\ldots n}(\underline{\theta}, \underline{z}).$$

After a lengthy calculation [11] we obtain for the case of the lowest breather the one-breather-$(n-2)$-soliton form factor

$$\mathcal{O}_{3\ldots n}(\xi, \underline{\theta}') = \prod_{2<i} F_{sb}(\xi - \theta_i) \prod_{2<i<j} F(\theta_{ij}) \sum_{l=0}^{1} (-1)^l \prod_{2<i} \rho(\xi - \theta_i, l) \\ \times \int_{\mathcal{C}_{\underline{\theta}}} dz_2 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \prod_{1<j} \chi(\xi - z_j, l) \\ \times \prod_{2<i} \prod_{1<j} \phi(\theta_i - z_j) \prod_{1<i<j} \tau(z_{ij})\, \tilde{p}^{\mathcal{O}}(\xi, \underline{\theta}', z^{(l)}, \underline{z}')\, \Psi_{3\ldots n}(\underline{\theta}', \underline{z}')$$

with $\underline{z}' = (z_2, \ldots, z_m)$. The soliton-breather form factor has been introduced as

$$\begin{aligned} F_{sb}(\theta) &= K_{sb}(\theta) \sin \tfrac{1}{2i}\theta\, f_{sb}^{(0)}(\theta) \\ f_{sb}^{(0)}(\theta) &= \exp \int_0^\infty \frac{dt}{t}\, 2 \frac{\cosh \frac{1}{2}\nu t}{\cosh \frac{1}{2}t} \frac{1 - \cosh t(1 - \theta/(i\pi))}{2 \sinh t} \\ K_{sb}(\theta) &= \frac{-\cos \frac{\pi}{4}(1-\nu)/E(\frac{1}{2}(1-\nu))}{\sinh \frac{1}{2}(\theta - \frac{i\pi}{2}(1+\nu)) \sinh \frac{1}{2}(\theta + \frac{i\pi}{2}(1+\nu))} \end{aligned}$$

The normalization has been chosen such that $F_{sb}(\infty) = 1$. The function $E(\nu)$ was used in [5, 1] Also we have introduced the short notations

$$\begin{aligned} \rho(\xi, l) &= (-1)^l \frac{\sinh \frac{1}{2}\left(\xi - \frac{i\pi}{2}(1 + (-1)^l \nu)\right)}{\sinh \frac{1}{2}\xi} \\ \chi(\xi, l) &= (-1)^l \frac{\sinh \frac{1}{2}(\xi + \frac{i\pi}{2}(1 + (-1)^l \nu))}{\sinh \frac{1}{2}(\xi - \frac{i\pi}{2}(1 + (-1)^l \nu))} \end{aligned}$$

The following identities have been used

$$\begin{aligned} F(\theta_1 - \theta_i) F(\theta_2 - \theta_i) \tilde{\phi}(\theta_i - z^{(l)}) &= F_{sb}(\xi - \theta_i) \rho(\xi - \theta_i, l) \\ \phi(\theta_1 - z_j) \phi(\theta_2 - z_j) S_{sb}(\xi - z_j) \tau(z^{(l)} - z_j) &= \chi(\xi - z_j, l) \end{aligned}$$

for $\theta_{1/2} = \xi \pm \frac{i\pi}{2}(1-\nu)$, $z^{(l)} = \xi - \frac{i\pi}{2}(1 - (-1)^l \nu)$. The new p-function is obtained from the old one by

$$\tilde{p}^{\mathcal{O}}(\xi, \underline{\theta}', z^{(l)}, \underline{z}') = m\, d(\nu)\, p^{\mathcal{O}}(\xi + \tfrac{1}{2}iu^{(1)}, \xi - \tfrac{1}{2}iu^{(1)}, \underline{\theta}', z^{(l)}, \underline{z}'\,)$$

where the constant $d(\nu)$ is given by

$$d(\nu) = \frac{\sqrt{E(\nu)}}{\varkappa\sqrt{\sin\frac{1}{2}\pi\nu}}$$

Iterating the procedure above we obtain the r-breather-s-soliton form factor with $2r+s=n$, the breather rapidities $\underline{\xi} = (\xi_1,\dots,\xi_r)$ and the soliton rapidities $\underline{\theta} = (\theta_1,\dots,\theta_s)$

