The nested $SU(N)$ off-shell Bethe Ansatz and exact form factors

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This work is dedicated to the 75th anniversary of H. Bethe's foundational work on the Heisenberg chain

Abstract

The form factor equations are solved for an $SU(N)$ invariant S-matrix under the assumption that the anti-particle is identified with the bound state of $N-1$ particles. The solution is obtained explicitly in terms of the nested off-shell Bethe ansatz where the contribution from each level is written in terms of multiple contour integrals.

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1 Introduction

The Bethe ansatz [1], was initially formulated by Bethe 75 years ago to solve the eigenvalue problem for the isotropic Heisenberg model. The approach has found applications in the context of several integrable systems in different areas, such as Statistical Mechanics, Quantum Field Theory, Condensed Matter Physics, Atom and Molecular Physics, among others. The original techniques have been refined into several directions: Lieb and Lininger [2] solved the one-dimensional bose gas problem with δ-function potential using the Bethe ansatz. The 6-vertex model was solved by Lieb [3, 4] with the same technique. C.N. Yang and C.P. Yang [5] proved 'Bethe's hypothesis'[1] for the ground state of the anisotropic Heisenberg spin chain. Due to Yang [7] and Baxter [8] we have

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[1] Yang and Yang decided to honor Bethe's insight by calling his assumption "Bethe's hypothesis" [6], now usually called "Bethe ansatz".

the fundamental Yang-Baxter equation for the two-particle S-matrix or for the matrix of the Boltzmann weights, which guarantees exact integrability of the system. Subsequently, Faddeev and collaborators [9, 10] formulated these ideas in an elegant algebraic way, known as the "algebraic Bethe ansatz". Yang [7] and Sutherland [11] generalized the technique of the Bethe ansatz for those cases where the underlying symmetry group is larger than $SU(2)$. This method is now called the "nested" Bethe ansatz. This technique was applied in [12] to derive the spectrum of the chiral $SU(N)$ Gross-Neveu model [13]. The algebraic nested Bethe ansatz was formulated in [14] for the $SU(N)$ and in [15] for the $O(2N)$ symmetric case, respectively. Another generalization of the Bethe ansatz is the "off-shell" Bethe ansatz[2], which was originally formulated by one of the authors (H.B.) [16, 17, 18, 19] to calculate correlation function in WZNW models (see also [20, 21]). This version of the Bethe ansatz paves the way to an analysis of off-shell quantities and opens up the intriguing possibility to merge the Bethe ansatz and the form factor approach. In this context we point out that recently the form factor program has received renewed interest in connection with condensed matter physics [22, 23, 24] and atomic physics [25]. In particular, applications to Mott insulators and carbon nanotubes [26, 22] doped two-leg ladders [27] and in the field of Bose-Einstein condensates of ultracold atomic and molecular gases [28, 25] have been discussed and in some instances correlation functions have been computed.

The *bootstrap program* to formulate particle physics in terms of the scattering data, i.e. in terms of the S-matrix goes back to Heisenberg [29] and Chew [30]. Remarkably, this approach works very well for integrable quantum field theories in 1+1 dimensions [31, 32, 33, 34, 35, 36]. One of the present authors (M.K.) et al. formulated the on-shell program [32] i.e. the exact determination of the scattering matrix using the Yang-Baxter equations and the off-shell program [35] i.e. the exact determination of form factors which are matrix elements of local operators. This approach was developed further and studied in the context of several explicit models by Smirnov [37] who proposed the form factor equations (i) – (v) (see below) as extensions of similar formulae in the original article [35]. The formulae were proven by two of the authors (H.B. and M.K.) et al. [38]. In this article the techniques of the "off-shell" Bethe ansatz was used to determine the form factors for the sine-Gordon model. There, however, the underlying group structure is simple and there was no need to use a nested version of the off-shell Bethe ansatz. In the present article we will focus on the determination of the form factors for an $SU(N)$ model. The procedure is similar as for the scaling $Z(N)$ Ising and affine $A(N-1)$ Toda models [39, 40] because the bound state structures of these models are similar. However, the algebraic structure of the form factors for the $SU(N)$ model is more complicated, because the S-matrix possesses backward scattering. Therefore we have to apply a nontrivial algebraic off-shell Bethe ansatz. For $N > 2$ we have to develop the nested version of this technique (see also [41]).

[2] "Off-shell" in the context of the Bethe ansatz means that the spectral parameters in the algebraic Bethe ansatz state are not fixed by Bethe ansatz equations in order to get an eigen state of a hamiltonian, but they are integrated over.

It is expected that the results of this paper apply to the chiral $SU(N)$ Gross-Neveu model [13, 42, 43, 44]. In a separate article [45] we will investigate these physical applications and compare our exact results with two different $1/N$-expansions of the chiral Gross-Neveu model [42] and [44]. We note that $SU(N)$ form factors were also calculated in [37, 46, 47] using other techniques, see also the related paper [48].

1.1 The $SU(N)$ S-matrix

The general solutions of the Yang-Baxter equations, unitarity and crossing relations for a $U(N)$ invariant S-matrix have been obtained in [49]. The S-matrix for the scattering of two particles belonging to the vector representation of $SU(N)$ can be written as

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = \delta_{\alpha\gamma}\delta_{\beta\delta}\, b(\theta) + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c(\theta)\,. \tag{1}$$

Unitarity reads as $S_i(-\theta)S_i(\theta) = 1$ for the S-matrix eigenvalues

$$S_+(\theta) = b(\theta) + c(\theta)\,, \quad S_-(\theta) = b(\theta) - c(\theta)\,.$$

The amplitude $S_+(\theta) = a(\theta)$ is the highest weight $w = (2, 0, \ldots, 0)$ S-matrix eigenvalue for the two particle scattering. It will be essential for the Bethe ansatz below.

As usual in this context we use in the notation

$$v^{1\ldots n} \in V^{1\ldots n} = V^1 \otimes \cdots \otimes V^n$$

for a vector in a tensor product space. The vector components are denoted by $v^{\underline{\alpha}} = v^{\alpha_1\ldots\alpha_n}$. Below we will also use co-vectors $v_{1\ldots n} \in \left(V^{1\ldots n}\right)^\dagger$ (the dual of $V^{1\ldots n}$) with components $v_{\underline{\alpha}}$. A linear operator connecting two such spaces with matrix elements $A_{\alpha_1\ldots\alpha_n}^{\alpha'_1\ldots\alpha'_{n'}}$ is denoted by

$$A_{1\ldots n}^{1'\ldots n'} : \; V^{1\ldots n} \to V^{1'\ldots n'}$$

where we omit the upper indices if they are obvious. All vector spaces V^i are isomorphic to a space V whose basis vectors label all kinds of particles (e.g. $V \cong \mathbb{C}^N$ for the vector representation of $SU(N)$). The vector spaces V^i is associated to a rapidity variable θ_i. An S-matrix such as $S_{ij}(\theta_{ij}) = S_{ij}^{ji}(\theta_i - \theta_j)$ acts nontrivially only on the factors $V_i \otimes V_j$ and exchanges these factors. Using this notation, the Yang-Baxter relation writes as

$$S_{12}(\theta_{12})S_{13}(\theta_{13})S_{23}(\theta_{23}) = S_{23}(\theta_{23})S_{13}(\theta_{13})S_{12}(\theta_{12}) \tag{2}$$

and implies here the relation between the amplitudes [49]

$$c(\theta) = -\frac{i\eta}{\theta} b(\theta)\,, \quad \eta = \frac{2\pi}{N}.$$

A solution [49, 42, 50] of all these equations writes as

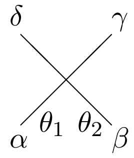
$$a(\theta) = b(\theta) + c(\theta) = -\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right)}\,. \tag{3}$$

This S-matrix possesses a bound state pole in $S_-(\theta)$ i.e. in the anti-symmetric tensor channel. It is consistent with Swieca's [51, 50, 44] picture that the anti-particle is a bound state of $N-1$ particles (see also [39, 40]).

For later convenience and in order to simplify the formulae we introduce

$$\tilde{S}(\theta) = \frac{S(\theta)}{a(\theta)} = \frac{\mathbf{1}\theta - \mathbf{P}i\eta}{\theta - i\eta} \tag{4}$$

where $\mathbf{1}$ is the unit, $\mathbf{P}$ the permutation operator. We depict this matrix as

$$\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta_{12}) =$$


$$= \delta_{\alpha\gamma}\delta_{\beta\delta}\,\tilde{b}(\theta_{12}) + \delta_{\alpha\delta}\delta_{\beta\gamma}\,\tilde{c}(\theta_{12})$$

and the amplitudes are explicitly

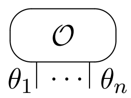
$$\tilde{b}(\theta) = \frac{\theta}{\theta - i\eta}\,, \quad \tilde{c}(\theta) = \frac{-i\eta}{\theta - i\eta}\,.$$

1.2 Generalized Form factors

For a state of n particles of kind α_i with rapidities θ_i and a local operator $\mathcal{O}(x)$ we define the form factor functions $F^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1,\dots,\theta_n)$, or using a short hand notation $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$, by

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, \theta_1,\dots,\theta_n \,\rangle^{in}_{\underline{\alpha}} = e^{-ix(p_1+\dots+p_n)} F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})\;, \quad \text{for } \theta_1 > \dots > \theta_n. \tag{5}$$

where $\underline{\alpha} = (\alpha_1,\dots,\alpha_n)$ and $\underline{\theta} = (\theta_1,\dots,\theta_n)$. For all other arrangements of the rapidities the functions $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ are given by analytic continuation. Note that the physical value of the form factor, i.e. the left hand side of (5), is given for ordered rapidities as indicated above and the statistics of the particles. The $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ are considered as the components of a co-vector valued function $F^{\mathcal{O}}_{1\dots n}(\underline{\theta}) \in V_{1\dots n} = \left(V^{1\dots n}\right)^\dagger$ which may be depicted as

$$F^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = \tag{6}$$


Now we formulate the main properties of form factors in terms of the functions $F^{\mathcal{O}}_{1\dots n}$. They follow from general LSZ-assumptions and "maximal analyticity", which means that $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ is a meromorphic function with respect to all θ's and in the 'physical' strips $0 < \operatorname{Im}\theta_{ij} < \pi$ ($\theta_{ij} = \theta_i - \theta_j\; i < j$) there are only poles of physical origin as for example bound state poles. The generalized form factor functions satisfy the following

Form factor equations: The co-vector valued auxiliary function $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ is meromorphic in all variables $\theta_1,\dots,\theta_n$ and satisfies the following relations:

(i) The Watson's equations describe the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces $i, j = i+1$ at the same time

$$F^{\mathcal{O}}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = F^{\mathcal{O}}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\, S_{ij}(\theta_{ij}) \tag{7}$$

for all possible arrangements of the θ's.

(ii) The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\begin{aligned} &{}^{\mathrm{out},\bar 1}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2,\dots,p_n \,\rangle^{\mathrm{in,conn.}}_{2\dots n} \\ &= F^{\mathcal{O}}_{1\dots n}(\theta_1+i\pi,\theta_2,\dots,\theta_n)\sigma^{\mathcal{O}}_1 \mathbf{C}^{\bar 1 1} = F^{\mathcal{O}}_{2\dots n1}(\theta_2,\dots,\theta_n,\theta_1-i\pi)\mathbf{C}^{1\bar 1} \end{aligned} \tag{8}$$

where $\sigma^{\mathcal{O}}_\alpha$ takes into account the statistics of the particle α with respect to $\mathcal{O}$. The charge conjugation matrix $\mathbf{C}^{\bar 1 1}$ will be discussed below.

(iii) There are poles determined by one-particle states in each sub-channel given by a subset of particles of the state in (5). In particular the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n) = 2i\,\mathbf{C}_{12}\, F^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n)\left(\mathbf{1} - \sigma^{\mathcal{O}}_2 S_{2n}\dots S_{23}\right)\,. \tag{9}$$

(iv) If there are also bound states in the model the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = i\eta,\ (0<\eta<\pi)$ such that

$$\operatorname*{Res}_{\theta_{12}=i\eta} F^{\mathcal{O}}_{12\dots n}(\theta_1,\theta_2,\dots,\theta_n) = F^{\mathcal{O}}_{(12)\dots n}(\theta_{(12)},\dots,\theta_n)\,\sqrt{2}\Gamma^{(12)}_{12} \tag{10}$$

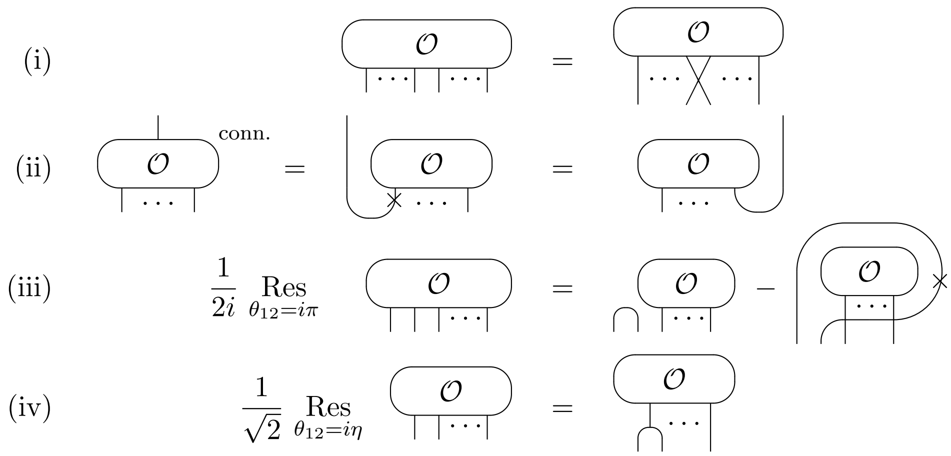
where the bound state intertwiner $\Gamma^{(12)}_{12}$ and the values of $\theta_1,\ \theta_2,\ \theta_{(12)}$ and η are given in [52, 53].

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F^{\mathcal{O}}_{1\dots n}(\theta_1+\mu,\dots,\theta_n+\mu) = e^{s\mu}\, F^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n) \tag{11}$$

if the local operator transforms under Lorentz transformations as $\mathcal{O} \to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

The property (i) - (iv) may be depicted as



where $\times$ denotes the statistics factor $\sigma^{\mathcal{O}}$. As was shown in [38] the properties (i) – (iii) follow from general LSZ-assumptions and "maximal analyticity".

We will now provide a constructive and systematic way of how to solve the form factor equations (i) – (v) for the co-vector valued function $F^{\mathcal{O}}_{1\dots n}$, once the scattering matrix is given.

Minimal form factor: The solutions of Watson's and the crossing equations (i) and (ii) for two particles with no poles in the physical strip $0 \leq \operatorname{Im}\theta \leq \pi$ and at most a simple zero at $\theta = 0$ are the minimal form factors. In particular those for highest weight states are essential for the construction of the off-shell Bethe ansatz. One easily finds the minimal solution of

$$F(\theta) = a(\theta)\, F(-\theta) = F(2\pi i - \theta)$$

using (3) as

$$F(\theta) = c \exp \int_0^\infty \frac{dt}{t \sinh^2 t} e^{\frac{t}{N}} \sinh t \left(1 - \frac{1}{N}\right) \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right) . \tag{12}$$

It belongs to the highest weight $w = (2, 0, \dots, 0)$. We define the corresponding 'Jost-function' as for the $Z(N)$ models [39, 40] by the equation

$$\prod_{k=0}^{N-2} \phi(\theta + ki\eta) \prod_{k=0}^{N-1} F(\theta + ki\eta) = 1 \ , \quad \eta = \frac{2\pi}{N} \tag{13}$$

which is typical for models where the bound state of $N-1$ particles is the anti-particle [40]. The solution is

$$\phi(\theta) = \Gamma\left(\frac{\theta}{2\pi i}\right) \Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right) \tag{14}$$

and satisfies the relations

$$\phi(\theta) = \phi(-\theta)a(-\theta) = \phi((N-1)i\eta - \theta) \\ = \frac{1}{-b(\theta)}\phi(2\pi i - \theta) = \frac{a(\theta - 2\pi i)}{-b(\theta)}\phi(\theta - 2\pi i)\,. \quad (15)$$

Notice that the equations (14) and (13) also determine the normalization constant c in (12) as

$$c = \Gamma^{-2(1-1/N)}\left(\frac{1}{2} - \frac{1}{2N}\right)\exp\left(-\int_0^\infty e^{\frac{1}{N}t}\left(\frac{\sinh\left(1-\frac{1}{N}\right)t}{t\sinh^2 t} - \frac{\left(1-\frac{1}{N}\right)}{t\sinh t}\right)dt\right).$$

Generalized form factors: The co-vector valued function (6) for n-particles can be written as [35]

$$F^{\mathcal{O}}_{1\ldots n}(\underline{\theta}) = K^{\mathcal{O}}_{1\ldots n}(\underline{\theta})\prod_{1\le i<j\le n} F(\theta_{ij}) \quad (16)$$

where $F(\theta)$ is the minimal form factor function (12). The K-function $K^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$ contains the entire pole structure and its symmetry is determined by the form factor equations (i) and (ii) where the S-matrix is replaced by $\tilde{S}(\theta) = S(\theta)/a(\theta)$

$$K^{\mathcal{O}}_{\ldots ij\ldots}(\ldots,\theta_i,\theta_j,\ldots) = K^{\mathcal{O}}_{\ldots ji\ldots}(\ldots,\theta_j,\theta_i,\ldots)\,\tilde{S}_{ij}(\theta_{ij}) \quad (17)$$

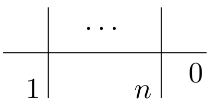
$$K^{\mathcal{O}}_{1\ldots n}(\theta_1 + i\pi, \theta_2, \ldots, \theta_n)\sigma^{\mathcal{O}}_1 \mathbf{C}^{\bar{1}1} = K^{\mathcal{O}}_{2\ldots n1}(\theta_2, \ldots, \theta_n, \theta_1 - i\pi)\mathbf{C}^{1\bar{1}} \quad (18)$$

for all possible arrangements of the θ's.

