$SU(N)$ and $O(N)$ off-shell nested Bethe ansatz and form factors

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The purpose of the "bootstrap program" for integrable quantum field theories in 1+1 dimensions is to construct explicitly a model in terms of its Wightman functions. In this article, this program is mainly illustrated in terms of the $SU(N)$ Gross-Neveu model and the $O(N)$ σ-model. Systems of $SU(N)$- and $O(N)$-matrix difference equations are solved by means of the off-shell version of the nested algebraic Bethe Ansatz. In the nesting process for the $O(N)$ case a new object, the Π-matrix, is introduced to overcome the complexities of the $O(N)$ group structure. Some explicit examples are discussed.

Keywords: Integrable quantum field theory, Form factors.

1. Introduction

The *bootstrap program* to formulate particle physics in terms of the scattering data, i.e. in terms of the S-matrix goes back to Heisenberg[1] and Chew.[2] Remarkably, this approach works very well for integrable quantum field theories in 1+1 dimensions.[3–8] The program does *not* start with any classical Lagrangian. Rather it classifies integrable quantum field theoretic models and in addition provides their explicit exact solutions in terms of all Wightman functions. We achieve contact with the classical models only, when at the end we compare our exact results with Feynman graph (or other) expansions which are usually based on Lagrangians.

One of the authors (M.K.) et al.[4] formulated the on-shell program i.e. the exact determination of the scattering matrix using the Yang-Baxter equations. The concept of generalized form factors was introduced by one of the authors (M.K.) et al.[7] In this article consistency equations were formulated which are expected to be satisfied by these quantities. Thereafter this approach was developed further and studied in the context of several explicit models by Smirnov.[9] In the present article we apply the form factor program for $SU(N)$ and $O(N)$ invariant S-matrices.[10–14] We have to apply the nested "off-shell"[a] Bethe ansatz to get the vectorial part of the form factors. We compare the $1/N$ expansions for the chiral $SU(N)$ Gross-Neveu model[15] and the nonlinear $O(N)$ σ-model[16,17] with our exact results for the form factors. The $SU(N)$ Gross-Neveu model is in particular interesting because the particles are anyons. Both models exhibits asymptotic freedom. Finally the Wightman functions should be obtained by taking integrals and sums over intermediate states. The explicit evaluation of all these integrals and sums remains an open challenge for almost all models, except the scaling Ising model.

2. The "bootstrap program"

The 'bootstrap program' for integrable quantum field theories in 1+1-dimensions provides the solution of a model in term of all its Wightman functions. The result is obtained in three steps:

(1) The S-matrix is calculated by means of general properties such as unitarity and crossing, the Yang-Baxter equations (which are a consequence of integrability) and the additional assumption of 'maximal analyticity'. This means that the two-particle S-matrix is an analytic function in the physical plane (of the Mandelstam variable $(p_1 + p_2)^2$) and possesses only those poles there which are of physical origin. The only input which depends on the model is the assumption of a particle spectrum with an underlining symmetry. A *classification* of all S-matrices obeying the given properties is obtained.

(2) Generalized form factors which are matrix elements of local operators

$$^{out}\langle\, \theta'_m, \ldots, \theta'_1\, |\mathcal{O}(x)|\, \theta_1, \ldots, \theta_n\, \rangle^{in}$$

are calculated by means of the S-matrix. More precisely, the "form factor equations" $(i) - (v)$ as listed in section 3 are solved.

(3) The Wightman functions are obtained by inserting a complete set of intermediate states. In particular the two point function for a hermitian operator $\mathcal{O}(x)$ reads

$$\langle\, 0\, |\mathcal{O}(x)\, \mathcal{O}(0)|\, 0\, \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{4\pi} \ldots \left|\langle\, 0\, |\mathcal{O}(0)|\, \theta_1, \ldots, \theta_n\, \rangle^{in}\right|^2 e^{-ix\sum p_i}.$$

Up to now a direct proof that these sums converge exists only for the scaling Ising model,[8,18–21] however, it was shown[22] that models with factorizing S-matrices exist within the framework of algebraic quantum field theory.