$$\begin{aligned}
\mathcal{O}_{1\dots s}(\underline{\xi},\underline{\theta}) &= \prod_{1\le i<j\le r} F_{bb}(\xi_{ij}) \prod_{i=1}^{r}\prod_{j=1}^{s} F_{sb}(\xi_i-\theta_j) \prod_{1\le i<j\le s} F(\theta_{ij}) \\
&\times \sum_{l_1=0}^{1}\cdots\sum_{l_r=0}^{1}(-1)^{l_1+\dots+l_r}\prod_{1\le i<j\le r}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\xi_{ij}}\right) \\
&\times \prod_{i=1}^{r}\prod_{j=1}^{s}\rho(\xi_i-\theta_j,l)\int dz_{r+1}\cdots\int dz_m \prod_{i=1}^{r}\prod_{j=r+1}^{m}\chi(\xi_i-z_j,l) \\
&\times \prod_{i=1}^{s}\prod_{j=r+1}^{m}\phi(\theta_i-z_j)\prod_{r<i<j\le m}\tau(z_{ij})\,\tilde p(\underline{\xi},\underline{\theta},\underline{z^{(l)}},\underline{z})\,\Psi_{1\dots s}(\underline{\theta},\underline{z})
\end{aligned}$$

again with $z_i^{(l_i)} = \xi_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu), (i=1,\dots,r)$. The two-breather form factor has been introduced as

$$\begin{aligned}
F_{bb}(\xi) &= K_{bb}(\xi)\,\sin\tfrac{1}{2i}\xi\; f_{bb}^{(0)}(\xi) \\
f_{bb}^{(0)}(\theta) &= \exp\int_0^\infty \frac{dt}{t}\,2\frac{\cosh(\frac{1}{2}-\nu)t}{\cosh\frac{1}{2}t}\,\frac{1-\cosh t(1-\theta/(i\pi))}{2\sinh t} \\
K_{bb}(\theta) &= \frac{-\cos\frac{1}{2}\pi\nu/E(\nu)}{\sinh\frac{1}{2}(\theta-i\pi\nu)\sinh\frac{1}{2}(\theta+i\pi\nu)}
\end{aligned}$$

The normalization has been chosen such that $F_{bb}(\infty)=1$. It has been used that

$$\begin{aligned}
&F_{sb}(\xi_1-\theta_3)F_{sb}(\xi_1-\theta_4)\,\rho(\xi_1-\theta_3,l_1)\rho(\xi_1-\theta_4,l_1)\chi(\xi_1-z_2^{(l_2)},l_1) \\
&\qquad = F_{bb}(\xi_{12})\left(1+(l_1-l_2)\frac{i\sin\pi\nu}{\sinh\xi_{12}}\right)
\end{aligned}$$

for $\theta_{3/4} = \xi_2 \pm \frac{i\pi}{2}(1-\nu)$. The new p-function is obtained from the old one by

$$\tilde p(\underline{\xi},\underline{\theta}'',\underline{z^{(l)}},\underline{z}'') = \binom{m}{r} r!\,d^r(\nu)\,p\left(\xi_1+\tfrac{1}{2}\theta^{(1)},\xi_1-\tfrac{1}{2}\theta^{(1)},\dots,\underline{\theta}'',z_1^{(l_1)},\dots,\underline{z}''\right).$$

In particular for $n=2r=2m$ we get the pure lowest breather form factor

$$\mathcal{O}(\underline{\xi}) = \prod_{i<j}F_{bb}(\xi_{ij})\sum_{l_1=0}^{1}\cdots\sum_{l_r=0}^{1}(-1)^{l_1+\dots+l_r}\prod_{1=i<j}^{r}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\xi_{ij}}\right)\tilde p(\underline{\xi},\underline{z^{(l)}}) \qquad (12)$$

and the pure breather p-function

$$\tilde p(\underline{\xi},\underline{z^{(l)}}) = r!\,d^r(\nu)\,p\left(\xi_1+\tfrac{1}{2}iu^{(1)},\xi_1-\tfrac{1}{2}iu^{(1)},\dots,z_1^{(l_1)},\dots\right).$$

6 The quantum sine-Gordon field equation

The classical sine-Gordon model is given by the wave equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0.$$

and two particle sine-Gordon S-matrix for the scattering of fundamental bosons (lowest breathers) [15]

$$S(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}$$

where θ is the rapidity difference defined by $p_1 p_2 = m^2\cosh\theta$ and ν is related to the coupling constant by $\nu = \beta^2/(8\pi - \beta^2)$.