1.3 Nested "off-shell" Bethe ansatz for $SU(N)$

We consider a state with n particles and define as usual in the context of the algebraic Bethe ansatz [9, 10] the monodromy matrix

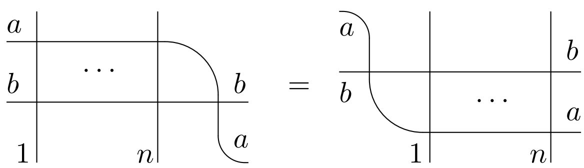
$$\tilde{T}_{1\ldots n,0}(\underline{\theta},\theta_0) = \tilde{S}_{10}(\theta_{10})\cdots\tilde{S}_{n0}(\theta_{n0}) =$$



. (19)

It is a matrix acting in the tensor product of the "quantum space" $V^{1\ldots n} = V^1\otimes\cdots\otimes V^n$ and the "auxiliary space" V^0. All vector spaces V^i are isomorphic to a space V whose basis vectors label all kinds of particles. Here we consider $V \cong \mathbb{C}^N$ as the space of the vector representation of $SU(N)$. The Yang-Baxter algebra relation for the S-matrix (2) yields

$$\tilde{T}_{1\ldots n,a}(\underline{\theta},\theta_a)\,\tilde{T}_{1\ldots n,b}(\underline{\theta},\theta_b)\,\tilde{S}_{ab}(\theta_a-\theta_b) = \tilde{S}_{ab}(\theta_a-\theta_b)\,\tilde{T}_{1\ldots n,b}(\underline{\theta},\theta_b)\,\tilde{T}_{1\ldots n,a}(\underline{\theta},\theta_a) \quad (20)$$



which implies the basic algebraic properties of the sub-matrices A, B, C, D with respect to the auxiliary space defined by

$$\tilde{T}_{1\ldots n,0}(\underline{\theta}, z) \equiv \begin{pmatrix} \tilde{A}_{1\ldots n}(\underline{\theta}, z) & \tilde{B}_{1\ldots n,\beta}(\underline{\theta}, z) \\ \tilde{C}^{\beta}_{1\ldots n}(\underline{\theta}, z) & \tilde{D}^{\beta'}_{1\ldots n,\beta}(\underline{\theta}, z) \end{pmatrix}, \quad 2 \leq \beta, \beta' \leq N \,. \tag{21}$$

We propose the following ansatz for the general form factor $F^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$ or the K-function defined by (16) in terms of a nested 'off-shell' Bethe ansatz and written as a multiple contour integral

$$\boxed{K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) = \frac{N_n}{m!} \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_1}{R} \cdots \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_m}{R} \, \tilde{h}(\underline{\theta}, \underline{z}) \, p^{\mathcal{O}}(\underline{\theta}, \underline{z}) \; \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})} \tag{22}$$

where $\tilde{h}(\underline{\theta}, \underline{z})$ is a scalar function which depends only on the S-matrix and not on the specific operator $\mathcal{O}(x)$

$$\tilde{h}(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n} \prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1\leq i<j\leq m} \tau(z_i - z_j) \tag{23}$$

$$\tau(z) = \frac{1}{\phi(\theta)\phi(-\theta)}\,, \quad \tilde{\phi}(\theta) = \phi(\theta)a(\theta) = \phi(-\theta)\,. \tag{24}$$

For the $SU(N)$ S-matrix the function $\phi(\theta)$ is given by (13) with the solution (14). The integration contour $\mathcal{C}_{\underline{\theta}}$ is depicted in Fig. 1. The constant R is defined by $R = \oint_{\theta} dz \tilde{\phi}(\theta - z)$ where the integration contour is a small circle around $z = \theta$ as part of $\mathcal{C}_{\underline{\theta}}$.

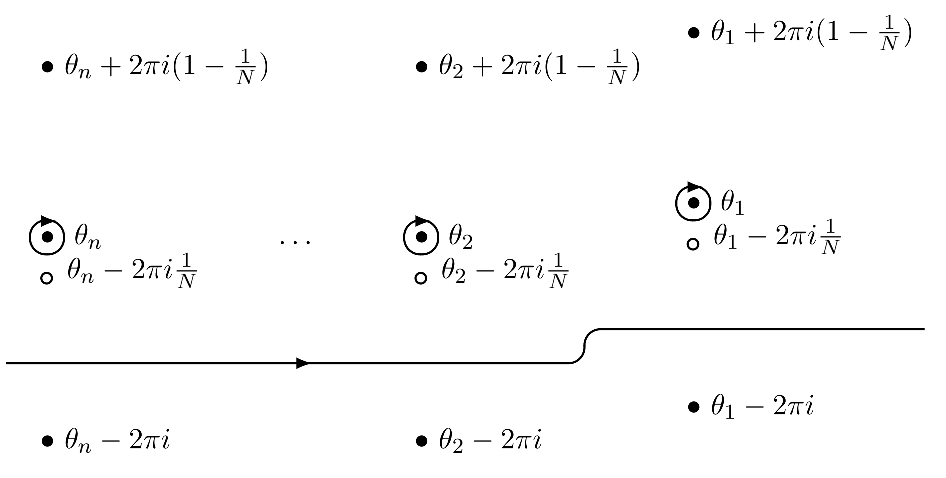


Figure 1: The integration contour $\mathcal{C}_{\underline{\theta}}$. The bullets refer to poles of the integrand resulting from $a(\theta_i - z_j)\,\phi(\theta_i - z_j)$ and the small open circles refer to poles originating from $b(\theta_i - z_j)$ and $c(\theta_i - z_j)$.

The dependence on the specific operator $\mathcal{O}(x)$ is encoded in the scalar p-function $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ which is in general a simple function of e^{θ_i} and e^{z_j} (see below).

By means of the ansatz (16) and (22) we have transformed the complicated form factor equations (i) - (v) (which are in general matrix equations) into much simpler scalar equations for the p-function (see below). The K-function is in general a linear combination of the *fundamental building blocks* [54, 39, 40] given by (22) - (24). We consider here cases where the sum consists only of one term.

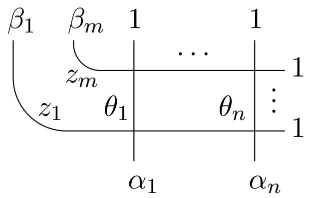
If in (21) the range of β's is non-trivial, i.e. if $N > 2$ the Bethe ansatz co-vectors are of the form

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = L_{\underline{\beta}}(\underline{z})\,\tilde{\Phi}^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z}) \tag{25}$$

where summation over all $\underline{\beta} = (\beta_1,\dots,\beta_m)$ with $\beta_i > 1$ is assumed. The basic Bethe ansatz co-vectors $\tilde{\Phi}^{\underline{\beta}}_{1\dots n} \in \left(V^{1\dots n}\right)^\dagger$ are defined as

$$\tilde{\Phi}^{\underline{\beta}}_{1\dots n}(\underline{\theta},\underline{z}) = \Omega_{1\dots n}\tilde{C}^{\beta_m}_{1\dots n}(\underline{\theta},z_m)\cdots\tilde{C}^{\beta_1}_{1\dots n}(\underline{\theta},z_1)$$

$$\tilde{\Phi}^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z}) =$$



$$, \quad \begin{array}{l} 2 \le \beta_i \le N \\ 1 \le \alpha_i \le N\,. \end{array} \tag{26}$$

Here the "pseudo-vacuum" is the highest weight co-vector (with weight $w = (n,0,\dots,0)$)

$$\Omega_{1\dots n} = e(1)\otimes\cdots\otimes e(1)$$

where the unit vectors $e(\alpha)$ $(\alpha = 1,\dots,N)$ correspond to the particle of type α which belong to the vector representation of $SU(N)$. The pseudo-vacuum vector satisfies

$$\begin{aligned} \Omega_{1\dots n}\,\tilde{B}^{\beta}_{1\dots n}(\underline{\theta},z) &= 0 \\ \Omega_{1\dots n}\,\tilde{A}_{1\dots n}(\underline{\theta},z) &= \Omega_{1\dots n} \\ \Omega_{1\dots n}\,\tilde{D}^{\beta'}_{1\dots n,\beta}(\underline{\theta},z) &= \delta^{\beta'}_{\beta}\prod_{i=1}^{n}\tilde{b}(\theta_i - z)\Omega_{1\dots n}\,. \end{aligned} \tag{27}$$

The amplitudes of the scattering matrices are given by eqs. (1) and (3). The technique of the **'nested Bethe ansatz'** means that one makes for the coefficients $L_{\underline{\beta}}(\underline{z})$ in (25) the analogous construction (22) - (24) as for $K_{\underline{\alpha}}(\underline{\theta})$ where now the indices $\underline{\beta}$ take only the values $2 \le \beta_i \le N$. This nesting is repeated until the space of the coefficients becomes one dimensional.

In this article we will focus on the determination of the form factors for an $SU(N)$ S-matrix. The paper is organized as follows: In Section 2 we construct the general form factor formula for the simplest $SU(2)$ case. In Section 3 we construct the general form factor formula for the $SU(3)$ case, which is more complex due to the presence of the nesting procedure (more explicitly, we have here two levels). We extend these results in Section 4, where the general form factors for $SU(N)$ are constructed and discussed in detail.

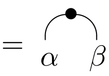
2 $SU(2)$ form factors

In this section we start with the simplest case. We perform the form factor program for the $SU(2)$ S-matrix. The results should apply to the well-known $SU(2)$ Gross-Neveu model [13], investigated by Bethe ansatz methods in [55, 56]. From the technical point of view the calculation of the form factors is very similar to the one done for the sine-Gordon model in [38, 53].

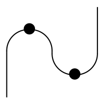
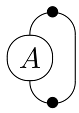
S-matrix: The $SU(2)$ S matrix[3] is given by (1) and (3) for $N = 2$. It turns out that the amplitudes satisfy the relations $b(\theta) = -a(i\pi - \theta)$ and $c(\theta) = c(i\pi - \theta)$ which may be written as

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = -\left(\delta_\alpha^\gamma \delta_\beta^\delta \, b(i\pi - \theta) + \epsilon^{\delta\gamma}\epsilon_{\alpha\beta} \, c(i\pi - \theta)\right) . \tag{28}$$

We have introduced

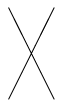
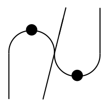
$\epsilon_{\alpha\beta} =$  , $\epsilon^{\alpha\beta} =$ 

which are antisymmetric and $\epsilon_{12} = \epsilon^{21} = 1$ such that

$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$:  $=$ and $\epsilon_{\alpha\beta}A_\gamma^\alpha\epsilon^{\gamma\beta} =$  $= -\operatorname{tr} A$ (29)

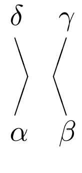
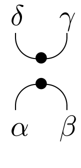
for a matrix A. Formula (28) may be understood as an unusual crossing relation (c.f. [51])

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = -\mathbf{C}_{\alpha\alpha'} S_{\beta\gamma'}^{\alpha'\delta}(i\pi - \theta)\mathbf{C}^{\gamma'\gamma} \tag{30}$$

 $= -$ 

if we define the "charge conjugation matrices" as $\mathbf{C}_{\alpha\beta} = \epsilon_{\alpha\beta}$ and $\mathbf{C}^{\alpha\beta} = \epsilon^{\alpha\beta}$. This means that particle 1 is the anti-particle of 2 and vice versa.

For $\theta \to 0$ and $i\pi$ the S-matrix yield the permutation matrix and the annihilation-creation operator, respectively, or in terms of $\tilde{S} = S/a$

$\tilde{S}_{\alpha\beta}^{\delta\gamma}(0) = \delta_\alpha^\delta \delta_\beta^\gamma =$  , $\operatorname*{Res}_{\theta=i\pi} \tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta) = i\pi\epsilon^{\delta\gamma}\epsilon_{\alpha\beta} = i\pi$  .

ansatz for form factors: Because of (29) the crossing equation (18) writes in components as

$$K_{\alpha_1\ldots\alpha_n}^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n)\sigma^{\mathcal{O}} = -K_{\alpha_2\ldots\alpha_n\alpha_1}^{\mathcal{O}}(\theta_2, \ldots, \theta_n, \theta_1 - 2\pi i) \tag{31}$$

[3]It is related to the sine-Gordon S-matrix for $\beta^2 = 8\pi$ by $(a, b, c)^{SU(2)} = (a, -b, c)^{S.G.}$ (see e.g. [38]).

The form factor equation (iii) here reads as

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n) = 2i\,\epsilon_{12}\,F^{\mathcal{O}}_{3\ldots n}(\theta_3,\ldots,\theta_n)\left(\mathbf{1}-\sigma^{\mathcal{O}}_2 S_{2n}\ldots S_{23}\right). \qquad (32)$$

Using the crossing relation (30) it turns out that the statistics factor in (18) and (32) has to satisfy $\left(\sigma^{\mathcal{O}}\right)^2 = (-1)^n$ with the solutions

$$\sigma^{\mathcal{O}} = i^{Q_{\mathcal{O}}} \quad \text{for} \quad Q_{\mathcal{O}} = n \bmod 2\,. \qquad (33)$$

We define the $SU(2)$ 'Jost-function' $\phi(\theta)$ and $\tau(z)$ by the same equations as for the sine-Gordon [38] and the $Z(2)$ models [39, 40]

$$\phi(\theta)F(\theta)F(\theta+i\pi) = 1 \qquad (34)$$

$$\tau(z)\phi(z)\phi(-z) = 1 \qquad (35)$$

with the solution for $\phi(\theta)$ given by (14). The functions $\phi\,(z)$ and $\tau(z)$ are for $SU(2)$ explicitly given as

$$\phi(z) = \tilde{\phi}(-z) = \Gamma\left(\frac{z}{2\pi i}\right)\Gamma\left(\frac{1}{2}-\frac{z}{2\pi i}\right)\,,\quad \tau(z) = \frac{z\sinh z}{4\pi^3}.$$

The scalar function $\tilde{h}(\underline{\theta},\underline{z})$ in (22) encodes only data from the scattering matrix. The **p-function** $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ on the other hand depends on the explicit nature of the local operator $\mathcal{O}$. It is analytic in all variables and in order that the form factors satisfy (i), (ii) and (iii) of eqs. (7) - (9) $p_{nm}(\underline{\theta},\underline{z})$ has to satisfy

$$\begin{array}{ll} (\mathrm{i}')_2 & p_{nm}(\underline{\theta},\underline{z}) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j \text{ and } z_i \leftrightarrow z_j \\ (\mathrm{ii}')_2 & \begin{cases} \sigma p_{nm}(\theta_1+2\pi i,\theta_2,\ldots,\underline{z}) = (-1)^{m-1}p_{nm}(\theta_1,\theta_2,\ldots,\underline{z}) \\ p_{nm}(\underline{\theta},z_1+2\pi i,z_2,\ldots) = (-1)^n p_{nm}(\underline{\theta},z_1,z_2,\ldots) \end{cases} \\ (\mathrm{iii}')_2 & \text{if } \theta_{12}=i\pi : p_{nm}(\underline{\theta},\underline{z})|_{z_1=\theta_1} = (-1)^{m-1}\sigma p_{nm}(\underline{\theta},\underline{z})|_{z_1=\theta_2} \\ & \qquad = \sigma p_{n-2m-1}(\theta_3,\ldots,\theta_n,z_2,\ldots,z_m)+\tilde{p} \end{array} \qquad (36)$$

where σ is the statistics factor of (31) and (32). In order to simplify the notation we have suppressed the dependence of the p-function $p^{\mathcal{O}}$ and the statistics factor $\sigma^{\mathcal{O}}$ on the operator $\mathcal{O}(x)$. By means of the ansatz (16) and (22) - (24) we have transformed the complicated form factor equations (i) - (iii) to the simple ones $(\mathrm{i}')_2$ - $(\mathrm{iii}')_2$ for the p-function.

Theorem 1 *The co-vector valued function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ given by the ansatz (16) and (22) satisfies the form factor equations* (i), (ii) *and* (iii) *(see (7) - (9)) if the p-function $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ satisfies the equations* $(\mathrm{i}')_2$, $(\mathrm{ii}')_2$, $(\mathrm{iii}')_2$ *of (36) and the normalization constants satisfies the recursion relation*

$$i\pi\tilde{\phi}(i\pi)F(i\pi)N_n = 2iN_{n-2}\,.$$

The proof of this theorem for $SU(2)$ is similar to that for the sine-Gordon model in [38] and may easily obtained from the ones for $SU(3)$ or $SU(N)$ established below.

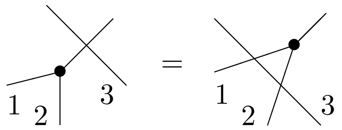
3 $SU(3)$ form factors

In this section we construct the form factors for the $SU(3)$ model, which corresponds to the simplest example where the nested Bethe ansatz technique has to be applied together with the off-shell Bethe ansatz. In comparison to the previous $SU(2)$ case, there are additional properties to be obeyed by the second-level Bethe ansatz function (See lemma 3 below for details).

3.1 S-matrix

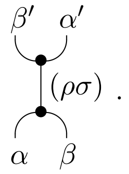
The $SU(3)$ S-matrix is given by (1) and (3) for $N = 3$. The eigenvalue S_- has a pole at $\theta = i\eta = \frac{2}{3}i\pi$ which means that there exist bound states of two fundamental particles $\alpha + \beta \to (\rho\sigma)$ (with $1 \le \rho < \sigma \le 3$) which transform as the anti-symmetric $SU(3)$ tensor representation. The general bound state S-matrix formula [52, 53] for the scattering of a bound state with another particle reads in particular for the $SU(3)$ case as

$$S_{(\rho\sigma)\gamma}^{\gamma'(\rho'\sigma')}(\theta_{(12)3})\Gamma_{\alpha\beta}^{(\rho\sigma)} = \Gamma_{\alpha'\beta'}^{(\rho'\sigma')} S_{\alpha\gamma''}^{\gamma'\alpha'}(\theta_{13}) S_{\beta\gamma}^{\gamma''\beta'}(\theta_{23}) \Big|_{\theta_{12}=i\eta} \tag{37}$$



where $\theta_{(12)}$ is the bound state rapidity and η the bound state fusion angle. The bound state fusion intertwiner $\Gamma_{\alpha\beta}^{(\rho\sigma)}$ is defined by

$$i \operatorname*{Res}_{\theta=i\eta} S_{\alpha\beta}^{\beta'\alpha'}(\theta) = \sum_{\rho<\sigma} \Gamma_{(\rho\sigma)}^{\beta'\alpha'} \Gamma_{\alpha\beta}^{(\rho\sigma)} =$$



With a convenient choice of an undetermined phase factor one obtains

$$\Gamma_{\alpha\beta}^{(\rho\sigma)} = \Gamma \left(\delta_\alpha^\rho \delta_\beta^\sigma - \delta_\alpha^\sigma \delta_\beta^\rho \right) , \quad \Gamma_{(\rho\sigma)}^{\beta'\alpha'} = \Gamma \left(\delta_\rho^{\beta'} \delta_\sigma^{\alpha'} - \delta_\sigma^{\beta'} \delta_\rho^{\alpha'} \right) \tag{38}$$

where $\Gamma = i\sqrt{ia(i\eta)i\eta}$ is a number. Choosing in (37) special cases for the external particles we calculate