[a] "Off-shell" in the context of the Bethe ansatz means that the spectral parameters in the algebraic Bethe ansatz state are not fixed by Bethe ansatz equations in order to get an eigenstate of a Hamiltonian, but they are integrated over.

Integrability

Integrability in (quantum) field theories means that there exist infinitely many local (or non-local) conservation laws

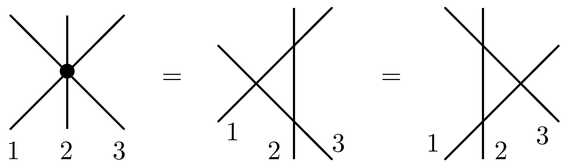
$$\partial_\mu J^\mu_L(t,x) = 0 \quad (L = \pm 1, \pm 3, \dots)\,.$$

A consequence of such conservation laws in 1+1 dimensions is that there is no particle production and the n-particle S-matrix is a product of 2-particle S-matrices

$$S^{(n)}(p_1,\dots,p_n) = \prod_{i<j} S_{ij}(p_i,p_j)\,.$$

If backward scattering occurs the 2-particle S-matrices will not commute and one has to specify the order. In particular for the 3-particle S-matrix there are two possibilities

$$S^{(3)} = S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$$



which yield the **"Yang-Baxter Equation".**

The two particle S-matrix is of the form $S_{\alpha\,\beta}^{\beta'\alpha'}(\theta_{12})$ where α,β etc. denote the type of the particles and the rapidity difference $\theta_{12} = \theta_1 - \theta_2$ is defined by $p_i = m_i(\cosh\theta_i, \sinh\theta_i)$. We also use the short hand notation $S_{12}(\theta_{12})$. It satisfies unitarity and crossing

$$S_{21}(\theta_{21})S_{12}(\theta_{12}) = 1 \tag{1}$$

$$S_{12}(\theta_1-\theta_2) = \mathbf{C}^{2\bar{2}}\,S_{\bar{2}1}(\theta_2+i\pi-\theta_1)\,\mathbf{C}_{\bar{2}2} = \mathbf{C}^{1\bar{1}}\,S_{2\bar{1}}(\theta_2-(\theta_1-i\pi))\,\mathbf{C}^{\bar{1}1} \tag{2}$$

where $\mathbf{C}^{1\bar{1}}$ and $\mathbf{C}_{1\bar{1}}$ are charge conjugation matrices.

Examples of integrable models in 1+1-dimensions are

- the $SU(N)$ Gross-Neveu[15] model described by the Lagrangian

$$\mathcal{L} = \bar\psi\, i\gamma\partial\,\psi + \frac{g^2}{2}\left((\bar\psi\psi)^2 - (\bar\psi\gamma^5\psi)^2\right)\,,$$

  where the Fermi fields form an $SU(N)$ multiplet.
- the nonlinear $O(N)$ σ-model defined by the Lagrangian and the constraint

$$\mathcal{L} = \frac{1}{2}\sum_{\alpha=1}^{N}(\partial_\mu\varphi_\alpha)^2 \qquad \text{with} \quad g\sum_{\alpha=1}^{N}\varphi_\alpha^2 = 1$$

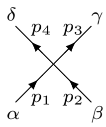
  where $\varphi_\alpha(x)$ is an isovector N-plett set of bosonic fields.

Further integrable quantum field theories are: the sine-Gordon, the Toda, the scaling Z_N-Ising, the $O(N)$ Gross-Neveu models etc.

The S-matrix

- The two particle S-matrix of the $SU(N)$ Gross-Neveu model is[23–26]

$$S_{\alpha\beta}^{\delta\gamma}(\theta) =$$



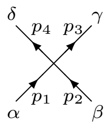
$$= \delta_{\alpha\gamma}\delta_{\beta\delta}\, b(\theta) + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c(\theta) \tag{3}$$

  where due to Yang-Baxter $c(\theta) = -\frac{2\pi i}{N\theta}\, b(\theta)$ holds and the highest weight amplitude is given as

$$a^{SU(N)}(\theta) = b(\theta) + c(\theta) = -\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right)}\,. \tag{4}$$