From the S-matrix off-shell quantities as arbitrary matrix elements of local operators are obtained by means of the “form factor program” [5]. In particular we provide exact expressions for all matrix elements of all powers of the fundamental bose field $\varphi(t,x)$ and its exponential $\mathcal{N}\exp i\gamma\varphi(t,x)$ for arbitrary γ. Here and in the following $\mathcal{N}$ denotes normal ordering with respect to the physical vacuum which means in particular for the vacuum expectation value $\langle\, 0\,|\,\mathcal{N}\exp i\gamma\varphi(t,x)|\,0\,\rangle = 1$. For the exceptional value $\gamma = \beta$ we find that the operator $\Box^{-1}\mathcal{N}\sin\beta\varphi(t,x)$ is local. Moreover the quantum sine-Gordon field equation[3]

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\mathcal{N}\sin\beta\varphi(t,x) = 0 \tag{13}$$

is fulfilled for all matrix elements, if the “bare” mass $\sqrt{\alpha}$ is related to the renormalized mass by[4]

$$\alpha = m^2\frac{\pi\nu}{\sin\pi\nu} \tag{14}$$

where m is the physical mass of the fundamental boson. The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction. For the sinh-Gordon model an analogous quantum field equation has been obtained in [29][5]. Note that in particular at the ‘free fermion point’ $\nu \to 1$ ($\beta^2 \to 4\pi$) this factor diverges, a phenomenon which is to be expected by short distance investigations [18]. For fixed bare mass square α and $\nu \to 2,3,4,\ldots$ the physical mass goes to zero. These values of the coupling are known to be specific: 1. the Bethe Ansatz vacuum in the language of the massive Thirring model shows phase transitions [19] and 2. the model at these points is related [20, 21, 22] to Baxters RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$ (see also [29]).

[3]In the framework of constructive quantum field theory quantum field equations where considered in [16, 17].

[4]Before such a formula was found in [25, 26].

[5]It should be obtained from (1) by the replacement $\beta \to ig$. However the relation between the bare and the renormalized mass in [29] differs from the analytic continuation of (14) by a factor which is $1 + O(\beta^4) \neq 1$.

Also we calculate all matrix elements of all higher local currents $J^\mu_M(t,x)$ ($M = \pm1, \pm3, \dots$) fulfilling $\partial_\mu J^\mu_M(t,x) = 0$ which is characteristic for integrable models. The higher charges fulfill the eigenvalue equation

$$\left(\int dx J^0_M(x) - \sum_{i=1}^n \left(p_i^+\right)^M\right) |\, p_1, \dots, p_n\rangle^{in} = 0. \tag{15}$$

In particular for $M = \pm1$ the currents yield the energy momentum tensor $T^{\mu\nu} = T^{\nu\mu}$ with $\partial_\mu T^{\mu\nu} = 0$. We find that its trace fulfills

$$T^\mu_{\ \mu}(t,x) = -2\frac{\alpha}{\beta^2}\left(1 - \frac{\beta^2}{8\pi}\right)\left(\mathcal{N}\cos\beta\varphi(t,x) - 1\right). \tag{16}$$

This formula is consistent with renormalization group arguments [14, 23]. In particular this means that $\beta^2/4\pi$ is the anomalous dimension of $\cos\beta\varphi$. Again this operator equation is modified by a quantum correction $(1 - \beta^2/8\pi)$. Obviously for fixed bare mass square α and $\beta^2 \to 8\pi$ the model will be conformal invariant which is related to a Berezinski-Kosterlitz-Thouless phase transition [24]. The proofs of the statements (13) – (16) is sketched in the following together with some checks in perturbation theory. The complete proofs will be published elsewhere [11].