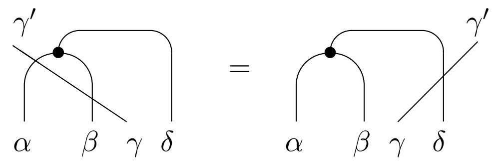
$$\begin{aligned} S_{(12)3}^{3(12)}(\theta) &= b(\theta + \tfrac{1}{3}i\pi)b(\theta - \tfrac{1}{3}i\pi) = a(i\pi - \theta) \\ S_{(12)2}^{2(12)}(\theta) &= b(\theta + \tfrac{1}{3}i\pi)a(\theta - \tfrac{1}{3}i\pi) = b(\pi i - \theta) \\ S_{(23)1}^{3(12)}(\theta) &= -b(\theta + \tfrac{1}{3}i\pi)c(\theta - \tfrac{1}{3}i\pi) = c(\pi i - \theta) \end{aligned}$$

which may be written as

$$S_{(\alpha\beta)\gamma}^{\gamma'(\alpha'\beta')}(\theta) = \delta_\gamma^{\gamma'} \delta_{(\alpha\beta)}^{(\alpha'\beta')} b(\pi i - \theta) + \epsilon^{\gamma'\alpha'\beta'} \epsilon_{\alpha\beta\gamma} c(\pi i - \theta)$$

with the total anti-symmetric tensors $\epsilon_{\alpha\beta\gamma}$ and $\epsilon^{\alpha\gamma\beta}$ ($\epsilon_{123} = \epsilon^{321} = 1$). These formulae may be again understood as a crossing relation (c.f. [51])

$$\epsilon_{\alpha'\beta'\delta} S_{\alpha\gamma''}^{\gamma'\alpha'}(\theta + \tfrac{1}{2}i\eta) S_{\beta\gamma}^{\gamma''\beta'}(\theta - \tfrac{1}{2}i\eta) = \epsilon_{\alpha\beta\delta'} S_{\gamma\delta}^{\delta'\gamma'}(i\pi - \theta) \tag{39}$$



or $S_{(\alpha\beta)\gamma}^{\gamma'(\alpha'\beta')}(\theta) = \mathbf{C}_{(\alpha\beta)\delta'} S_{\gamma\delta}^{\delta'\gamma'}(i\pi - \theta) \mathbf{C}^{\delta(\alpha'\beta')}$ if we write the charge conjugation matrices as

$$\mathbf{C}_{(\alpha\beta)\gamma} = \mathbf{C}_{\alpha(\beta\gamma)} = \epsilon_{\alpha\beta\gamma}\,, \quad \mathbf{C}^{\alpha(\beta\gamma)} = \mathbf{C}^{(\alpha\beta)\gamma} = \epsilon^{\alpha\beta\gamma}\,. \tag{40}$$

Therefore we have the relations (c.f. (29))

$$\mathbf{C}_{\alpha(\rho\sigma)} \mathbf{C}^{(\rho\sigma)\beta} = \delta_\alpha^\beta\,, \quad \mathbf{C}_{\alpha(\rho\sigma)} A_\beta^\alpha \mathbf{C}^{\beta(\rho\sigma)} = \operatorname{tr} A \tag{41}$$

and

$$\mathbf{C}_{\alpha(\rho\sigma)} \Gamma_{\beta\gamma}^{(\rho\sigma)} = \mathbf{C}_{(\rho\sigma)\gamma} \Gamma_{\alpha\beta}^{(\rho\sigma)} = \epsilon_{\alpha\beta\gamma} \Gamma\,. \tag{42}$$

These results[4] are consistent with the picture that the bound state of particles 1 and 2 is to be identified with the anti-particle of 3. For later convenience we consider the total 3-particle S-matrix in the neighborhood of its poles at $\theta_{12}, \theta_{23} = i\eta$

$$\begin{aligned} \tilde{S}_{\alpha\beta\gamma}^{\gamma'\beta'\alpha'}(\theta_1, \theta_2, \theta_3) &= \tilde{S}_{\alpha''\beta''}^{\beta'\alpha'}(\theta_{12}) \tilde{S}_{\alpha\gamma''}^{\gamma'\alpha''}(\theta_{13}) \tilde{S}_{\beta\gamma}^{\gamma''\beta''}(\theta_{23}) \\ &\approx 2 \frac{i\eta}{\theta_{12} - i\eta} \frac{i\eta}{\theta_{23} - i\eta} \epsilon^{\gamma'\beta'\alpha'} \epsilon_{\alpha\beta\gamma}\,. \end{aligned} \tag{43}$$

3.2 Form factors

In order to obtain a recursion relation where only form factors for the fundamental particles of type $\alpha = 1, 2, 3$ (which transform as the $SU(3)$ vector representation) are involved, we have to apply the bound state relation (iv) to get the anti-particle and then the creation annihilation equation (iii)

$$\begin{aligned} \operatorname*{Res}_{\theta_{12}=i\eta} F_{123\dots n}^{\mathcal{O}}(\theta_1, \theta_2, \theta_3, \dots, \theta_n) &= F_{(12)3\dots n}^{\mathcal{O}}(\theta_{(12)}, \theta_3, \dots, \theta_n) \sqrt{2}\,\Gamma_{12}^{(12)} \\ \operatorname*{Res}_{\theta_{(12)3}=i\pi} F_{(12)3\dots n}^{\mathcal{O}}(\theta_{(12)}, \theta_3, \dots, \theta_n) &= 2i\mathbf{C}_{(12)3} F_{4\dots n}^{\mathcal{O}}(\theta_4, \dots, \theta_n) \\ &\quad \times \left(\mathbf{1} - \sigma_3^{\mathcal{O}} S_{3n} \dots S_{34}\right) \end{aligned}$$

where $\Gamma_{12}^{(12)}$ is the bound state intertwiner (38) (see [52, 53]), $\theta_{(12)} = \frac{1}{2}(\theta_1 + \theta_2)$ is the bound state rapidity, $\eta = \frac{2}{3}\pi$ is the bound state fusion angle and $\mathbf{C}_{(12)3}$ is the charge conjugation matrix (40). We obtain with the short notation $\underline{\check{\theta}} = (\theta_4, \dots, \theta_n)$

$$\operatorname*{Res}_{\theta_{23}=i\eta} \operatorname*{Res}_{\theta_{12}=i\eta} F_{123\dots n}^{\mathcal{O}}(\underline{\theta}) = 2i\varepsilon_{123} \sqrt{2}\,\Gamma F_{4\dots n}^{\mathcal{O}}(\underline{\check{\theta}}) \left(\mathbf{1} - \sigma_3^{\mathcal{O}} S_{3n} \dots S_{34}\right) \tag{44}$$

where (42) has been used.

[4]The physical aspects of these facts will we discussed elsewhere [45].

ansatz for form factors: We propose that the n-particle form factor of an operator $\mathcal{O}(x)$ is given by the same formula (16) as for $SU(2)$ where the form factor equations (i) and (ii) for K-function write again as (17) and (18). Consistency of (44), (18) and the crossing relation (39) means that the statistics factors are of the form $\sigma_\alpha^{\mathcal{O}} = \sigma^{\mathcal{O}}(r_\alpha)$ if the particle of type α belongs to a $SU(3)$ representation of rank $r_\alpha = 1, 2$ and

$$\sigma^{\mathcal{O}}(r) = e^{i\pi\frac{2}{3}rQ_{\mathcal{O}}} \quad \text{for} \quad Q_{\mathcal{O}} = n \bmod 3 \tag{45}$$

as an extension of (33). As for $SU(2)$, we propose again for the K-function the ansatz in form of the integral representation (22) with (23). However, the Bethe ansatz co-vector is here for $SU(3)$ of the form

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) = L_{\underline{\beta}}(\underline{z})\tilde{\Phi}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z}) \tag{46}$$

where the basic Bethe states $\tilde{\Phi}_{\underline{\alpha}}^{\underline{\beta}}(\underline{\theta}, \underline{z})$ are given by (26) and the indices α_i run over $1 \leq \alpha_i \leq 3$ and the β_i over $2 \leq \beta_i \leq 3$. For the function $L_{\underline{\beta}}(\underline{z})$ one makes an analogous ansatz (22) as for $K_{\underline{\alpha}}(\underline{\theta})$ where the indices run here over a set with one element less. By this procedure one obtains the nested Bethe ansatz. The next level function $L_{\underline{\beta}}(\underline{z})$ is assumed to satisfy

(i)$^{(1)}$

$$L_{\dots ij\dots}(\dots, z_i, z_j, \dots) = L_{\dots ji\dots}(\dots, z_j, z_i, \dots)\, \tilde{S}_{ij}(z_{ij}) \tag{47}$$

(ii)$^{(1)}$

$$\mathbf{C}^{\bar{1}1} L_{1\dots m}(z_1 + i\pi, z_2, \dots, z_m) = L_{2\dots m1}(z_2, \dots, z_m, z_1 - i\pi)\mathbf{C}^{1\bar{1}} \tag{48}$$

(iii)$^{(1)}$ there is a pole at $z_{12} = i\eta$ such that

$$\operatorname*{Res}_{z_{12}=i\eta} L_{\underline{\beta}}(\underline{z}) = c_1 \prod_{i=3}^{m} \tilde{\phi}(z_{i2}) \operatorname*{Res}_{z_{12}=i\eta} \tilde{S}_{\beta_1\beta_2}^{32}(z_{12}) L_{\underline{\check{\beta}}}(\underline{\check{z}}) \tag{49}$$

with $c_1 = \tilde{\phi}(i\eta)$.

These properties of the next level Bethe ansatz function $L_{\underline{\beta}}(\underline{z})$ are discussed in lemma 3.

The minimal two particle form factor function

$$F(\theta) = c \exp \int_0^\infty \frac{dt}{t \sinh^2 t} e^{\frac{t}{3}} \sinh \frac{2}{3} t \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right) \tag{50}$$

belongs to the highest weight $w = (2, 0, 0)$. The $SU(3)$ and the $Z(3)$ model [39, 40] possess the same bound state structure, namely the anti-particle is to be identified with the bound state of two particles. Therefore we define the $SU(3)$ 'Jost-function' $\phi(z)$ by the same equation as for the $Z(3)$ model

$$\phi(\theta)\phi(\theta + i\eta)F(\theta)F(\theta + i\eta)F(\theta + 2i\eta) = 1 \tag{51}$$

with the solution

$$\phi(z) = \Gamma\left(\frac{z}{2\pi i}\right)\Gamma\left(\frac{2}{3}-\frac{z}{2\pi i}\right).$$

These equations determine also the constant c in (50). The function $\tau(z)$ is again defined by (24).

The function $\tilde{h}(\underline{\theta},\underline{z})$ is scalar and encodes only data from the scattering matrix. The **p-function** $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ on the other hand depends on the explicit nature of the local operator $\mathcal{O}$. It is analytic in all variables and, in order that the form factors satisfy (i) - (iii) of eqs. (7) - (9), we assume

$$\begin{array}{lll} (\mathrm{i}')_3 & p(\underline{\theta},\underline{z}) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j & \\ (\mathrm{ii}')_3 & \begin{cases} \sigma p(\theta_1+2\pi i,\theta_2,\dots,\underline{z}) = (-1)^m p(\theta_1,\theta_2,\dots,\underline{z}) \\ p(\underline{\theta},z_1+2\pi i,z_2,\dots) = (-1)^n p(\underline{\theta},z_1,z_2,\dots) \end{cases} & \\ (\mathrm{iii}')_3 & \text{for } \theta_{12}=\theta_{23}=i\eta \begin{cases} p(\underline{\theta};\theta_2,\theta_3,\underline{\check{z}}) = (-1)^m\, p(\underline{\check{\theta}},\underline{\check{z}}) \\ p(\underline{\theta};\theta_1,\theta_2,\underline{\check{z}}) = \sigma\, p(\underline{\check{\theta}},\underline{\check{z}}) \end{cases} & \end{array} \tag{52}$$

with $\underline{\check{\theta}} = (\theta_4,\dots,\theta_n)$, $\underline{\check{z}} = (z_3,\dots,z_m)$. In $(\mathrm{ii}')_3$ and $(\mathrm{iii}')_3$ σ is the statistics factor of the operator $\mathcal{O}$ with respect to the fundamental particle belonging to the vector representation of $SU(3)$.

Theorem 2 *Let the co-vector valued function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ be defined by the ansatz (16), (22), (23) and (46). Let the p-function satisfy* $(\mathrm{i}')_3$, $(\mathrm{ii}')_3$ *and* $(\mathrm{iii}')_3$ *of (52) and let the function $L_{\underline{\beta}}(\underline{z})$ satisfy* $(\mathrm{i})^{(1)}$, $(\mathrm{ii})^{(1)}$ *and* $(\mathrm{iii})^{(1)}$ *of (47) - (49). Let the normalization constants satisfy the recursion relation*

$$2(i\eta)^2\tilde{\phi}^2(i\eta)\tilde{\phi}(2i\eta)F^2(i\eta)F(2i\eta)N_n = 2i\sqrt{2}\Gamma N_{n-3}\,. \tag{53}$$

Then the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ satisfies the form factor equations (i), (ii) *and* (iii) *(see (7) - (9)). In particular (44) is satisfied.*

Proof. Property (i) in the form of (17) follows directly from $(\mathrm{i}')_3$, the Yang-Baxter equations and the action of the S-matrix on the pseudo-ground state Ω

$$\begin{aligned} \Omega_{\dots ji\dots}\,\tilde{C}_{\dots ji\dots}(\dots\theta_j,\theta_i\dots)\,\tilde{S}_{ij}(\theta_{ij}) &= \Omega_{\dots ji\dots}\,\tilde{S}_{ij}(\theta_{ij})\,\tilde{C}_{\dots ij\dots}(\dots\theta_i,\theta_j\dots) \\ &= \Omega_{\dots ij\dots}\,\tilde{C}_{\dots ij\dots}(\dots\theta_i,\theta_j\dots)\,. \end{aligned}$$

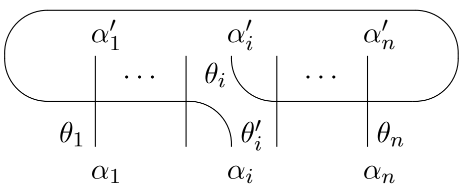
because $\tilde{S}_{11}^{11}(\theta) = S_{11}^{11}(\theta)/a(\theta) = 1$ and $F(\theta) = F(-\theta)a(\theta)$.

Using (i) and (41) the property (ii) in the form of (18) may be rewritten as a matrix difference equation [57, 58, 41]

$$K_{1\dots n}(\underline{\theta}')\sigma = K_{1\dots n}(\underline{\theta})\,Q_{1\dots n}(\underline{\theta}) \tag{54}$$

where $\underline{\theta}' = (\theta_1+2\pi i,\theta_2\dots,\theta_n)$ and σ is a statistics factor. The matrix $Q(\underline{\theta}) = Q(\underline{\theta},1)$ is a special case (for $i=1$) of the trace

$$Q_{1\dots n}(\underline{\theta},i) = \mathrm{tr}_0\,\tilde{T}_{Q,1\dots n,0}(\underline{\theta},i) =$$



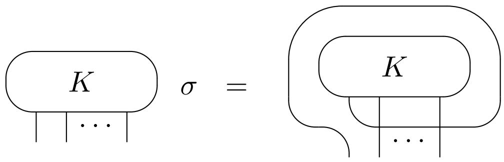
of a modified monodromy matrix

$$\tilde{T}_{Q,1\dots n,0}(\underline{\theta},i) = \tilde{S}_{10}(\theta_1-\theta_i')\cdots\mathbf{P}_{i0}\cdots\tilde{S}_{n0}(\theta_n-\theta_i)$$

where $\mathbf{P}$ is the permutation matrix. For the special case $i=1$ the matrix $Q(\underline{\theta},1)=Q(\underline{\theta})$ may be written as a trace of the ordinary monodromy matrix (19) over the auxiliary space for the specific value of the spectral parameter $\theta_0=\theta_1$

$$Q_{1\dots n}(\underline{\theta}) = \mathrm{tr}_0\,\tilde{T}_{1\dots n,0}(\underline{\theta},\theta_1) \tag{55}$$

because $\tilde{S}(0)=\mathbf{P}$. The difference equation (54) may be depicted as



where we use the rule that the rapidity of a line changes by $2\pi i$ if the line bends by 360^0 in the positive sense.

In the following we will suppress the indices $1\dots n$. The Yang-Baxter relations (20) imply the typical commutation rules for the matrices $\tilde{A},\tilde{C},\tilde{D}$ defined in eq. (21)

$$\tilde{C}^\beta(\underline{\theta},z)\tilde{A}(\underline{\theta},\theta) = \frac{1}{\tilde{b}(\theta-z)}\tilde{A}(\underline{\theta},\theta)\tilde{C}^\beta(\underline{\theta},z) - \frac{\tilde{c}(\theta-z)}{\tilde{b}(\theta-z)}\tilde{A}(\underline{\theta},z)\tilde{C}^\beta(\underline{\theta},\theta) \tag{56}$$

$$\begin{aligned}\tilde{C}^\beta(\underline{\theta},z)\tilde{D}_\gamma^{\gamma'}(\underline{\theta},\theta) &= \frac{1}{\tilde{b}(z-\theta)}\tilde{S}_{\beta'\gamma''}^{\gamma'\beta}(z-\theta)\tilde{D}_\gamma^{\gamma''}(\underline{\theta},\theta)\tilde{C}^{\beta'}(\underline{\theta},z)\\ &\quad-\frac{\tilde{c}(z-\theta)}{\tilde{b}(z-\theta)}\tilde{D}_\gamma^\beta(\underline{\theta},z)\tilde{C}^{\gamma'}(\underline{\theta},\theta)\end{aligned}$$

where $\beta,\beta',\gamma,\gamma',\gamma''\in\{2,3\}$. In addition there are the Zapletal commutation rules [57, 58, 41] where also the matrices A_Q,C_Q,D_Q defined by

$$\tilde{T}_Q(\underline{\theta}) = \begin{pmatrix}\tilde{A}_Q(\underline{\theta}) & \tilde{B}_Q^\beta(\underline{\theta})\\ \tilde{C}_Q^\beta(\underline{\theta}) & \tilde{D}_{Q\beta}^{\ \beta'}(\underline{\theta})\end{pmatrix}$$

are involved [57]

$$\tilde{C}^\beta(\underline{\theta},z)\tilde{A}_Q(\underline{\theta}) = \frac{1}{\tilde{b}(\theta_1'-z)}\tilde{A}_Q(\underline{\theta})\tilde{C}^\beta(\underline{\theta}',z) - \frac{\tilde{c}(\theta_1'-z)}{\tilde{b}(\theta_1'-z)}\tilde{A}(\underline{\theta},z)\tilde{C}_Q^\beta(\underline{\theta}) \tag{57}$$

$$\begin{aligned}\tilde{C}^\beta(\underline{\theta},z)\tilde{D}_{Q\gamma}^{\ \gamma'}(\underline{\theta}) &= \frac{1}{\tilde{b}(z-\theta_1')}\tilde{S}_{\beta'\gamma''}^{\gamma'\beta}(z-\theta_1')\tilde{D}_{Q\gamma}^{\ \gamma''}(\underline{\theta})\tilde{C}^{\beta'}(\underline{\theta}',z)\\ &\quad-\frac{\tilde{c}(z-\theta_1)}{\tilde{b}(z-\theta_1)}\tilde{D}_\gamma^\beta(\underline{\theta},z)\tilde{C}_Q^{\gamma'}(\underline{\theta})\,.\end{aligned} \tag{58}$$

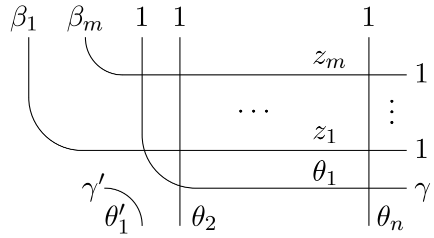
Note that we assign to the auxiliary space of $\tilde{T}_Q(\underline{\theta})$ corresponding to the horizontal line the spectral parameter θ_1 on the right hand side and $\theta_1'=\theta_1+2\pi i$ on the left hand side.