  There is a bound state pole at $\theta = i\eta = 2\pi i/N$ in the antisymmetric tensor sector which agrees with Swieca's[27] picture that the bound state of $N-1$ particles is to be identified with the anti-particle.
- The two particle S-matrix of the nonlinear $O(N)$ σ-model is[6]

$$S_{\alpha\beta}^{\delta\gamma}(\theta) =$$



$$= \delta_\alpha^\gamma\delta_\beta^\delta\, b(\theta) + \delta_\alpha^\delta\delta_\beta^\gamma\, c(\theta) + \delta^{\delta\gamma}\delta_{\alpha\beta}\, d(\theta) \tag{5}$$

  where due to Yang-Baxter $c(\theta) = -\frac{2\pi i}{(N-2)\theta}\, b(\theta)$ and crossing $b(\theta) = b(i\pi - \theta)$, $d(\theta) = c(i\pi - \theta)$ hold. The highest weight amplitude $b(\theta) + c(\theta)$ is given as

$$a^{O(N)}(\theta) = -\frac{\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} + \frac{1}{N-2} - \frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{2} + \frac{1}{N-2} + \frac{\theta}{2\pi i}\right)} \frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{N-2} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{1}{N-2} - \frac{\theta}{2\pi i}\right)}\,. \tag{6}$$

  For the Bethe ansatz used below it is more convenient to use instead of the real basis $|\alpha\rangle_r$, $(\alpha = 1, 2, \dots, N)$ a complex basis

$$\begin{cases} |\alpha\rangle = \frac{1}{\sqrt{2}}\left(|2\alpha - 1\rangle_r + i|2\alpha\rangle_r\right), \ \alpha = 1, \dots, [N/2] \\ |\bar\alpha\rangle = \frac{1}{\sqrt{2}}\left(|2\alpha - 1\rangle_r - i|2\alpha\rangle_r\right), \ \bar\alpha = \bar 1, \dots, \overline{[N/2]}\,. \end{cases}$$

  For N odd there is in addition $|0\rangle = |\bar 0\rangle = |N\rangle_r$. In (5) $\delta^{\delta\gamma}\delta_{\alpha\beta}$ is then replaced by $\mathbf{C}^{\delta\gamma}\mathbf{C}_{\alpha\beta}$ with the charge conjugation matrix $\mathbf{C}$.

3. Form factors

For a local operator $\mathcal{O}(x)$ the generalized form factors[7] are defined as

$$F_{\alpha_1\dots\alpha_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n) = \langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1, \dots, p_n\,\rangle_{\alpha_1\dots\alpha_n}^{in} \tag{7}$$

for $\theta_1 > \cdots > \theta_n$. For other orders of the rapidities they are defined by analytic continuation. The index α_i denotes the type of the particle with momentum p_i. We also use the short notations $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ or $F^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$.

We assume 'maximal analyticity' for the form factors which means that they are meromorphic and all poles in the 'physical strips' $0 \leq \operatorname{Im}\theta_{ij} \leq \pi$ have a physical interpretation. Together with the usual LSZ-assumptions[28] of local quantum field theory the following form factor equations can be derived:

(i) The Watson's equations describe the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces $i, j = i+1$ at the same time

$$F^{\mathcal{O}}_{\ldots ij\ldots}(\ldots,\theta_i,\theta_j,\ldots) = F^{\mathcal{O}}_{\ldots ji\ldots}(\ldots,\theta_j,\theta_i,\ldots)\, S_{ij}(\theta_{ij}) \tag{8}$$

for all possible arrangements of the θ's.

(ii) The crossing relation which implies a periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\begin{aligned} &{}^{\mathrm{out},\bar{1}}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, p_2,\ldots,p_n\,\rangle^{\mathrm{in,conn.}}_{2\ldots n} \\ &= \mathbf{C}^{\bar{1}1}\sigma^{\mathcal{O}}_1 F^{\mathcal{O}}_{1\ldots n}(\theta_1 + i\pi,\theta_2,\ldots,\theta_n) = F^{\mathcal{O}}_{2\ldots n1}(\theta_2,\ldots,\theta_n,\theta_1 - i\pi)\mathbf{C}^{1\bar{1}} \end{aligned} \tag{9}$$

where $\sigma^{\mathcal{O}}_{\alpha}$ takes into account the statistics of the particle α with respect to $\mathcal{O}$ (e.g., $\sigma^{\mathcal{O}}_{\alpha} = -1$ if α and $\mathcal{O}$ are both fermionic, these numbers can be more general for anyonic or order and disorder fields[29]).