A form factor of n fundamental bosons (lowest breathers) is of the form [5]

$$f_n^{\mathcal{O}}(\underline{\theta}) = N_n^{\mathcal{O}} K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij})$$

where $N_n^{\mathcal{O}}$ is a normalization constant, $\theta_{ij} = \theta_i - \theta_j$ and $F(\theta)$ is the two particle form factor function. It fulfills Watson's equations

$$F(\theta) = F(-\theta)S(\theta) = F(2\pi i - \theta)$$

with the S-matrix given above. Explicitly it is given by the integral representation [5]

$$F(\theta) = N \exp\int_0^\infty \frac{dt}{t}\, \frac{\left(\cosh\frac{1}{2}t - \cosh(\frac{1}{2}+\nu)t\right)\left(1 - \cosh t(1 - \frac{\theta}{i\pi})\right)}{\cosh\frac{1}{2}t \sinh t}$$

normalized such that $F(\infty) = 1$. The K-function $K_n^{\mathcal{O}}(\underline{\theta})$ is meromorphic, symmetric and periodic (under $\theta_i \to \theta_i + 2\pi i$).

From (12) we have

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1}\cdots\sum_{l_n=0}^{1}(-1)^{l_1+\dots+l_n} \prod_{1\le i<j\le n}\left(1 + (l_i - l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta}, \underline{z}) \tag{17}$$

where $z_i = \theta_i - \frac{i\pi}{2}(1 + (2l_i - 1)\nu)$. The dependence on the operator is encoded in the 'p-function' $p_n^{\mathcal{O}}$. It is separately symmetric with respect to the variables $\underline{\theta}$ and $\underline{z}$ and has to fulfill some simple conditions in order that the form factor function $f_n^{\mathcal{O}}$ fulfill some properties [5, 6]. These properties follow (see [1]) from general LSZ-assumptions and in additions specific features typical for integrable field theories. In particular the recursion relation holds

$$\operatorname*{Res}_{\theta_{12}=i\pi} f_n^{\mathcal{O}}(\theta_1,\dots,\theta_n) = 2i\, f_{n-2}^{\mathcal{O}}(\theta_3,\dots,\theta_n)\,(\mathbf{1} - S(\theta_{2n})\dots S(\theta_{23}))\,. \tag{18}$$

Here we will not provide more details but only give some examples of operators and their corresponding p-functions:

1. The correspondence of exponentials of the field and their p-function[6] is

$$\mathcal{N} e^{i\gamma\varphi} \leftrightarrow \prod_{i=1}^{n} e^{(2l_i-1)i\pi\nu\gamma/\beta} \tag{19}$$

   for an arbitrary constant γ.

2. Taking derivatives of this formula with respect to γ we get for the field and its powers

$$\mathcal{N}\varphi^N \leftrightarrow \left(\sum_{i=1}^{r}(2l_i-1)\right)^N . \tag{20}$$

3. Higher currents (for $M = \pm1, \pm3, \dots$) correspond to the p-functions

$$J_M^{\pm} \leftrightarrow \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{i=1}^{n} e^{Mz_i}$$

   for $n =$ even and zero for $n =$ odd. For $M = \pm1$ we get the light cone components of the energy momentum tensor $T^{\rho\sigma} = J_\sigma^\rho$ with $\rho, \sigma = \pm$ (see also [29]).

In order to prove equations as for example (13) and (16) we consider the corresponding p-functions and their K-functions defined by (17). The K-functions are rational functions of the $x_i = e^{\theta_i}$. We analyze the poles and the asymptotic behavior and find identities by using induction and Liouville's theorem. Transforming these identities to the corresponding form factors one finds the field equation (13) and the trace equation (16) up to normalizations.