We are now going to prove (54) in the form

$$K(\underline{\theta})\left(\tilde{A}_Q(\underline{\theta})+\sum_{\beta=2}^{3}\tilde{D}_{Q}{}_{\beta}^{\beta}(\underline{\theta})\right)=K(\underline{\theta})\,Q(\underline{\theta})=K(\underline{\theta}')\sigma \tag{59}$$

where $K(\underline{\theta})$ is a co-vector valued function as given by eq. (22) and the Bethe ansatz state (26). To analyze the left hand side of eq. (59) we proceed as follows: We apply the trace of $\tilde{T}_Q$ which is $\tilde{A}_Q+\sum_{\beta=2}^{3}\tilde{D}_{Q}{}_{\beta}^{\beta}$ to the co-vector $K(\underline{\theta})$

$$\Omega\,\tilde{C}^{\beta_m}(\underline{\theta},z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta},z_1)\tilde{T}_{Q}{}_{\gamma}^{\gamma'}(\underline{\theta})=$$



In the contribution from $\tilde{A}_Q(\underline{\theta})$ which means $\gamma'=\gamma=1$ one may use Yang-Baxter relations to observe that only the amplitudes $\tilde{S}_{11}^{11}(\theta_1-z_j)=1$ appear in the S-matrices $\tilde{S}(\theta_1-z_j)$ which are constituents of the C-operators. Therefore we may shift all z_j-integration contours $\mathcal{C}_{\underline{\theta}}$ to $\mathcal{C}_{\underline{\theta}'}$ without changing the values of the integrals, because there are no singularities inside $\mathcal{C}_{\underline{\theta}}\cup-\mathcal{C}_{\underline{\theta}'}$ (c.f. Fig. 1). Using a short notation we have

$$K(\underline{\theta})\,\tilde{A}_Q(\underline{\theta})=\int_{\mathcal{C}_{\underline{\theta}'}}d\underline{z}\,\tilde{h}(\underline{\theta},\underline{z})p(\underline{\theta},\underline{z})\,\tilde{\Psi}(\underline{\theta},\underline{z})\,\tilde{A}_Q(\underline{\theta})$$

(with $\int_{\mathcal{C}_{\underline{\theta}}}d\underline{z}=\frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz_1}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}}}\frac{dz_m}{R}$). We now proceed as usual in the algebraic Bethe ansatz and push the $\tilde{A}_Q(\underline{\theta})$ and $\tilde{D}_Q(\underline{\theta})$ through all the $\tilde{C}$-operators using the commutation rules (57) and (58) and obtain

$$\begin{aligned}\tilde{C}^{\beta_m}(\underline{\theta},z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta},z_1)\,\tilde{A}_Q(\underline{\theta})&=\prod_{j=1}^{m}\frac{1}{\tilde{b}(\theta_1'-z_j)}\,\tilde{A}_Q(\underline{\theta})\\&\quad\times\tilde{C}^{\beta_m}(\underline{\theta}',z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta}',z_1)+\sum uw_A\,,\end{aligned}$$

$$\begin{aligned}\tilde{C}^{\beta_m}(\underline{\theta},z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta},z_1)\,\tilde{D}_{Q}{}_{\beta}^{\beta'}(\underline{\theta})&=\prod_{j=1}^{m}\frac{1}{\tilde{b}(z_j-\theta_1')}\tilde{T}^{(1)}{}_{\underline{\beta}\,\beta''}^{\underline{\beta}'\beta'}(\underline{z},\theta_1')\tilde{D}_{Q}{}_{\beta}^{\beta''}(\underline{\theta})\\&\quad\times\tilde{C}^{\beta_m'}(\underline{\theta},z_m)\cdots\tilde{C}^{\beta_1'}(\underline{\theta},z_1)+\sum uw_D\,.\end{aligned}$$

The "wanted terms" written out explicitly originate from the first term in the commutations rules (57); all other contributions yield the so-called "unwanted terms". The next level monodromy matrix is

$$\tilde{T}^{(1)}{}_{\underline{\beta}\,\beta}^{\underline{\beta}'\beta'}(\underline{z},\theta)=\left(\tilde{S}_{10}(z_1-\theta)\cdots\tilde{S}_{m0}(z_m-\theta)\right)_{\underline{\beta}\,\beta}^{\beta'\,\underline{\beta}'}$$

where the β's and also the internal summation indices take the values $2,3$. If we insert these equations into the representation (22) of $K(\underline{\theta})$ we first find that the wanted contribution from $\tilde{A}_Q$ already gives the result we are looking for. Secondly the wanted contribution from $\tilde{D}_Q$ applied to Ω gives zero. Thirdly the unwanted contributions from $\tilde{A}_Q$ and $\tilde{D}_Q$ cancel after integration over the z_j. All these three facts can be seen as follows. We have

$$\Omega\,\tilde{A}_Q(\underline{\theta}) = \Omega\,, \quad \Omega\,\tilde{D}_{Q\beta}^{\beta'}(\underline{\theta}) = \delta_\beta^{\beta'}\prod_{i=1}^n \tilde{b}(\theta_i-\theta_1)\Omega = 0 \tag{60}$$

which follow from eq. (27). Therefore the wanted term from $\tilde{A}_Q$ yields

$$\begin{aligned}\left[K(\underline{\theta})\,\tilde{A}_Q(\underline{\theta})\right]^w &= \int_{\mathcal{C}_{\underline{\theta}'}} d\underline{z}\prod_{j=1}^m \frac{1}{\tilde{b}(\theta_1'-z_j)}\,\tilde{h}(\underline{\theta},\underline{z})p(\underline{\theta},\underline{z})\tilde{\Psi}(\underline{\theta}',\underline{z})\\ &= \sigma\int_{\mathcal{C}_{\underline{\theta}'}} d\underline{z}\,\tilde{h}(\underline{\theta}',\underline{z})p(\underline{\theta}',\underline{z})\tilde{\Psi}(\underline{\theta}',\underline{z})\\ &= \sigma K(\underline{\theta}')\,.\end{aligned}$$

It has been used that the 'shift relation' of the function $\phi(\theta)$ in (15) and (ii') of (52) for the function p imply that

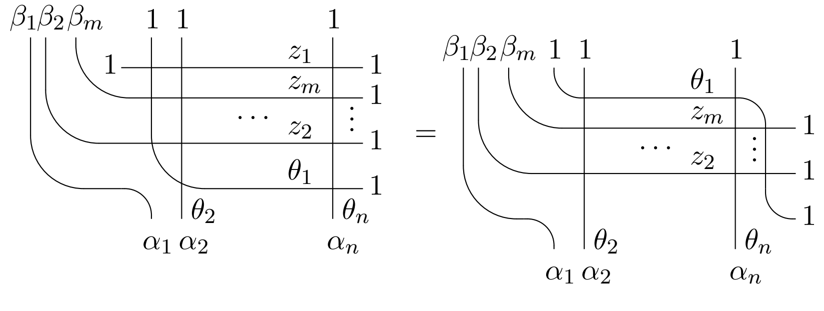
$$\prod_{j=1}^m \frac{1}{\tilde{b}(\theta_1'-z_j)}\,\tilde{h}(\underline{\theta},\underline{z})p(\underline{\theta},\underline{z}) = \sigma\tilde{h}(\underline{\theta}',\underline{z})p(\underline{\theta}',\underline{z})\,.$$

The wanted contribution from $\tilde{D}_Q$ vanish, since $\tilde{b}(0)=0$. Therefore it remains to prove that the unwanted terms cancel. The commutation relations (57) and (58) imply that the unwanted terms are proportional to a product of $\tilde{C}$-operators, where one $\tilde{C}(\underline{\theta},z_j)$ is replaced by $\tilde{C}_Q(\underline{\theta})$. Because of the symmetry (i)[1] of $L_{\underline{\beta}}(\underline{z})$ it is sufficient to consider only the unwanted terms for $j=1$ which are denoted by uw_A^1 and uw_D^1. They originate from the second term in the commutation rules (57) when $\tilde{A}_Q(\underline{\theta})$ is commuted with $\tilde{C}(\underline{\theta},z_1)$. Then the resulting $\tilde{A}(\underline{\theta},z_1)$ pushed through the other C's. Taking into account only the first terms in (56) we arrive at

$$uw_A^1(\underline{z}) = -\frac{\tilde{c}(\theta_1'-z_1)}{\tilde{b}(\theta_1'-z_1)}\prod_{j=2}^m\frac{1}{\tilde{b}(z_1-z_j)}\,\Omega\,\tilde{A}(\underline{\theta},z_1)\,\tilde{C}^{\beta_m}(\underline{\theta},z_m)\dots\tilde{C}_Q^{\beta_1}(\underline{\theta})$$

Using (27) and Yang-Baxter relations (always taking into account the '$SU(N)$ ice rule' which means 'color conservation') and $\tilde{a}(\theta)=1$ one obtains

$$\left[\Omega\,\tilde{A}(\underline{\theta},z_1)\,\tilde{C}^{\beta_m}(\underline{\theta},z_m)\dots\tilde{C}_Q^{\beta_1}(\underline{\theta})\right]_{\underline{\alpha}} = \delta_{\alpha_1}^{\beta_1}\check{\Phi}_{\underline{\tilde{\alpha}}}^{\underline{\tilde{\beta}}}(\check{\underline{\theta}},\check{\underline{z}})$$

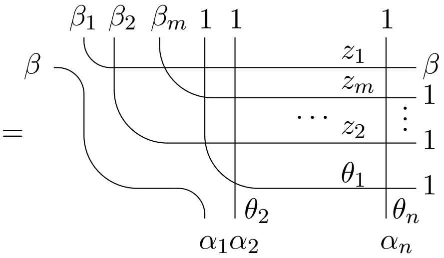


where $\check{\Phi}^{\underline{\tilde\beta}}_{\underline{\tilde\alpha}}(\underline{\check\theta},\underline{\check z})$ does not depend on $\alpha_1,\beta_1,\theta_1,z_1$. Similarly, we obtain the first unwanted contribution from $\sum_\beta \tilde D_{Q}{}^{\beta}_{\beta}$ as

$$uw_D^1 = -\frac{\tilde c(z_1-\theta_1)}{\tilde b(z_1-\theta_1)}\prod_{j=2}^m \frac{1}{\tilde b(z_j-z_1)}\,\left(\tilde T^{(1)}{}^{\beta\,\underline{\beta}}_{\underline{\beta}'\beta''}(\underline z,z_1)\right) \times \Omega \tilde D^{\beta''}_{\beta}(\underline\theta,z_1)\tilde C^{\beta'_m}(\underline\theta,z_m)\cdots \tilde C^{\beta'_1}_{Q}(\underline\theta)$$

which may be depicted as

$$\left(\tilde T^{(1)}{}^{\beta\,\underline{\beta}}_{\underline{\beta}'\,\beta''}(\underline z,z_1)\right)\Omega\tilde D^{\beta''}_{\beta}(\underline\theta,z_1)\tilde C^{\beta'_m}(\underline\theta,z_m)\cdots\tilde C^{\beta'_1}_{Q}(\underline\theta)$$

$$= $$ 

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Again (27) and Yang-Baxter relations yield

$$\left[\Omega\,\tilde D^{\beta''}_{\beta}(\underline\theta,z_1)\tilde C^{\beta'_m}(\underline\theta,z_m)\cdots\tilde C^{\beta'_1}_{Q}(\underline\theta)\right]_{\underline\alpha} = \delta^{\beta''}_{\beta}\prod_{i=1}^n \tilde b(\theta_i-z_1)\,\delta^{\beta'_1}_{\alpha_1}\check\Phi^{\underline{\tilde\beta}'}_{\underline{\tilde\alpha}}(\underline{\check\theta},\underline{\check z})$$

while assumption (ii)$^{(1)}$ for $L_{\underline\beta}(\underline z)$ means

$$L_{\underline\beta}(\underline z)Q^{(1)}{}^{\underline\beta}_{\underline\beta'}(\underline z) = L_{\underline\beta}(\underline z')$$

where analogously to eq. (55) $Q^{(1)}{}^{\underline\beta}_{\underline\beta'}(\underline z) = \tilde T^{(1)}{}^{\beta\,\underline\beta}_{\underline\beta'\,\beta}(\underline z,z_1)$ is the next level Q-matrix and $\underline z' = (z_1' = z_1+2\pi i,\ldots,z_m)$. Therefore we finally obtain

$$\left[K(\underline\theta)\,\tilde A_Q(\underline\theta)\right]^{uw}_{\underline\alpha}$$

$$= -\int_{\mathcal C_{\underline\theta'}} d\underline z\,\tilde h(\underline\theta,\underline z)\,p(\underline\theta,\underline z)\frac{\tilde c(\theta_1'-z_1)}{\tilde b(\theta_1'-z_1)}\prod_{j=2}^m\frac{1}{\tilde b(z_1-z_j)}\,L_{\underline\beta}(\underline z)\delta^{\beta_1}_{\alpha_1}\check\Phi^{\underline{\tilde\beta}}_{\underline{\tilde\alpha}}(\underline{\check\theta},\underline{\check z})$$

$$\left[K(\underline\theta)\,\tilde D_{Q}{}^{\beta}_{\beta}\right]^{uw}_{\underline\alpha}$$

$$= -\int_{\mathcal C_{\underline\theta}} d\underline z\,\tilde h(\underline\theta,\underline z')\,p(\underline\theta,\underline z')\frac{\tilde c(z_1-\theta_1)}{\tilde b(z_1-\theta_1)}\prod_{j=2}^m\frac{1}{\tilde b(z_1'-z_j)}\,L_{\underline\beta}(\underline z')\delta^{\beta_1}_{\alpha_1}\check\Phi^{\underline{\tilde\beta}}_{\underline{\tilde\alpha}}(\underline{\check\theta},\underline{\check z})\,.$$

It has been used that the 'shift relation' of the function $\phi(\theta)$ in (15) and (ii') of (52) for the function p imply

$$\prod_{i=1}^n \tilde b(\theta_i-z_1)\prod_{j=2}^m\frac{1}{\tilde b(z_j-z_1)}\,\tilde h(\underline\theta,\underline z)p(\underline\theta,\underline z) = \prod_{j=2}^m\frac{1}{\tilde b(z_1'-z_j)}\,\tilde h(\underline\theta,\underline z')p(\underline\theta,\underline z')\,.$$

For the $\tilde{D}_Q$-term we rewrite the z_1-integral by replacing $z_1 \to z_1 - 2\pi i$ (such that $z_1' \to z_1$ and $\mathcal{C}_{\underline{\theta}} \to \mathcal{C}_{\underline{\theta}} + 2\pi i$) and obtain for the sum of the unwanted term from $\tilde{A}_Q$ and $\tilde{D}_Q$ (using $\tilde{c}(z)/\tilde{b}(z) = -\tilde{c}(-z)/\tilde{b}(-z)$)

$$\left[K(\underline{\theta})\left(\tilde{A}_Q(\underline{\theta}) + \tilde{D}_Q(\underline{\theta})\right)\right]^{uw}_{\underline{\alpha}} = \left(\int_{\mathcal{C}_{\underline{\theta}'}} d\underline{z} - \int_{\mathcal{C}_{\underline{\theta}}+2\pi i} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} d\underline{\check{z}}\right) \times \frac{\tilde{c}(z_1-\theta_1')}{\tilde{b}(z_1-\theta_1')}\tilde{h}(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z}) L_{\underline{\beta}}(\underline{z}) \prod_{j=2}^{m} \frac{1}{\tilde{b}(z_1-z_j)}\, \delta^{\beta_1}_{\alpha_1} \check{\Phi}^{\tilde{\underline{\beta}}}_{\underline{\tilde{\alpha}}}(\underline{\tilde{\theta}},\underline{\tilde{z}})\,.$$

Taking into account the pole structure of $\frac{\tilde{c}(z_1-\theta_1')}{\tilde{b}(z_1-\theta_1')}\tilde{h}(\underline{\theta},\underline{z})$ we may replace for the $\tilde{A}_Q$ -term

$$\int_{\mathcal{C}_{\underline{\theta}'}} d\underline{z} \cdots = \left(-2\pi i \operatorname*{Res}_{z_1=\theta_1'} + \int_{\mathcal{C}_{\underline{\theta}}} dz_1\right) \int_{\mathcal{C}_{\underline{\theta}'}} d\underline{\check{z}} \cdots$$

where again $\left(\int_{\mathcal{C}_{\underline{\theta}'}} - \int_{\mathcal{C}_{\underline{\theta}}}\right) d\underline{\check{z}} \cdots = 0$ and for the $\tilde{D}_Q$-term

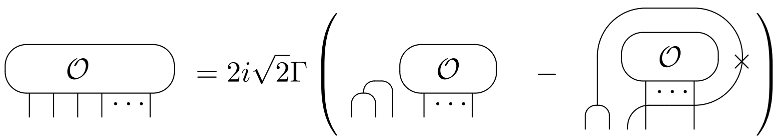
$$\int_{\mathcal{C}_{\underline{\theta}}+2\pi i} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} d\underline{\check{z}} \cdots = \left(-2\pi i \operatorname*{Res}_{z_1=\theta_1'} + \int_{\mathcal{C}_{\underline{\theta}}} dz_1\right) \int_{\mathcal{C}_{\underline{\theta}}} d\underline{\check{z}} \cdots$$

which proves the cancellation of the unwanted terms.

The proof of (iii) is similar to that for the $Z(3)$ model in [40]. We use the short notations

$$\begin{aligned}
\underline{\theta} &= (\theta_1,\dots,\theta_n),\ \underline{\hat{\theta}} = (\theta_1,\theta_2,\theta_3),\ \underline{\check{\theta}} = (\theta_4,\dots,\theta_n),\\
\underline{\alpha} &= (\alpha_1,\dots,\alpha_n),\ \underline{\hat{\alpha}} = (\alpha_1,\alpha_2,\alpha_3),\ \underline{\check{\alpha}} = (\alpha_4,\dots,\alpha_n),\\
\underline{z} &= (z_1,\dots,z_m),\ \underline{\hat{z}} = (z_1,z_2),\ \underline{\check{z}} = (z_3,\dots,z_m),\\
\underline{\beta} &= (\beta_1,\dots,\beta_m),\ \underline{\hat{\beta}} = (\beta_1,\beta_2),\ \underline{\check{\beta}} = (\beta_3,\dots,\beta_m)
\end{aligned}$$

and prove (44) which may be depicted as

$\operatorname*{Res}_{\theta_{23}=i\eta}\operatorname*{Res}_{\theta_{12}=i\eta}$ 

.

We will show that the K-function given by the integral representation (22) with (23) satisfies

$$\operatorname*{Res}_{\theta_{23}=i\eta}\operatorname*{Res}_{\theta_{12}=i\eta} K_{1\dots n}(\underline{\theta}) = c_0 \prod_{i=4}^{n}\prod_{j=2}^{3} \tilde{\phi}(\theta_{ij})\, \varepsilon_{123}\, K_{4\dots n}(\underline{\check{\theta}})\, (\mathbf{1} - \sigma_3 S_{3n}\dots S_{34}) \tag{61}$$
$$c_0 = 2i\sqrt{2}\Gamma F^{-2}(i\eta)F^{-1}(2i\eta)\,.$$

This is equivalent to (44) because of the ansatz (16) for $F_{1\dots n}(\underline{\theta})$ and the relation of $F(z)$ and $\phi(z)$ given by (51). The residues of $K_{1\dots n}(\underline{\theta})$ consists of three terms

$$\operatorname*{Res}_{\theta_{12}=i\eta}\operatorname*{Res}_{\theta_{23}=i\eta} K_{1\dots n}(\underline{\theta}) = R^{(1)}_{1\dots n} + R^{(2)}_{1\dots n} + R^{(3)}_{1\dots n}\,.$$

This is because each pair of the z integration contours will be "pinched" at three points. Due to symmetry it is sufficient to determine the contribution from the z_1- and the z_2-integrations and multiply the result by $m(m-1)$. The pinching points are

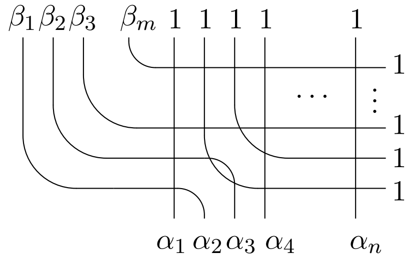
(1) $z_1 = \theta_2,\ z_2 = \theta_3,$

(2) $z_1 = \theta_1,\ z_2 = \theta_2,$

(3) $z_1 = \theta_2 - i\eta,\ z_2 = \theta_3 - i\eta.$

The contribution of (1) is given by z_1-and z_2-integrations $\oint_{\theta_2} dz_1 \oint_{\theta_3} dz_2 \cdots$ along small circles around $z_1 = \theta_2$ and $z_2 = \theta_3$ (see figure 1). The S-matrices $\tilde S(\theta_2 - z_1)$ and $\tilde S(\theta_3 - z_2)$ yield the permutation operator $\tilde S(0) = \mathbf{P}$

$$\left(\Omega\, \tilde C^{\beta_m}(\underline\theta, z_m) \cdots \tilde C^{\beta_2}(\underline\theta, \theta_3) \tilde C^{\beta_1}(\underline\theta, \theta_2)\right)_{\underline\alpha} =$$



$$= \tilde S^{\beta_2 1}_{\alpha_1' \alpha_3}(\theta_{13}) \tilde S^{\beta_1 \alpha_1'}_{\alpha_1 \alpha_2}(\theta_{12}) \prod_{j=3}^{m} \tilde b(\theta_1 - z_j) \left(\Omega \tilde C^{\beta_m}(\check{\underline\theta}, z_m) \cdots \tilde C^{\beta_3}(\check{\underline\theta}, z_3)\right)_{\check{\underline\alpha}} .$$

We have used the fact that because of the $SU(N)$ ice rule only the amplitude $\tilde b$ contributes to the S-matrices $\tilde S(\theta_1 - z_j)$ and only $\tilde a = 1$ to the S-matrices $\tilde S(\theta_2 - z_j), \tilde S(\theta_3 - z_j), \tilde S(\theta_i - z_1), \tilde S(\theta_i - z_2)$ after having applied Yang-Baxter relations. Further we use that for $z_{12} \to \theta_{23} \to i\eta$ by assumption (iii)$^{(1)}$ (c.f. (49))

$$L_{\underline\beta}(\underline z) \approx c_1 \prod_{i=3}^{m} \tilde\phi(z_{i2}) \tilde S^{32}_{\beta_1\beta_2}(z_{12}) L_{\check{\underline\beta}}(\check{\underline z})$$
$$c_1 = \tilde\phi(z_{12}) = \tilde\phi(\theta_{23}) = \tilde\phi(i\eta)$$

and that because of (43)

$$\operatorname*{Res}_{\theta_{12}=i\eta} \operatorname*{Res}_{\theta_{23}=i\eta} \tilde\phi(\theta_{12})\tilde\phi(\theta_{13}) \tilde S^{321}_{\alpha_1\alpha_2\alpha_3}(\theta_1,\theta_2,\theta_3) = \tilde\phi(i\eta)\tilde\phi(2i\eta) 2\,(i\eta)^2\, \epsilon_{\alpha_1\alpha_2\alpha_3} .$$

We combine this with the scalar functions $\tilde h$ and p and after having performed the remaining z_j-integrations we obtain

$$R^{(1)}_{\underline\alpha} = c_0 \prod_{i=4}^{n} \prod_{j=2}^{3} \tilde\phi(\theta_{ij})\, \epsilon_{\hat{\underline\alpha}} K_{\check{\underline\alpha}}(\check{\underline\theta})$$

because of $\tilde b(\theta_1 - z)\tilde\phi(\theta_1 - z)\tilde\phi(\theta_2 - z)\tilde\phi(z - \theta_3)\tilde\phi(\theta_3 - z)\tau(\theta_2 - z)\tau(\theta_3 - z) = -1$, the relation (iii')$_3$ of (52) for the p-functions $p(\underline\theta, \theta_2, \theta_3, \check{\underline z})|_{\theta_{12}=\theta_{23}=i\eta} = (-1)^m\, p(\check{\underline\theta}, \check{\underline z})$

and the relation (53) for the normalization constants $N_n c_1 2(i\eta)^2\tilde{\phi}(i\eta)\tilde{\phi}(2i\eta) = N_{n-3}c_0$.

The remaining contribution to (61) is due to $R^{(2)}_{1\dots n} + R^{(3)}_{1\dots n}$. It is convenient to shift the particle with momentum θ_3 to the right by applying S-matrices using (i). Then we have to prove that

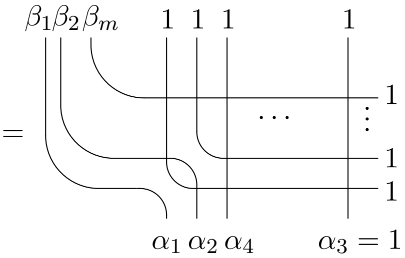
$$\left(\left(R^{(2)}_{1\dots n} + R^{(3)}_{1\dots n}\right) S_{43}\dots S_{n3} + c_0 \prod_{i=4}^{n}\prod_{j=2}^{3} \tilde{\phi}(\theta_{ij})\,\varepsilon_{123}\,K_{4\dots n}(\underline{\check{\theta}})\sigma_3\right)_{\alpha_1\alpha_2\underline{\check{\alpha}}\alpha_3} = 0\,. \tag{62}$$

Note that because of (i) (c.f. (17))

$$K_{1\dots n}(\underline{\theta})S_{43}\dots S_{n3} = \prod_{i=4}^{n} a(\theta_{i3})K_{1\dots 4\dots n3}(\theta_1,\dots,\theta_4,\dots,\theta_n,\theta_3)\,.$$

Due to Lemma 6 in appendix A it is sufficient to prove equation (62) only for $\alpha_3 = 1$ since the left hand side of this equation is a highest weight co-vector. This is because Bethe ansatz vectors, in particular also off-shell Bethe ansatz vectors are of highest weight (see [57]). Therefore we consider this equation for the components with $\alpha_3 = 1$. The contribution of pinching (2) is given by the z_1-, z_2-integrations along the small circles around $z_1 = \theta_1$ and $z_2 = \theta_2$ (see again figure 1). Now $\tilde{S}(\theta_1 - z_1)$ and $\tilde{S}(\theta_2 - z_2)$ yield permutation operators $\mathbf{P}$ and the co-vector part of this contribution for $\alpha_3 = 1$ is

$$\left(\Omega\,\tilde{C}^{\beta_m}(\underline{\theta}, z_m)\cdots\tilde{C}^{\beta_2}(\underline{\theta},\theta_2)\tilde{C}^{\beta_1}(\underline{\theta},\theta_1)P_3(1)\right)_{\underline{\alpha}}$$

$=$ 

$$= \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\left(\Omega\tilde{C}^{\beta_m}(\underline{\check{\theta}}, z_m)\cdots\tilde{C}^{\beta_3}(\underline{\check{\theta}}, z_3)\right)_{\underline{\check{\alpha}}}\delta^1_{\alpha_3} \tag{63}$$

where $P_3(1)$ projects onto the components with $\alpha_3 = 1$. We have used the fact that because of the $SU(N)$ ice rule the amplitude a only contributes to the S-matrices $S(\theta_1 - z_j), S(\theta_2 - z_j), S(\theta_3 - z_j), S(\theta_i - z_1)$ after having applied Yang-Baxter relations. One derives the formula

$$\operatorname*{Res}_{\theta_{23}=i\eta}\operatorname*{Res}_{\theta_{12}=i\eta}\tilde{\phi}(\theta_{31})\tilde{\phi}(\theta_{32})\tilde{S}^{32}_{\alpha_1\alpha_2}(\theta_{12}) = -\tilde{\phi}(i\eta)\tilde{\phi}(2i\eta)2(i\eta)^2\epsilon_{\alpha_1\alpha_2 1}$$

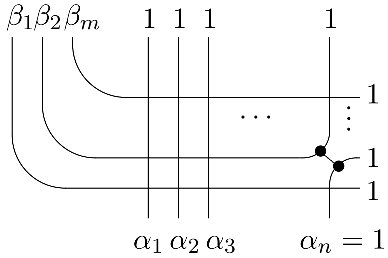
using (43), $\epsilon^{321} = 1$, $\tilde{S}^{321}_{1\alpha_1\alpha_2}(\theta_3 + 2\pi i, \theta_1, \theta_2) = b(\theta_{31} + 2\pi i)b(\theta_{32} + 2\pi i)\tilde{S}^{32}_{\alpha_1\alpha_2}(\theta_{12})$ (because of the $SU(3)$ ice rule) and $\tilde{\phi}(\theta) = -\tilde{b}(\theta + 2\pi i)\tilde{\phi}(\theta + 2\pi i)$ (see (15)). We combine this and (63) with the scalar functions $\tilde{h}$ and p taking into account the property (iii)$^{(1)}$ of (43) and after having performed the remaining z_j-integrations we obtain

$$R^{(2)}_{1\dots n} S_{43}\dots S_{n3} P_3(1) \;=\; -c_0 \prod_{i=4}^{n}\prod_{j=2}^{3} \tilde\phi(\theta_{ij})\,\varepsilon_{123}\, K_{4\dots n}(\underline{\check\theta})\sigma_3 P_3(1)\,. \tag{64}$$

We have used the following equations $a(\theta_{i3})\tilde\phi(\theta_{i1}) = \tilde\phi(\theta_{i3})$ (because of (15)), the definition (24) of $\tau(z)$ which implies $\tilde\phi(z-\theta_2)\tilde\phi(\theta_1-z)\tilde\phi(\theta_2-z)\tilde\phi(\theta_3-z)\tau(\theta_1-z)\tau(\theta_2-z) = 1$, the second relation (iii')$_3$ of (52) for the p-functions $p(\underline\theta,\theta_1,\theta_2,\underline{\check z})|_{\theta_{12}=\theta_{23}=i\eta} = \sigma\, p(\underline{\check\theta},\underline{\check z})$ and the relation (53) for the normalization constants $N_n c_1 \tilde\phi(i\eta)\tilde\phi(2i\eta)2(i\eta)^2 = N_{n-3}c_0$.

The contribution of pinching (3) is given by the z_1-, z_2-integrations along the small circles around $z_1=\theta_2-i\eta$ and $z_2=\theta_3-i\eta$, (see again figure 1). Now $\tilde S(\theta_3-z_2)$ yields $\tilde S^{\delta\gamma}_{\alpha\beta}(\theta_3-z_2)\to\Gamma^{\delta\gamma}_{(\rho\sigma)}\Gamma^{(\rho\sigma)}_{\alpha\beta}$ and the residue of the Bethe ansatz state

$$i \operatorname*{Res}_{z_2=\theta_3-i\eta}\Big(\Omega\,\tilde C(\underline\theta,z_m)\cdots\tilde C(\underline\theta,z_1)P_n(1)\Big)_{\underline\alpha} =$$



vanishes for $\alpha_3=1$ because $\Gamma^{(\rho\sigma)}_{\alpha\beta}$ is antisymmetric with respect to α,β. Therefore equation (62) is proved for $\alpha_3=1$ and because of Lemma 6 also in general. ■

The higher level Bethe ansatz

Lemma 3 *Let the constant R and the contour $\mathcal C_{\underline z}$ be defined as in the context of (22). Then the higher level function*

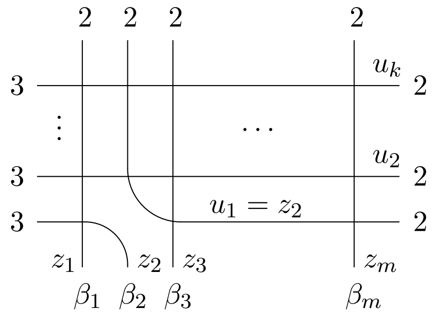
$$L_{\underline\beta}(\underline z) = \frac{1}{k!}\int_{\mathcal C_{\underline z}}\frac{du_1}{R}\cdots\int_{\mathcal C_{\underline z}}\frac{du_k}{R}\,\tilde h(\underline z,\underline u)p^{(1)}(\underline z,\underline u)\,\tilde\Psi^{(1)}_{\underline\beta}(\underline z,\underline u)$$
$$\tilde\Psi^{(1)}_{\underline\beta}(\underline z,\underline u) = \Big(\Omega^{(1)}\,\tilde C^{(1)}(\underline z,u_k)\cdots\tilde C^{(1)}(\underline z,u_1)\Big)_{\underline\beta}$$
$$\tilde h(\underline z,\underline u) = \prod_{i=1}^{m}\prod_{j=1}^{k}\tilde\phi(z_i-u_j)\prod_{1\le i<j\le k}\tau(u_i-u_j)\,.$$

satisfies (i)$^{(1)}$ - (iii)$^{(1)}$ *of the equations (47) - (49) if*

$$\begin{array}{ll} (\mathrm{i}')^{(1)}_3 & p^{(1)}_{mk}(\underline z,\underline u) \text{ is symmetric under } z_i\leftrightarrow z_j\\ (\mathrm{ii}')^{(1)}_3 & \begin{cases} p^{(1)}_{mk}(z_1+2\pi i,z_2,\dots,\underline u) = (-1)^k p^{(1)}_{mk}(z_1,z_2,\dots,\underline u)\\ p^{(1)}_{mk}(\underline z,u_1+2\pi i,u_2,\dots) = (-1)^m p^{(1)}_{mk}(\underline z,u_1,u_2,\dots)\end{cases}\\ (\mathrm{iii}')^{(1)}_3 & p^{(1)}_{mk}(\underline z,\underline u)|_{z_{12}=i\eta,u_1=z_2} = (-1)^{k-1}p^{(1)}_{m-2k-1}(\underline{\check z},\underline{\check u})\,.\end{array} \tag{65}$$

Proof. The proofs of the second level equations (i)$^{(1)}$ and (ii)$^{(1)}$ are similar to the ones for the first level. For the proof of (iii)$^{(1)}$ we observe that for $z_{12} \to i\eta$ there is a pinching at $u_j = z_2$ which means that in $\tilde{\Psi}^{(1)}_{\underline{\beta}}(\underline{z}, \underline{u})$ we may replace $\tilde{S}(z_2 - u_1) \to \mathbf{P}$ and we have to consider (for $z_{12} \to i\eta$ and $u_1 = z_2$)

$$\left(\Omega^{(1)}\, \tilde{C}^{(1)}(\underline{z}, u_k) \cdots \tilde{C}^{(1)}(\underline{z}, z_2)\right)_{\underline{\beta}} =$$



$$= \tilde{S}^{32}_{\beta_1\beta_2}(z_{12}) \prod_{j=2}^{k} \tilde{b}(z_1 - u_j) \check{\Psi}_{\underline{\check{\beta}}}(\underline{\check{z}}, \underline{\check{u}})$$

with $\underline{\check{\beta}} = (\beta_3, \dots, \beta_m)$, $\underline{\check{z}} = (z_3, \dots, z_m)$, $\underline{\check{u}} = (u_2, \dots, u_k)$. Therefore because of $\tilde{b}(z_1 - u)\tilde{\phi}(z_1 - u)\tilde{\phi}(z_2 - u)\tau(z_2 - u) = -1$ and $(\mathrm{iii}')^{(1)}_3$ for $z_{12} \to i\eta$

$$L_{\underline{\beta}}(\underline{z}) \approx \tilde{\phi}(z_{12}) \prod_{i=3}^{m} \tilde{\phi}(z_{i2}) \tilde{S}^{23}_{\beta_1\beta_2}(z_{12}) L_{\underline{\check{\beta}}}(\underline{\check{z}})$$

which proves the claim (iii)$^{(1)}$ of (49). ■

4 $SU(N)$ form factors

In this section we perform the form factor program for the general $SU(N)$ S-matrix. For this purpose, we extend the procedures of the previous section, i.e., the nested Bethe ansatz method now with $N-1$ levels combined with the off-shell Bethe ansatz. Applications of the results to the $SU(N)$ Gross-Neveu model [13] will be investigated in a separate article [45].

4.1 S-matrix

The $SU(N)$ S-matrix is given by (1) and (3). Again, the eigenvalue $S_-(\theta)$ has a pole at $\theta = i\eta = \frac{2}{N} i\pi$ which means that there exist bound states of r fundamental particles $\alpha_1 + \cdots + \alpha_r \to (\rho_1 \dots \rho_r)$ (with $\rho_1 < \cdots < \rho_r$) which transform as the anti-symmetric $SU(N)$ tensor representation of rank r, $(0 < r < N)$. The masses of the bound states satisfy $m_r = m_{N-r}$ which suggests Swieca's [51, 50, 44] picture that the antiparticle of a particle of rank r is to be identified with the particle of rank $N - r$ (see also [39, 40]).