Normalization constants are obtained in the various cases by the following observations:

a) The normalization a field annihilating a one-particle state is given by the vacuum one-particle matrix element, in particular for the fundamental bose field one has

$$\langle\, 0 \,|\, \varphi(0) \,|\, p \,\rangle = \sqrt{Z^\varphi}$$

[6]For the sinh-Gordon model an analogous representation as (17) together with this p-function was obtained in [30] by different methods.

with the finite wave function renormalization constant calculated in [5] as

$$Z^{\varphi} = (1+\nu)\frac{\frac{\pi}{2}\nu}{\sin\frac{\pi}{2}\nu}\exp\left(-\frac{1}{\pi}\int_0^{\pi\nu}\frac{t}{\sin t}dt\right)$$

where $Z^{\varphi} = 1 + O(\beta^4)$. This gives the normalization constant

$$N_1^{(1)} = \sqrt{Z^{\varphi}}/2 \tag{21}$$

for the form factors of the fundamental bose field and which are obtained from the p-function of (20) for $N = 1$.

b) If a local operator is connected to an observable like a charge $Q = \int dx\, \mathcal{O}(x)$ we use the relation

$$\langle\, p' \,|\, Q \,|\, p \,\rangle = q\langle\, p' \mid p \,\rangle.$$

For example for the higher charges we obtain

$$N_2^{J_M} = \frac{i^M m^{M+1}}{2\sin\pi\nu\sin\frac{M}{2}\pi\nu F(i\pi)} \quad \text{with} \quad \frac{1}{F(i\pi)} = Z^{\varphi}\frac{\beta^2}{8\pi\nu}\frac{\sin\pi\nu}{\pi\nu}.$$

c) We use Weinberg's power counting theorem for bosonic Feynman graphs [11][7]. For the exponentials of the boson field $\mathcal{O} = \mathcal{N} e^{i\gamma\varphi}$ this yields in particular the asymptotic behavior

$$f_n^{\mathcal{O}}(\theta_1,\theta_2,\dots) = f_1^{\mathcal{O}}(\theta_1)\, f_{n-1}^{\mathcal{O}}(\theta_2,\dots) + O(e^{-\operatorname{Re}\theta_1})$$

as $\operatorname{Re}\theta_1 \to \infty$ in any order of perturbation theory. This behavior is also assumed to hold for the exact form factors. Applying this formula iteratively we obtain from (17) with (19) the following relation for the normalization constants of the operators $\mathcal{N} e^{i\gamma\varphi}$

$$N_n^{\gamma} = (N_1^{\gamma})^n \quad (n \geq 1).$$

d) The recursion relation (18) relates N_{n+2} and N_n. For all p-functions mentioned above we obtain

$$N_{n+2} = N_n \frac{2}{\sin\pi\nu F(i\pi)} \quad (n \geq 1).$$

Using c) and d) we calculate the normalization constants for the exponential of the field $\mathcal{N} e^{i\gamma\varphi}$ and obtain

$$N_1^{\gamma} = \sqrt{Z^{\varphi}}\frac{\beta}{2\pi\nu}. \tag{22}$$

The normalization constants (21) and (22) now yield the field equation (13) with the mass relation (14). The statement (16) is proved similarly. The eigen value equation (15)

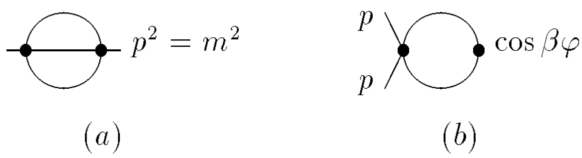


Figure 2: Feynman graphs

is obtained by taking the scalar products with arbitrary out-states and by using a general crossing formula [11].

All the results may be checked in perturbation theory by Feynman graph expansions. In particular in lowest order the relation between the bare and the renormalized mass (14) is given by Figure 1 (a). It had already been calculated in [5] and yields

$$m^2 = \alpha\left(1 - \frac{1}{6}\left(\frac{\beta^2}{8}\right)^2 + O(\beta^6)\right)$$

which agrees with the exact formula above. Similarly we check the quantum corrections of the trace of the energy momentum tensor (16) by calculating the Feynman graph of Figure 1 (b) with the result again taken from [5] as

$$\langle\, p\,|\,\mathcal{N}\cos\beta\varphi - 1\,|\,p\,\rangle = -\beta^2\left(1 + \frac{\beta^2}{8\pi}\right) + O(\beta^6).$$

This again agrees with the exact formula above since the normalization given by eq. (15) implies $\langle\, p\,|\,T^\mu{}_\mu|\,p\,\rangle = 2m^2$. All other equations have also been checked in perturbation theory [13].

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