Iterating $N-1$ times the general bound state formula one obtains for the scattering of a bound state $(\beta_1\beta_2 \dots \beta_{N-1})$ with another particle δ (analogously to (37) for the $SU(3)$ case)

$$S^{\delta'(\gamma_1\gamma_2\dots\gamma_{N-1})}_{(\beta_1\beta_2\dots\beta_{N-1})\delta}(\theta)\Gamma^{(\beta_1\beta_2\dots\beta_{N-1})}_{\alpha_1\alpha_2\dots\alpha_{N-1}}$$
$$= \Gamma^{(\gamma_1\gamma_2\dots\gamma_{N-1})}_{\alpha'_1\alpha'_2\dots\alpha'_{N-1}} S^{\delta'\alpha'_1}_{\alpha_1\delta_1}(\theta+i\pi-i\eta)\dots S^{\delta_{N-2}\alpha'_{N-1}}_{\alpha_{N-1}\delta}(\theta-i\pi+i\eta)$$

with the total bound state fusion intertwiner

$$\Gamma^{(\beta_1\beta_2\dots\beta_{N-1})}_{\alpha_1\alpha_2\dots\alpha_{N-1}} = \Gamma^{(\beta_1\beta_2\dots\beta_{N-1})}_{(\beta_1\beta_2\dots\beta_{N-2})\alpha_{N-1}}\dots\Gamma^{(\beta_1\beta_2\beta_3)}_{(\beta_1\beta_2)\alpha_3}\Gamma^{(\beta_1\beta_2)}_{\alpha_1\alpha_2}\,.$$

Taking special cases for the external particles we obtain

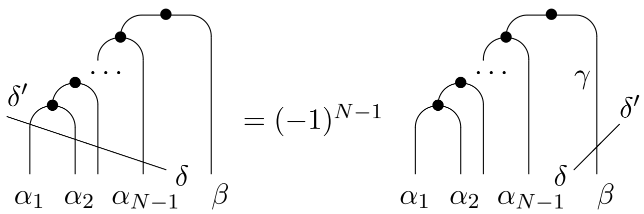
$$S^{N(1,2\dots,N-1)}_{(1,2\dots,N-1)N}(\theta) = b(\theta+i\pi-i\eta)\dots b(\theta-i\pi+i\eta) = (-1)^{N-1}a(\pi i-\theta)$$
$$S^{N-1(1,2\dots,N-1)}_{(1,2\dots,N-1)N-1}(\theta) = b(\theta+i\pi-i\eta)\dots a(\theta-i\pi+i\eta) = (-1)^{N-1}b(\pi i-\theta)$$
$$S^{N(1,2\dots,N-1)}_{(2,3\dots,N)1}(\theta) = -b(\theta+i\pi-i\eta)\dots c(\theta-i\pi+i\eta) = (-1)^{N-1}c(\pi i-\theta)\,.$$

which may be written as

$$S^{\beta_N(\beta_1\dots\beta_{N-1})}_{(\alpha_1\dots\alpha_{N-1})\alpha_N}(\theta)$$
$$= (-1)^{N-1}\left(\delta^{(\beta_1\dots\beta_{N-1})}_{(\alpha_1\dots\alpha_{N-1})}\delta^{\beta_N}_{\alpha_N}\,b(\pi i-\theta) + \epsilon^{\beta_N\beta_1\dots\beta'_{N-1}}\epsilon_{\alpha_1\dots\alpha_{N-1}\alpha_N}\,c(\pi i-\theta)\right)$$

with the total anti-symmetric tensors $\epsilon_{\alpha_1\dots\alpha_N}$ and $\epsilon^{\alpha_1\dots\alpha_N}$ $(\epsilon_{1\dots N} = \epsilon^{N\dots 1} = 1)$. These results may be interpreted as an unusual crossing relation

$$\epsilon_{\beta_1\beta_2\dots\beta_{N-1}\beta}\,S^{\delta'\beta_1}_{\alpha_1\delta_1}(\theta_1)\dots S^{\delta_{N-2}\beta_{N-1}}_{\alpha_{N-1}\delta}(\theta_{N-1}) = (-1)^{N-1}\epsilon_{\alpha_1\alpha_2\dots\alpha_{N-1}\gamma}\,S^{\gamma\delta'}_{\delta\beta}(i\pi-\theta) \tag{66}$$



with $\theta_j = \theta + i\pi - ji\eta$ if we write the charge conjugation matrices as

$$\mathbf{C}_{(\alpha_1\dots\alpha_{N-1})\alpha_N} = \mathbf{C}_{\alpha_1(\alpha_2\dots\alpha_N)} = \epsilon_{\alpha_1\dots\alpha_N}\,,$$
$$\mathbf{C}^{(\alpha_1\dots\alpha_{N-1})\alpha_N} = \mathbf{C}^{\alpha_1(\alpha_2\dots\alpha_N)} = \epsilon^{\alpha_1\dots\alpha_N}\,.$$

Therefore we have the relations (c.f. (29) and (41))

$$\mathbf{C}_{\alpha(\alpha_1\dots\alpha_{N-1})}\mathbf{C}^{(\alpha_1\dots\alpha_{N-1})\beta} = \delta^\beta_\alpha\,,\ \mathbf{C}_{\alpha(\alpha_1\dots\alpha_{N-1})}A^\alpha_\beta\mathbf{C}^{\beta(\alpha_1\dots\alpha_{N-1})} = (-1)^{N-1}\,\mathrm{tr}\,A$$
$$\text{and } \mathbf{C}_{(\beta_1\dots\beta_{N-1})\gamma}\Gamma^{(\beta_1\dots\beta_{N-1})}_{\alpha_1\dots\alpha_{N-1}} = \epsilon_{\alpha_1\dots\alpha_{N-1}\gamma}\Gamma \tag{67}$$

where the constant Γ is

$$\Gamma = i\sqrt{\frac{1}{2\pi}N\Gamma^N\left(1-\frac{1}{N}\right)}\,.$$

These results are consistent with the picture (c.f. [51]) that the bound state of $N-1$ particles of rank 1 is to be identified with the anti-particle of rank 1. As already remarked, the physical aspects of these facts will be discussed elsewhere [45].

For later convenience we consider as a generalization of (43) the total N-particle S-matrix (consisting of $N(N-1)/2$ factors) in terms of $\tilde{S}$

$$\tilde{S}_{12\ldots N} = \left(\tilde{S}_{12}\tilde{S}_{13}\ldots\tilde{S}_{1N}\right)\left(\tilde{S}_{23}\ldots\tilde{S}_{2N}\right)\ldots\tilde{S}_{N-1N} \tag{68}$$

in the limit $\theta_{jj+1} \to i\eta$ ($j = 1,\ldots N-1$) which behaves as

$$\tilde{S}_{\alpha_1\alpha_2\ldots\alpha_N}^{\beta_N\ldots\beta_2\beta_1} \approx (N-1)!\frac{i\eta}{\theta_{12}-i\eta}\cdots\frac{i\eta}{\theta_{N-1N}-i\eta}\,\epsilon^{\beta_N\ldots\beta_2\beta_1}\epsilon_{\alpha_1\alpha_2\ldots\alpha_N}\,. \tag{69}$$

The algebraic structure of this relation follows because one can use the Yang-Baxter equations to shift in (68) any factor $\tilde{S}_{ii+1} \sim (1-\mathbf{P})_{ii+1}$ to the right or the left. Therefore the expression is totally anti-symmetric with respect to the α_i and the β_i. The factor follows from (4).

4.2 The general form factor formula

In order to obtain a recursion relation where only form factors for the fundamental particles of type α (which transform as the $SU(N)$ vector representation) are involved, we have to apply iteratively the bound state relation (iv) to get the $(N-1)$-bound state which is to be identified with the anti-particle

$$\begin{aligned}&\operatorname*{Res}_{\theta_{12}=i\eta}\cdots\operatorname*{Res}_{\theta_{N-2N-1}=i\eta} F_{1\ldots n}^{\mathcal{O}}(\underline{\theta})\\ &\qquad= F_{(1\ldots N-1)N\ldots n}^{\mathcal{O}}(\theta_{(1\ldots N-1)},\theta_N,\ldots,\theta_n)\sqrt{2}^{N-2}\Gamma_{1\ldots N-1}^{(1\ldots N-1)}\end{aligned}$$

and finally the annihilation residue equation (iii)

$$\begin{aligned}&\operatorname*{Res}_{\theta_{(1\ldots N-1)N}=i\pi} F_{(1\ldots N-1)N\ldots n}^{\mathcal{O}}(\theta_{(1\ldots N-1)},\theta_N,\ldots,\theta_n)\\ &\qquad= 2i\mathbf{C}_{(1\ldots N-1)N}F_{N+1\ldots n}^{\mathcal{O}}(\theta_{N+1},\ldots,\theta_n)\left(\mathbf{1}-\sigma_N^{\mathcal{O}}S_{Nn}\ldots S_{NN+1}\right).\end{aligned}$$

Similar as for $N=3$ we obtain with (67)

$$\begin{aligned}&\operatorname*{Res}_{\theta_{N-1N}=i\eta}\cdots\operatorname*{Res}_{\theta_{12}=i\eta} F_{123\ldots n}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)\\ &\qquad= 2i\varepsilon_{1\ldots N}\sqrt{2}^{N-2}\Gamma F_{N+1\ldots n}^{\mathcal{O}}(\theta_{N+1},\ldots,\theta_n)\left(\mathbf{1}-\sigma_N^{\mathcal{O}}S_{Nn}\ldots S_{NN+1}\right).\end{aligned} \tag{70}$$

ansatz for form factors: We propose the n-particle form factors of an operator $\mathcal{O}(x)$ as given by the same formula (16) as for $SU(2)$ and $SU(3)$ in terms of the K-function and the minimal form factor function $F(\theta)$ given by (12) which belongs to the highest weight $w = (2,0,\ldots,0)$. The form factor equations (i) and (ii) for K-function write again as (17) and (18). Consistency of (70), (18) and the crossing relation (66) means that the statistics factors are of the form

$\sigma_\alpha^{\mathcal{O}} = \sigma^{\mathcal{O}}(r_\alpha)$ if the particle of type α belongs to an $SU(N)$ representation of rank $r_\alpha = 1, \dots, N-1$ and

$$\sigma^{\mathcal{O}}(r) = e^{i\pi(1-1/N)rQ_{\mathcal{O}}} \quad \text{for} \quad Q_{\mathcal{O}} = n \bmod N \tag{71}$$

as an extension of (33) and (45). For the K-function $K_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ we propose again the ansatz in form of the integral representation (22) with (23) and (24). The Bethe ansatz co-vector $\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ is again of the form (46) and for the function $L_{\underline{\beta}}(\underline{z})$ one makes again an analogous ansatz as for the K-function (22) where the indices run over sets with one element less. For the $SU(N)$ case we have to iterate this $N-2$ times

$$\begin{aligned} \tilde{\Psi}_{\underline{\alpha}}^{(l-1)}(\underline{\theta}, \underline{z}) &= L_{\underline{\beta}}^{(l)}(\underline{z}) \tilde{\Phi}_{\underline{\alpha}}^{(l-1)\underline{\beta}}(\underline{\theta}, \underline{z}) \\ L_{\underline{\beta}}^{(l)}(\underline{z}) &= \frac{1}{k!} \int_{\mathcal{C}_{\underline{z}}} \frac{du_1}{R} \cdots \int_{\mathcal{C}_{\underline{z}}} \frac{du_k}{R} \, \tilde{h}(\underline{z}, \underline{u}) p^{(l)}(\underline{z}, \underline{u}) \, \tilde{\Psi}_{\underline{\beta}}^{(l)}(\underline{z}, \underline{u}) \\ \tilde{h}(\underline{z}, \underline{u}) &= \prod_{i=1}^{m} \prod_{j=1}^{k} \tilde{\phi}(\theta_i - z_j) \prod_{1 \le i < j \le k} \tau(z_i - z_j) \end{aligned} \tag{72}$$

for $l = 1, \dots, N-2$, $\alpha_i = l, \dots, N$, $\beta_i = l+1, \dots, N$, $(\tilde{\Psi}^{(0)} = \tilde{\Psi})$. By this procedure one obtains the nested Bethe ansatz[5].

As for the $Z(N)$ and $A(N-1)$ models [39, 40] the 'Jost-function' $\phi(\theta) = \tilde{\phi}(\theta)/a(\theta) = \tilde{\phi}(-\theta)$ is a solution of the equation

$$\prod_{k=0}^{N-2} \phi(\theta + ki\eta) \prod_{k=0}^{N-1} F(\theta + ki\eta) = 1 \tag{73}$$

which is typical for models where the bound state of $N-1$ particles is the anti-particle. The solution is

$$\phi(z) = \Gamma\left(\frac{z}{2\pi i}\right) \Gamma\left(1 - \frac{1}{N} - \frac{z}{2\pi i}\right)$$

and again we define $\tau(z) = 1/(\phi(z)\phi(-z))$.

The higher level Bethe ansatz: In order that the form factor $F^{\mathcal{O}}(\underline{\theta})$ given by (16) and (72) satisfies the form factor equations (i) - (iii) the higher level L-functions $L^{(l)}_{\beta_1,\beta_2,\dots,\beta_m}(z_1, z_2, \dots, z_m)$, $(l < \beta_i \le N, \ m = n_l)$ have to satisfy:

(i)$^{(l)}$ Watson's equations

$$L^{(l)}_{\dots ij \dots}(\dots, z_i, z_j, \dots) = L^{(l)}_{\dots ji \dots}(\dots, z_j, z_i, \dots) \tilde{S}_{ij}(z_{ij}),$$

(ii)$^{(l)}$ the crossing relation

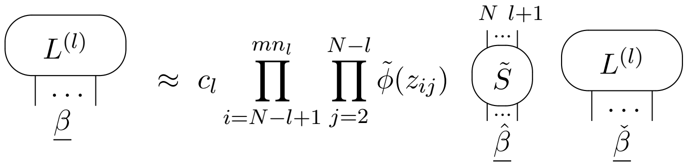
$$L^{(l)}_{\beta_1,\beta_2,\dots,\beta_m}(z_1, z_2, \dots, z_m) = L^{(l)}_{\beta_2,\dots,\beta_m,\beta_1}(z_2, \dots, z_m, z_1 - 2\pi i)$$

and

[5] In [41] the nested off-shell Bethe ansatz was formulated in terms of "Jackson-type integrals" instead of contour integrals.

(iii)$^{(l)}$ the function $L^{(l)}_{\underline{\beta}}(\underline{z})$ has to possess simple poles at $z_{12},\ldots,z_{N-l-1,N-l} = i\eta$ and it has to factorize in the neighborhood of these poles as

$$L^{(l)}_{\underline{\beta}}(\underline{z}) \approx c_l \prod_{i=N-l+1}^{m} \prod_{j=2}^{N-l} \tilde{\phi}(z_{ij})\, \tilde{S}^{N\ldots l+1}_{\underline{\hat{\beta}}}(\underline{\hat{z}})\, L_{\underline{\check{\beta}}}(\underline{\check{z}}) \tag{74}$$



where we have used the short notations $\underline{\hat{\beta}} = (\beta_1,\ldots,\beta_{N-l})$, $\underline{\check{\beta}} = (\beta_{N-l+1},\ldots,\beta_m)$ and $\underline{\hat{z}} = (z_1,\ldots,z_{N-l})$, $\underline{\check{z}} = (z_{N-l+1},\ldots,z_m)$. The S-matrix $\tilde{S}^{N\ldots l+1}_{\underline{\hat{\beta}}}(\underline{\hat{z}})$ describes the total scattering of the $N-l$ particles with rapidities $\underline{\hat{z}}$ for the initial quantum numbers $\underline{\hat{\beta}}$ and the final configuration of quantum numbers $(N,\ldots,l+1)$.

The constants satisfy the recursion relation

$$c_l = c_{l+1} \prod_{j=1}^{N-l-1} \tilde{\phi}(ji\eta) \tag{75}$$

with the solution

$$c_1 = \tilde{\phi}^{N-2}(i\eta)\tilde{\phi}^{N-3}(2i\eta)\cdots\tilde{\phi}((N-2)i\eta) \tag{76}$$

if $c_{N-1} = 1$ (see appendix C).

Lemma 4 *The higher level function* $L^{(l)}_{\underline{\beta}}(\underline{z})$ *satisfies* (i)$^{(l)}$ - (iii)$^{(l)}$*, if the higher level p-function in the integral representation (72) satisfies*

$$\begin{array}{ll} (\mathrm{i}')^{(l)} & p^{(l)}(\underline{z},\underline{u}) \text{ is symmetric under } z_i \leftrightarrow z_j \\ (\mathrm{ii}')^{(l)} & \begin{cases} p^{(l)}_{mk}(z_1+2\pi i, z_2,\ldots,\underline{u}) = (-1)^k p^{(l)}_{mk}(z_1,z_2,\ldots,\underline{u}) \\ p^{(l)}_{mk}(\underline{z}, u_1+2\pi i, u_2,\ldots) = (-1)^m p^{(l)}_{mk}(\underline{z},u_1,u_2,\ldots) \end{cases} \\ (\mathrm{iii}')^{(l)} & p^{(l)}_{mk}(\underline{z}, z_2,\ldots z_{N-l},\underline{\check{u}})|_{z_{12}=\cdots=z_{z_{N-l-1,N-l}}=i\eta} = (-1)^{k-N-l+1}\, p^{(l)}_{\check{m}\check{k}}(\underline{\check{z}},\underline{\check{u}}) \end{array} \tag{77}$$

with $\check{m} = M-N+l$, $\check{k} = M-N+l-1$.

This lemma is proved in appendix C.

The p-function: In order that the form factor $F^{\mathcal{O}}(\theta)$ given by (16) and (22) - (24) satisfies the form factor equations (i) - (iii) the p-function $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ in (22) which depends on the explicit nature of the local operator $\mathcal{O}$ is assumed to satisfy

$$\begin{array}{ll}(\mathrm{i}') & p_{nm}(\underline{\theta},\underline{z}) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j \\ (\mathrm{ii}') & \begin{cases} \sigma p_{nm}(\theta_1+2\pi i,\theta_2,\dots,\underline{z}) = (-1)^{m+N-1}p_{nm}(\theta_1,\theta_2,\dots,\underline{z}) \\ p_{nm}(\underline{\theta},z_1+2\pi i,z_2,\dots) = (-1)^n p_{nm}(\underline{\theta},z_1,z_2,\dots) \end{cases} \\ (\mathrm{iii}') & p_{nm}(\underline{\theta},\theta_2,\dots,\theta_N,\underline{\check{z}})|_{\theta_{12}=\dots=\theta_{N-1N}=i\eta} = (-1)^{m-N+1}\,p_{n-Nm-N+1}(\underline{\check{\theta}},\underline{\check{z}}) \\ & p_{nm}(\underline{\theta},\theta_1,\dots,\theta_{N-1},\underline{\check{z}})|_{\theta_{12}=\dots=\theta_{N-1N}=i\eta} = \sigma\, p_{n-Nm-N+1}(\underline{\check{\theta}},\underline{\check{z}}) \end{array} \tag{78}$$

where $\underline{\theta} = (\theta_1,\dots,\theta_n)$, $\underline{\check{\theta}} = (\theta_{N+1},\dots,\theta_n)$, $\underline{z} = (z_1,\dots,z_m)$ and $\underline{\check{z}} = (z_N,\dots,z_m)$. In order to simplify the notation, we have in these equations suppressed the dependence of the p-function $p^{\mathcal{O}}$ and the statistics factor $\sigma^{\mathcal{O}}$ on the operator $\mathcal{O}(x)$.

Theorem 5 *The co-vector valued function $F_{\underline{\alpha}}(\underline{\theta})$ given by the ansatz (16) and the integral representation (22) satisfies the form factor equations* (i), (ii) *and* (iii) *of (7) – (9), in particular (70) if*

1. *$L_{\underline{\beta}}(\underline{z})$ satisfies the equation* $(\mathrm{i})^{(1)}$, $(\mathrm{ii})^{(1)}$, $(\mathrm{iii})^{(1)}$ *and (76) of lemma 4,*
2. *$p^{\mathcal{O}}(\underline{\theta},\underline{z})$ satisfies the equation* (i′), (ii′) *and* (iii′) *of (78) and*
3. *the normalization constants satisfy*

$$(N-1)!(i\eta)^{N-1}\left(\prod_{j=1}^{N-1}\left(\tilde{\phi}(ji\eta)F(ji\eta)\right)^{N-j}\right)N_n = 2i\sqrt{2}^{N-2}\Gamma N_{n-N}\,. \tag{79}$$

The proof of this theorem can be found in appendix B.

4.3 Examples

To illustrate our general results we present some simple examples.

The energy momentum tensor: For the local operator $\mathcal{O}(x) = T^{\rho\sigma}(x)$ (where $\rho,\sigma = \pm$ denote the light cone components) the p-function is, as for the sine-Gordon model in [53]

$$p^{T^{\rho\sigma}}(\underline{\theta},\underline{z}) = \sum_{i=1}^{n} e^{\rho\theta_i}\sum_{i=1}^{m} e^{\sigma z_i}\,.$$

For the $n=N$ particle form factor there are $n_l = N-l$ integrations in the l-th level of the off-shell Bethe ansatz. The $SU(N)$ weights are (see [41])

$$w = (n-n_1, n_1-n_2,\dots,n_{N-2}-n_{N-1},n_{N-1}) = (1,1,\dots,1,1)\,.$$

We calculate the form factor of the particle α and the bound state $(\underline{\beta}) = (\beta_1,\dots,\beta_{N-1})$ of $N-1$ particles. In each level all integrations up to one may be performed iteratively using the bound state relation (iv) (similar as in the proof of theorem 5). Then all remaining integrations in the higher levels can be done by means of the formula

$$\int_{-i\infty}^{i\infty} ds\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) = 2\pi i \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\,.$$

The result for the form factor of the particle α and the bound state $(\underline{\beta})$ writes as

$$\begin{aligned} F_{\alpha(\underline{\beta})}^{T^{\rho\sigma}}(\theta_1,\theta_2) &= K_{\alpha(\underline{\beta})}^{T^{\rho\sigma}}(\theta_1,\theta_2)\,G(\theta_{12}) \\ K_{\alpha(\underline{\beta})}^{T^{\rho\sigma}}(\theta_1,\theta_2) &= N_2^{T^{\rho\sigma}}\left(e^{\rho\theta_1}+e^{\rho\theta_2}\right)\int_{\mathcal{C}_{\underline{\theta}}} \frac{dz}{R}\tilde{\phi}(\theta_1-z)e^{\sigma z}L(\theta_2-z) \\ &\quad\times \epsilon_{\delta\underline{\gamma}}\tilde{S}_{\alpha\epsilon}^{\delta 1}(\theta_1-z)\tilde{S}_{(\underline{\beta})1}^{\epsilon(\underline{\gamma})}(\theta_2-z) \end{aligned} \tag{80}$$

where the summation is over $\underline{\gamma}$ and $\delta>1$ and $G(\theta)$ is the minimal form factor function of two particles of rank $r=1$ and $r=N-1$. The functions $G(\theta)$ and $L(\theta)$ are given by

$$\begin{aligned} G(i\pi-\theta)F(\theta)\phi(\theta) &= 1 \\ L(\theta) &= \Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(-\frac{1}{2}+\frac{1}{N}-\frac{\theta}{2\pi i}\right). \end{aligned}$$

The remaining integral in(80) may be performed (similar as in [53]) with the result[6]

$$\langle\, 0\,|\, T^{\rho\sigma}(0)\,|\,\theta_1,\theta_2\,\rangle_{\alpha(\underline{\beta})}^{in} = 4M^2\epsilon_{\alpha\underline{\beta}}e^{\frac{1}{2}(\rho+\sigma)(\theta_1+\theta_2+i\pi)}\frac{\sinh\frac{1}{2}(\theta_{12}-i\pi)}{\theta_{12}-i\pi}G(\theta_{12})\,.$$

Similar as in [53] one can prove the eigenvalue equation

$$\left(\int dx T^{\pm 0}(x) - \sum_{i=1}^{n} p_i^{\pm}\right)|\,\theta_1,\dots,\theta_n\rangle_{\underline{\alpha}}^{in} = 0$$

for arbitrary states.

The fields $\psi_\alpha(x)$: Because the Bethe ansatz yields highest weight states we obtain the matrix elements of the spinor field $\psi(x)=\psi_1(x)$. The p-function for the local operator $\psi^{(\pm)}(x)$ is (see also [38])

$$p^{\psi^{(\pm)}}(\underline{\theta},\underline{z}) = \exp\pm\frac{1}{2}\left(\sum_{i=1}^{m} z_i - \left(1-\frac{1}{N}\right)\sum_{i=1}^{n}\theta_i\right).$$

For example the 1-particle form factor is

$$\langle\, 0\,|\,\psi^{(\pm)}(0)\,|\,\theta\,\rangle_\alpha = \delta_{\alpha 1}\, e^{\mp\frac{1}{2}(1-\frac{1}{N})\theta}\,.$$

[6] In [58, 41] this result has been obtained using Jackson type integrals.

The last two formulae are consistent with the proposal of Swieca et al. [51, 44] that the statistics of the fundamental particles in the chiral $SU(N)$ Gross-Neveu model should be $\sigma = \exp(2\pi i s)$, where $s = \frac{1}{2}\left(1 - \frac{1}{N}\right)$ is the spin (see also (71)). For the $n = N + 1$ particle form factor there are again $n_l = N - l$ integrations in the l-th level of the off-shell Bethe ansatz and the $SU(N)$ weights are $w = (2, 1, \ldots, 1, 1)$. Similar as above one obtains the 2-particle and 1-bound state form factor

$$F^{\psi(\pm)}_{\alpha\beta(\underline{\gamma})}(\theta_1, \theta_2, \theta_3) = K^{\psi(\pm)}_{\alpha\beta(\underline{\gamma})}(\theta_1, \theta_2, \theta_3) F(\theta_{12}) G(\theta_{13}) G(\theta_{23})$$

$$K^{\psi(\pm)}_{\alpha\beta(\underline{\gamma})} = N^{\psi} e^{\mp\frac{1}{2}\left(1-\frac{1}{N}\right)\sum\theta_i} \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz}{R} \tilde{\phi}(\theta_1 - z)\tilde{\phi}(\theta_2 - z) L(\theta_3 - z) e^{\pm\frac{1}{2}z}$$

$$\times\, \epsilon_{\delta\underline{\gamma}} \tilde{S}^{\delta 1}_{\alpha_1\epsilon}(\theta_1 - z) \tilde{S}^{\epsilon 1}_{\alpha_2\zeta}(\theta_2 - z) \tilde{S}^{\zeta(\underline{\gamma})}_{(\underline{\beta})1}(\theta_3 - z)\,.$$

We were not able to perform this integration. In [45] we will discuss the $1/N$ expansion. There we will also discuss the physical interpretation of the results for the chiral Gross-Neveu model.

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Appendix

A A Lemma

Lemma 6 *Let $v^{1\ldots n} \in V^{1\ldots n}$ be a highest weight vector, i.e. $E_{1\ldots n} v^{1\ldots n} = 0$. Then $v^{1\ldots n}$ vanishes if the components $v^{\alpha_1\alpha_2\ldots\alpha_n}$ for $\alpha_1 = 1$ vanish.*

Proof. By definition E acts on the basis vectors $e^{1\ldots n}_{\underline{\alpha}} = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n}$ as

$$E_{1\ldots n} e^{1\ldots n}_{\underline{\alpha}} = \sum_{i=1}^{n} e_{\alpha_1} \otimes \cdots \otimes E e_{\alpha_i} \otimes \cdots \otimes e_{\alpha_n}$$

$$E e_{\alpha} = e_{\alpha-1} \quad \text{where } e_0 = 0\,.$$

Therefore we may write

$$\begin{aligned}0 = E_{1\dots n} v^{1\dots n} &= \sum_{\underline{\alpha},\alpha_1>1} v^{\alpha_1\alpha_2\dots\alpha_n} E_{1\dots n} e^{1\dots n}_{\underline{\alpha}} \\ &= \sum_{\underline{\alpha},\alpha_1>1} v^{\alpha_1\alpha_2\dots\alpha_n} \sum_{i=1}^{n} e_{\alpha_1}\otimes\dots\otimes e_{\alpha_i-1}\otimes\dots\otimes e_{\alpha_n} \\ &= \sum_{\underline{\alpha},\alpha_1=2} v^{2\alpha_2\dots\alpha_n} e_1\otimes\dots\otimes e_{\alpha_n} + \sum_{\underline{\alpha},\alpha_1>2} w^{\alpha_1\alpha_2\dots\alpha_n} e_{\alpha_1}\otimes\dots\otimes e_{\alpha_n}\end{aligned}$$

for some vector $w^{1\dots n}$. Because the second term $w^{1\dots n}$ is orthogonal to the first one all components $v^{2\alpha_2\dots\alpha_n}$ vanish. Iterating this procedure proves the claim. ■

B Proof of Theorem 5

The proof of the main theorem of this article for $SU(N)$ is a straightforward extension of the one for $SU(3)$ above.

Proof. The form factor equations (i) and (ii) may be proved quite analogously as for $SU(3)$. The proof of (iii) is similar to that for the $Z(N)$ model in [40]. We use the short hand notations

$$\begin{aligned}&\underline{\theta} = (\theta_1,\dots,\theta_n),\ \underline{\hat\theta} = (\theta_1,\dots,\theta_N),\ \underline{\check\theta} = (\theta_{N+1},\dots,\theta_n), \\ &\underline{\alpha} = (\alpha_1,\dots,\alpha_n),\ \underline{\hat\alpha} = (\alpha_1,\dots,\alpha_N),\ \underline{\check\alpha} = (\alpha_{N+1},\dots,\alpha_n), \\ &\underline{z} = (z_1,\dots,z_m),\ \underline{\hat z} = (z_1,\dots,z_{N-1}),\ \underline{\check z} = (z_N,\dots,z_m), \\ &\underline{\beta} = (\beta_1,\dots,\beta_m),\ \underline{\hat\beta} = (\beta_1,\dots,\beta_{N-1}),\ \underline{\check\beta} = (\beta_N,\dots,\beta_m)\,.\end{aligned}$$

That $F^{\mathcal{O}}_{1\dots n}(\underline{\theta})$ given by (16), (22), (23) and (72) satisfies (iii) in the form of (70) is equivalent to that $K_{1\dots n}(\underline{\theta})$ satisfies

$$\operatorname*{Res}_{\theta_{N-1N}=i\eta}\dots\operatorname*{Res}_{\theta_{12}=i\eta} K_{1\dots n}(\underline{\theta}) = c_0 \prod_{i=N+1}^{n}\prod_{j=2}^{N}\tilde\phi(\theta_{ij})\,\varepsilon_{1\dots N} K_{N+1\dots n}(\underline{\check\theta}) \times (\mathbf{1} - \sigma_N S_{Nn}\dots S_{NN+1}) \quad (81)$$

$$c_0 = 2i\sqrt{2}^{N-2}\Gamma\prod_{j=1}^{N-1} F^{-(N-j)}(ji\eta)$$

where the relation of $F(z)$ and $\phi(z)$ given by (73) has been used. The residues of $K_{1\dots n}(\underline{\theta})$ consists again of three terms

$$\operatorname*{Res}_{\theta_{N-1N}=i\eta}\dots\operatorname*{Res}_{\theta_{12}=i\eta} K_{1\dots n}(\underline{\theta}) = R^{(1)}_{1\dots n} + R^{(2)}_{1\dots n} + R^{(3)}_{1\dots n}$$

because $N-1$ of the z integration contours will be "pinched" at three points. Again due to symmetry it is sufficient to determine the contribution from the $z_1,\dots,\ z_{N-1}$-integrations and multiply the result by $m\dots(m-N+2)$. The pinching points are

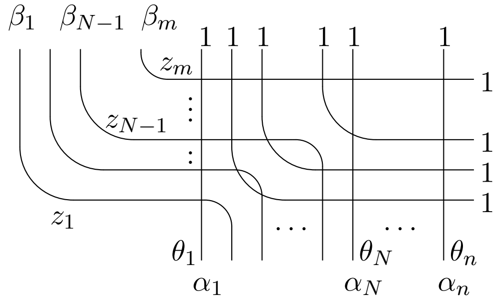
(1) $z_1 = \theta_2, \dots, z_{N-1} = \theta_N$,

(2) $z_1 = \theta_1, \dots, z_{N-1} = \theta_{N-1}$,

(3) $z_1 = \theta_2 - i\eta, \dots, z_{N-1} = \theta_N - i\eta$,

The contribution of (1) is given by $N-1$ integrations along small circles around $z_1 = \theta_2, z_2 = \theta_3, \dots, z_{N-1} = \theta_N$ (see figure 1). The S-matrices $\tilde S(\theta_2 - z_1), \dots, \tilde S(\theta_N - z_{N-1})$ yield the permutation operator $\tilde S(0) = \mathbf{P}$. Therefore for $\theta_{12}, \dots, \theta_{N-2N-1}, \theta_{N-1N} \to i\eta$

$$\left(\Omega\, \tilde C^{\beta_m}(\underline\theta, z_m) \cdots \tilde C^{\beta_N}(\underline\theta, z_N) \tilde C^{\beta_{N-1}}(\underline\theta, \theta_N) \cdots \tilde C^{\beta_1}(\underline\theta, \theta_2)\right)_{\underline\alpha}$$

$=$ 

$$= \prod_{j=N}^{m} \tilde b(\theta_1 - z_j) \left(\tilde S_{1N} \dots \tilde S_{12}\right)_{\hat{\underline\alpha}}^{\hat{\underline\beta},1} \left(\Omega \tilde C^{\beta_m}(\check{\underline\theta}, z_m) \cdots \tilde C^{\beta_N}(\check{\underline\theta}, z_N)\right)_{\check{\underline\alpha}} . \quad (82)$$

It has been used that due to the $SU(N)$ ice rule only the amplitude $b(\cdot)$ contributes to the S-matrices $S(\theta_1 - z_j)$ and $a(\cdot)$ to the S-matrices $S(\theta_2 - z_j), \dots, S(\theta_N - z_j), S(\theta_i - z_1), \dots, S(\theta_i - z_{N-2})$ after having applied Yang-Baxter relations. One observes that the product of S-matrices in (74) together with the one in (82) yields the total $N-1$ S-matrix

$$\tilde S_{\underline{\hat\beta}}^{N\dots 2}(\hat{\underline z}) \left(\tilde S_{1N} \dots \tilde S_{12}\right)_{\hat{\underline\alpha}}^{\hat{\underline\beta},1} = \tilde S_{\hat{\underline\alpha}}^{N\dots 21}(\hat{\underline\theta})$$

for which the residue formula (69) applies

$$\operatorname*{Res}_{\theta_{12}=i\eta} \dots \operatorname*{Res}_{\theta_{N-1N}=i\eta} \tilde S_{\alpha_1\dots\alpha_N}^{N\dots 21} = (N-1)!(i\eta)^{N-1} \epsilon^{N\dots 21} \epsilon_{\alpha_1\dots\alpha_N} .$$

We combine (82) with the function $L_{\underline\beta}(\underline z)$ with the property (74) for ($l = 1$) and the scalar functions $\tilde h$ and p and after having performed the remaining z_j-integrations we obtain

$$R^{(1)}_{1\dots n} = c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde\phi(\theta_{ij})\, \varepsilon_{1\dots N}(\hat{\underline\theta}) K_{N+1\dots n}(\check{\underline\theta}) .$$

We have used the following equations $\tilde b(\theta_1 - z)\tilde\phi(\theta_1 - z) = -\tilde\phi(z - \theta_2)$ (for $\theta_{12} = i\eta$) together with the definition (35) of $\tau(z)$, the relation (iii') of (78) for the p-function $p(\underline\theta, \theta_2, \dots, \theta_N, \check{\underline z}) = (-1)^{m-N+1} p(\check{\underline\theta}, \check{\underline z})$ (for $\theta_{12} = \dots =$

$\theta_{N-1N} = i\eta$), the relation (79) for the normalization constants $N_n c_1 (N-1)!(i\eta)^{N-1} \prod_{j=1}^{N-1} \tilde{\phi}(ji\eta) = N_{n-N} c_0$ and the recursion relation (75) for the constants c_l with the solution (86).

The remaining contribution to (81) is due to R_2 and R_3

$$R^{(2)}_{1\dots n} + R^{(3)}_{1\dots n} = -c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde{\phi}(\theta_{ij})\, \varepsilon_{1\dots N} K_{N+1\dots n}(\underline{\check{\theta}}) \sigma_N S_{Nn} \dots S_{NN+1}$$

It is convenient to shift the particle with momentum θ_N to the right by applying S-matrices and write

$$\left(R^{(2)}_{1\dots n} + R^{(3)}_{1\dots n} \right) S_{N+1N} \dots S_{nN} + c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde{\phi}(\theta_{ij})\, \varepsilon_{1\dots N} K_{N+1\dots n}(\underline{\check{\theta}}) \sigma_N = 0 \tag{83}$$

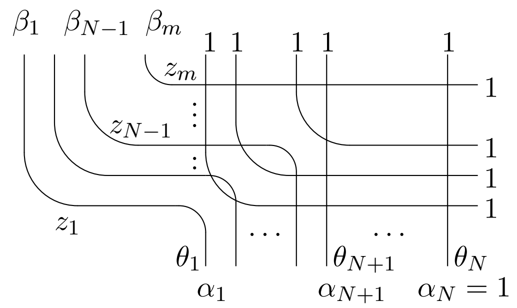
where the components of this co-vector are now denoted by $v_{\alpha_1 \dots \alpha_{N-1 N+1} \dots \alpha_n \alpha_N}$. Note that because of (i) (17)

$$K_{1\dots n}(\underline{\theta}) S_{N+1N} \dots S_{nN} = \prod_{i=N+1}^{n} a(\theta_{iN}) K_{1\dots N+1\dots nN}(\theta_1, \dots, \theta_{N+1}, \dots, \theta_n, \theta_N)\,.$$

Again due to Lemma 6 in appendix A it is sufficient to prove equation (83) only for $\alpha_N = 1$ since the left hand side of this equation is a highest weight co-vector. This is because Bethe ansatz vectors, in particular also off-shell Bethe ansatz vectors are of highest weight (see [57]). Therefore we consider this equation for the components with $\alpha_N = 1$ only. The contribution of $R^{(2)}_{1\dots n}$ is given by the $z_1, \dots, z_{N-1}$-integrations along the small circles around $z_1 = \theta_1, \dots, z_{N-1} = \theta_{N-1}$ (see again figure 1). Now $\tilde{S}(\theta_1 - z_1)_1, \dots, \tilde{S}(\theta_{N-1} - z_{N-1})$ yield permutation operators $\mathbf{P}$ and the co-vector part of this contribution for $\alpha_N = 1$ is

$$\left(\Omega\, \tilde{C}^{\beta_m}(\underline{\theta}, z_m) \cdots \tilde{C}^{\beta_{N-1}}(\underline{\theta}, \theta_{N-1}) \cdots \tilde{C}^{\beta_1}(\underline{\theta}, \theta_1) P_N(1) \right)_{\alpha_1 \dots \alpha_{N-1 N+1} \dots \alpha_n \alpha_N}$$

$=$



$$= \delta^{\beta_1}_{\alpha_1} \cdots \delta^{\beta_{N-1}}_{\alpha_{N-1}} \left(\Omega \tilde{C}^{\beta_m}(\underline{\check{\theta}}, z_m) \cdots \tilde{C}^{\beta_N}(\underline{\check{\theta}}, z_N) \right)_{\underline{\check{\alpha}}} \delta^1_{\alpha_N} \tag{84}$$

where $P_N(1)$ projects onto the components with $\alpha_N = 1$. We have used the fact that because of the $SU(N)$ ice rule the amplitude $a(\cdot)$ only contributes to the S-matrices $S(\theta_1 - z_j), S(\theta_2 - z_j), S(\theta_N - z_j), S(\theta_i - z_1)$ after having applied

Yang-Baxter relations. We use $\tilde{\phi}(\theta) = -\tilde{b}(\theta + 2\pi i)\tilde{\phi}(\theta + 2\pi i)$ to replace for $i = 1, \dots, N-1$ and $\beta \neq 1$

$$\tilde{\phi}(\theta_{Ni}) = \tilde{S}_{11}^{11}(\theta_{Ni})\tilde{\phi}(\theta_{Ni}) = -\tilde{S}_{1\beta}^{\beta 1}(\theta_{Ni} + 2\pi i)\tilde{\phi}(\theta_{Ni} + 2\pi i)$$

therefore using again (69) we obtain

$$\begin{aligned}
&\operatorname*{Res}_{\theta_{N-1N}=i\eta} \cdots \operatorname*{Res}_{\theta_{12}=i\eta} \tilde{\phi}(\theta_{N1}) \cdots \tilde{\phi}(\theta_{NN-1})\tilde{S}^{N\dots 2}_{\alpha_1\dots\alpha_{N-1}}(\theta_1, \cdots, \theta_{N-1}) \\
&= (-1)^{N-1}\tilde{\phi}(i\eta) \cdots \tilde{\phi}((N-1)i\eta) \\
&\times \operatorname*{Res}_{\theta_{N-1N}=i\eta} \cdots \operatorname*{Res}_{\theta_{12}=i\eta} \tilde{S}^{N\cdots 21}_{1\alpha_1\cdots\alpha_{N-1}}(\theta_N + 2\pi i, \theta_1, \cdots, \theta_{N-1}) \\
&= -\tilde{\phi}(i\eta) \cdots \tilde{\phi}((N-1)i\eta)(N-1)!(i\eta)^{N-1}\epsilon_{\alpha_1\cdots\alpha_{N-1}1}\,.
\end{aligned}$$

We combine (84) with the function $L_{\underline{\beta}}(\underline{z})$ with the property (74) (for $l = 1$) and the scalar functions $\tilde{h}$ and p and after having performed the remaining z_j-integrations we obtain

$$R^{(2)}_{1\dots n}S_{N+1N}\dots S_{nN}P_N(1) = -c_0 \prod_{i=N+1}^{n}\prod_{j=2}^{N} \tilde{\phi}(\theta_{ij})\, \varepsilon_{1\dots N}(\underline{\hat{\theta}})K_{N+1\dots n}(\underline{\check{\theta}})\sigma_N P_N(1)\,.$$

We have used the following equations: $a(\theta_{iN})\tilde{\phi}(\theta_{i1}) = \tilde{\phi}(\theta_{iN})$ (because of (15)), the definition (35) of $\tau(z)$, the relation (iii') of (78) for the p-function $p(\underline{\theta}, \theta_1, \dots, \theta_{N-1}, \underline{\check{z}}) = \sigma p(\underline{\check{\theta}}, \underline{\check{z}})$ (for $\theta_{12} = \cdots = \theta_{N-1N} = i\eta$), the relation (79) for the normalization constants $N_n c_1 (N-1)!(i\eta)^{N-1}\prod_{j=1}^{N-1}\tilde{\phi}(ji\eta) = N_{n-N}c_0$ and the recursion relation (75) for the constants c_l.

The contribution of pinching (3) is given by the $z_1, \dots, z_{N-1}$-integrations along the small circles around $z_1 = \theta_2 - i\eta, \dots, z_{N-2} = \theta_{N-1} - i\eta$, $z_{N-1} = \theta_N - i\eta$, (see again figure 1). As for $N = 3$ the S-matrix $\tilde{S}^{\delta\gamma}_{\alpha\beta}(\theta_N - z_{N-1})$ yields $\Gamma^{\delta\gamma}_{(\rho\sigma)}\Gamma^{(\rho\sigma)}_{\alpha\beta}$ and the residue of the Bethe ansatz state vanishes,

$$\operatorname*{Res}_{z_{N-1}=\theta_N - i\eta}\Big(\Omega\, \tilde{C}(\underline{\theta}, z_m)\cdots\tilde{C}(\underline{\theta}, z_1)P_N(1)\Big)_{\underline{\alpha}} = 0\,,$$

because $\Gamma^{(\rho\sigma)}_{\alpha\beta}$ is antisymmetric with respect to α, β. Therefore equation (83) is proved for $\alpha_N = 1$ and because of Lemma 6 also in general. ∎

C The higher level Bethe ansatz

Proof of lemma 3: For the higher level functions $L^{(l)}_{\underline{\beta}}(\underline{z})$ one may verify the equations $(\mathrm{i})^{(l)}$ and $(\mathrm{ii})^{(l)}$ quite analogously to the corresponding ones for the main theorem (e.g. for $N = 3$).

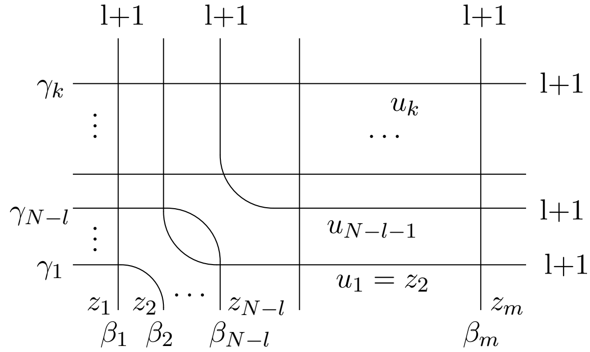
We prove $(\mathrm{iii})^{(l)}$ by induction and assume:

$$L^{(l+1)}_{\underline{\gamma}}(\underline{u}) \approx c_{l+1}\tilde{S}^{N\dots l+2}_{\underline{\hat{\gamma}}}(\underline{\hat{u}}) \prod_{i=N-l}^{k}\prod_{j=2}^{N-l-1}\tilde{\phi}(u_{ij})L^{(l+1)}_{\underline{\check{\gamma}}}(\underline{\check{u}}) \tag{85}$$

for $u_{12},\dots,u_{N-l-2,N-2}\to i\eta$. In the integral representation (72) of $L^{(l)}_{\underline{\beta}}(\underline{z})$ there are pinchings at $u_1=z_2,\dots,u_{N-l-1}=z_{N-l}$ if $z_{12},\dots,z_{N-l-1,N-l}\to i\eta$. Therefore in $\tilde\Psi^{(l)}_{\underline{\beta}}(\underline{z},\underline{u})$ the S-matrices $\tilde S(z_2-u_1),\dots,\tilde S(z_{N-l}-u_{N-l-1})$ yield the permutation operator $\mathbf{P}$ and we have to consider

$$\tilde\Phi^{(l)\underline{\gamma}}_{\underline{\beta}}(\underline{z},\underline{u})=\left(\Omega^{(l)}\,\tilde C^{(l)\gamma_k}(\underline{z},u_k)\cdots\tilde C^{(l)\gamma_{N-l-1}}(\underline{z},z_{N-l})\cdots\tilde C^{(l)\gamma_1}(\underline{z},z_2)\right)_{\underline{\beta}}$$

$$=$$



$$=\left(\tilde S_{12}(z_{12})\dots\tilde S_{1N-l}(z_{1N-l})\right)^{\gamma_1\dots\gamma_{N-l-1}l+1}_{\beta_1\dots\beta_{N-l}}\prod_{j=N-l}^{k}\tilde b(z_1-u_j)\tilde\Phi^{(l)\check{\underline{\gamma}}}_{\check{\underline{\beta}}}(\check{\underline{z}},\check{\underline{u}})$$

where $\check{\underline{\beta}}=(\beta_{N-l+1},\dots,\beta_m)$, $\check{\underline{z}}=(z_{N-l+1},\dots,z_m)$, $\check{\underline{u}}=(u_{N-l},\dots,u_k)$. We may write for $\hat{\underline{u}}=(u_1,\dots,u_{N-l-1})=(z_2,\dots,z_{N-l})$

$$\tilde S^{N\dots l+2}_{\hat{\underline{\gamma}}}(\hat{\underline{u}})\left(\tilde S_{12}(z_{12})\dots\tilde S_{1N-l}(z_{1N-l})\right)^{\gamma_1\dots\gamma_{N-l-1}l+1}_{\beta_1\dots\beta_{N-l}}=\tilde S^{N\dots l+1}_{\hat{\underline{\beta}}}(\hat{\underline{z}})$$

with the notation $\hat{\underline{z}}=(z_1,\dots,z_{N-l})$. Therefore using the assumption (85) we obtain when $z_{12},\dots,z_{N-l-1,N-l}\to i\eta$

$$L^{(l)}_{\underline{\beta}}(\underline{z})\approx\frac{1}{\check k!}\oint_{z_2}\frac{du_1}{R}\cdots\oint_{z_{N-l}}\frac{du_{N-l-1}}{R}\int_{\mathcal{C}_{\underline{z}}}\frac{du_{N-l}}{R}\cdots\int_{\mathcal{C}_{\underline{z}}}\frac{du_k}{R}\,\tilde h(\underline{z},\underline{u})$$
$$\times L^{(l+1)}_{\underline{\gamma}}(\underline{u})\,\tilde\Phi^{(l)\underline{\gamma}}_{\underline{\beta}}(\underline{z},\underline{u})=c_l\tilde S^{N\dots l+1}_{\hat{\underline{\beta}}}(\hat{\underline{z}})\prod_{i=N-l+1}^{m}\prod_{j=2}^{N-l}\tilde\phi(z_{ij})L^{(l)}_{\check{\underline{\beta}}}(\check{\underline{z}})$$

where $\check k=k-N+l+1$. The following formulae have been used: $\tilde b(z_1-u_j)\prod_{i=3}^{N-l}\tilde\phi(u_j-z_i)\prod_{i=1}^{N-l}\tilde\phi(z_i-u_j)\prod_{i=2}^{N-l}\tau(z_i-u_j)=-1$, the relation $(\mathrm{iii}')^{(l)}$ of (77) for the p-function $p^{(l)}_{mk}(\underline{z},z_2,\dots z_{N-l},\check{\underline{u}})=(-1)^{k-N-l+1}p^{(l)}_{\check m\check k}(\check{\underline{z}},\check{\underline{u}})$ (if $z_{12}=\cdots=z_{z_{N-l-1,N-l}}=i\eta$) and the recursion relation $c_l=c_{l+1}\prod_{j=1}^{N-l-1}\tilde\phi(ji\eta)$. The solution to this recursion relation with $c_{N-1}=1$ is

$$c_1=\tilde\phi^{N-2}(i\eta)\tilde\phi^{N-3}(2i\eta)\cdots\tilde\phi((N-2)i\eta)\qquad(86)$$

■

References

[1] H. Bethe, On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain, Z. Phys. **71**, 205–226 (1931).

[2] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The General solution and the ground state, Phys. Rev. **130**, 1605–1616 (1963).

[3] E. H. Lieb, Exact Solution of the Problem of the Entropy of Two-Dimensional Ice, Phys. Rev. Lett. **18**, 692–694 (1967).

[4] E. H. Lieb, Exact solution of the F model of an antiferroelectric, Phys. Rev. Lett. **18**, 1046–1048 (1967).

[5] C. N. Yang and C. P. Yang, One-dimensional chain of anisotropic spin-spin interactions. I: Proof of Bethe's hypothesis for ground state in a finite system, Phys. Rev. **150**, 321–327 (1966).

[6] C. N. Yang, Selected papers, 1945 – 1980, with commentary, W. H. Freeman and Company, San Francisco (1983).

[7] C. N. Yang, Some exact results for the many body problems in one dimension with repulsive delta function interaction, Phys. Rev. Lett. **19**, 1312–1314 (1967).

[8] R. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press (1982).

[9] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, The quantum inverse problem method. 1, Theor. Math. Phys. **40**, 688–706 (1980).

[10] L. A. Takhtajan and L. D. Faddeev, The Quantum method of the inverse problem and the Heisenberg XYZ model, Russ. Math. Surveys **34**, 11–68 (1979).

[11] B. Sutherland, Model for a multicomponent quantum system, Phys. Rev. B **12**(9), 3795–3805 (Nov 1975).

[12] N. Andrei and J. H. Lowenstein, Derivation of the chiral Gross-Neveu spectrum for arbitrary SU(N) symmetry, Phys. Lett. **B90**, 106 (1980).

[13] D. J. Gross and A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. **D10**, 3235 (1974).

[14] O. Babelon, H. J. de Vega, and C. M. Viallet, Exact solution of the Z(N+1) x Z(N+1) symmetric generalization of the XXZ model, Nucl. Phys. **B200**, 266 (1982).

[15] H. J. de Vega and M. Karowski, Exact Bethe ansatz solution of O(2N) symmetric theories, Nucl. Phys. **B280**, 225 (1987).

[16] H. M. Babujian, Correlation function in WZNW model as a Bethe wave function for the Gaudin magnetics, In 'Gosen 1990, Proceedings, Theory of elementary particles' 12-23. (see high energy physics index 29 (1991) No. 12257) , 12–23 (1990).

[17] H. M. Babujian, Correlation function in WZNW model as a Bethe wave for the Gaudin magnetics, YERPHI-1261-47-90 (1990).

[18] H. M. Babujian, Off-shell Bethe Ansatz equation and N point correlators in SU(2) WZNW theory, J. Phys. **A26**, 6981–6990 (1993).

[19] H. M. Babujian and R. Flume, Off-shell Bethe Ansatz equation for Gaudin magnets and solutions of Knizhnik-Zamolodchikov equations, Mod. Phys. Lett. **A9**, 2029–2040 (1994).

[20] I. B. Frenkel and N. Y. Reshetikhin, Quantum affine algebras and holonomic difference equations, Commun. Math. Phys. **146**, 1–60 (1992).

[21] V. Schechtman and A. Varchenko, Arrangements of hyerplanes and Lie algebra homology, Invent. Math. **106**, 139–194 (1991).

[22] F. H. L. Essler and R. M. Konik, Applications of massive integrable quantum field theories to problems in condensed matter physics, In 'Shifman, M. (ed.) et al.: From fields to strings' **1**, 684–830 (2004).

[23] A. M. Tsvelik, Quantum field theory in condensed matter physics, Cambridge, UK: Univ. Pr. , 332 (1995).

[24] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization in Strongly Correlated Systems, Cambridge, UK: Univ. Pr. (1999).

[25] J. Links, H. Zhou, R. McKenzie, and M. Gould, Algebraic Bethe ansatz method for the exact calculation of energy spectra and form factors: applications to models of Bose-Einstein condensates and metallic nanograins, J. Phys. **A36**, R63–R104 (2003).

[26] D. Controzzi and F. H. L. Essler, Dynamical density correlation function of 1D Mott insulators in a magnetic field, Phys. Rev. **B66**, 165112 (2002).

[27] F. H. L. Essler and R. M. Konik, Dynamical Spin Response of Doped Two-Leg Hubbard-like Ladders, cond-mat/0607783 (2004).

[28] J. Links and H. Zhou, Exact form factors for the Josephson tunneling current and relative particle number fluctuations in a model of two coupled Bose-Einstein condensates, Lett. Math. Phys **60**, 275–282 (2002).

[29] W. Heisenberg, Der mathematische Rahmen der Quantentheorie der Wellenfelder, Zeitschrift für Naturforschung **1**, 608–622 (1946).

[30] G. Chew, The S-Matrix Theory of Strong Interaction, W. A. Benjamin, Inc. New York (1961).

[31] B. Schroer, T. T. Truong, and P. Weisz, Towards an explicit construction of the sine-Gordon theory, Phys. Lett. **B63**, 422–424 (1976).

[32] M. Karowski, H. J. Thun, T. T. Truong, and P. Weisz, On the uniqueness of a purely elastic S matrix in 1+1 dimensions, Phys. Lett. **B67**, 321–322 (1977).

[33] M. Karowski and H. J. Thun, Complete S matrix of the massive Thirring model, Nucl. Phys. **B130**, 295–308 (1977).

[34] A. B. Zamolodchikov and A. B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models, Annals Phys. **120**, 253–291 (1979).

[35] M. Karowski and P. Weisz, Exact form factors in (1+1)-dimensional field theoretic models with soliton behavior, Nucl. Phys. **B139**, 455–476 (1978).

[36] B. Berg, M. Karowski, and P. Weisz, Construction of Green functions from an exact S matrix, Phys. Rev. **D19**, 2477–2479 (1979).

[37] F. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory, Adv. Series in Math. Phys. **14**, World Scientific (1992).

[38] H. M. Babujian, A. Fring, M. Karowski, and A. Zapletal, Exact form factors in integrable quantum field theories: The sine-Gordon model, Nucl. Phys. **B538**, 535–586 (1999).

[39] H. Babujian and M. Karowski, Exact form factors for the scaling Z(N)-Ising and the affine A(N-1) Toda quantum field theories, Phys. Lett. **B575**, 144–150 (2003).

[40] H. Babujian, A. Foerster, and M. Karowski, Exact form factors in integrable quantum field theories: The scaling Z(N)-Ising model, Nucl. Phys. **B736**, 169–198 (2006).

[41] H. Babujian, M. Karowski, and A. Zapletal, Matrix Difference Equations and a Nested Bethe Ansatz, J. Phys. **A30**, 6425–6450 (1997).

[42] B. Berg and P. Weisz, Exact S-matrix of the chiral invariant SU(N) Thirring model, Nucl. Phys. **B146**, 205–214 (1978).

[43] E. Abdalla, B. Berg, and P. Weisz, More about the S-matrix of the chiral SU(N) Thirring model, Nucl. Phys. **B157**, 387–391 (1979).

[44] R. Köberle, V. Kurak, and J. A. Swieca, Scattering theory and 1/N expansion in the chiral Gross- Neveu model, Phys. Rev. **D20**, 897–902 (1979).

[45] H. M. Babujian, A. Foerster, and M. Karowski, Exact form factors of the SU(N) Gross-Neveu model and 1/N expansion, Nucl. Phys. **B825**, 396–425 (2010).

[46] A. Nakayashiki and Y. Takeyama, On form factors of SU(2) invariant Thirring model, math-ph/0105040 (2001).

[47] Y. Takeyama, Form factors of SU(N) invariant Thirring model, Publ. Res. Inst. Math. Sci. Kyoto **39**, 59–116 (2003).

[48] E. Mukhin, V. Tarasov, and A. Varchenko, Bethe Eigenvectors of Higher Transfer Matrices, math.QA/0605015 (2006).

[49] B. Berg, M. Karowski, V. Kurak, and P. Weisz, Factorized U(n) symmetric S matrices in two-dimensions, Nucl. Phys. **B134**, 125–132 (1978).

[50] R. Köberle and J. A. Swieca, Factorizable Z(N) models, Phys. Lett. **B86**, 209–210 (1979).

[51] V. Kurak and J. A. Swieca, Anti-particles as bound states of particles in the factorized S-matrix framework, Phys. Lett. **B82**, 289–291 (1979).

[52] M. Karowski, On the bound state problem in (1+1) dimensional field theories, Nucl. Phys. **B153**, 244–252 (1979).

[53] H. Babujian and M. Karowski, Exact form factors in integrable quantum field theories: The sine-Gordon model. II, Nucl. Phys. **B620**, 407–455 (2002).

[54] H. Babujian and M. Karowski, Sine-Gordon breather form factors and quantum field equations, J. Phys. **A35**, 9081–9104 (2002).

[55] N. Andrei and J. H. Lowenstein, Diagonalization of the chiral invariant Gross-Neveu hamiltonian, Phys. Rev. Lett. **43**, 1698 (1979).

[56] A. A. Belavin, Exact solution of the two-dimensional model with asymptotic freedom, Phys. Lett. **B87**, 117–121 (1979).

[57] H. Babujian, M. Karowski, and A. Zapletal, SU(N) Matrix Difference Equations and a Nested Bethe Ansatz, hep-th/9611005 (1996).

[58] H. Babujian, M. Karowski, and A. Zapletal, U(N) Matrix Difference Equations and a Nested Bethe Ansatz, hep-th/9611006 (1996).