(iii) There are poles determined by one-particle states in each sub-channel given by a subset of particles of the state in (7).
In particular the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n) = 2i\,\mathbf{C}_{12}\,F^{\mathcal{O}}_{3\ldots n}(\theta_3,\ldots,\theta_n)\left(\mathbf{1} - \sigma^{\mathcal{O}}_2 S_{2n}\ldots S_{23}\right). \tag{10}$$

(iv) If there are also bound states in the model the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = i\eta$ $(0 < \eta < \pi)$ such that

$$\operatorname*{Res}_{\theta_{12}=i\eta} F^{\mathcal{O}}_{12\ldots n}(\theta_1,\theta_2,\ldots,\theta_n) = F^{\mathcal{O}}_{(12)\ldots n}(\theta_{(12)},\ldots,\theta_n)\,\sqrt{2}\Gamma^{(12)}_{12} \tag{11}$$

where $\Gamma^{(12)}_{12}$ is the bound state intertwiner.[30,31]

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F^{\mathcal{O}}_{1\ldots n}(\theta_1 + \mu,\ldots,\theta_n + \mu) = e^{s\mu}\,F^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n) \tag{12}$$

if the local operator transforms under Lorentz transformations as $F^{\mathcal{O}} \to e^{s\mu}F^{\mathcal{O}}$ where s is the "spin" of $\mathcal{O}$.

These equations have been proposed by Smirnov[9] as generalizations of equations derived in the original articles.[7,8,32] They have been proven[33] by means of the LSZ-assumptions and 'maximal analyticity'.

We will now provide a constructive and systematic way of how to solve the equations (i) – (v) for the co-vector valued function $F^{\mathcal{O}}_{1\ldots n}$ once the scattering matrix is given.

3.1. *Two-particle form factors*

For the two-particle form factors the form factor equations (i) and (ii) are

$$\begin{aligned} F(\theta) &= F(-\theta)\, S(\theta) \\ F(i\pi-\theta) &= F(i\pi+\theta) \end{aligned} \tag{13}$$

for all eigenvalues of the two-particle S-matrix. For general theories Watson's[34] equations only hold below the particle production thresholds. However, for integrable theories there is no particle production and therefore they hold for all complex values of θ. It has been shown[7] that these equations together with "maximal analyticity" have a unique solution.

As an example we write the (highest weight) $SU(N)$ and $O(N)$ form factor functions[10,14]

$$F^{SU(N)}(\theta) = \exp \int_0^\infty \frac{dt}{t\sinh^2 t} e^{\frac{t}{N}} \sinh t\,(1-1/N)\,(1-\cosh t\,(1-\theta/(i\pi))) \tag{14}$$

$$F^{O(N)}(\theta) = \exp \int_0^\infty \frac{dt}{t\sinh t} \frac{1-e^{-2t/(N-2)}}{1+e^{-t}}\,(1-\cosh t\,(1-\theta/(i\pi))) \tag{15}$$

which are the minimal solution of (13) with $S(\theta) = a(\theta)$ as given by (4) and (6), respectively. In particular for $O(3)$ we have $F^{O(3)}(\theta) = (\theta - i\pi)\tanh\frac{1}{2}\theta$.

3.2. *The general form factor formula*

As usual[7] we split off the minimal part and write the form factor for n particles as

$$F^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1,\ldots,\theta_n) = K^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij})\,. \tag{16}$$

By means of the following "off-shell Bethe ansatz" for the (co-vector valued) K-function

$$\boxed{K^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\, \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z})} \tag{17}$$

we transform the complicated form factor equations $(i)-(v)$ into simple ones for the p-functions which are scalar and simple functions of $e^{\pm z_i}$. The **"off-shell Bethe ansatz"** state $\Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z})$ is obtained as a product of S-matrix elements and the integration contour $\mathcal{C}_{\underline{\theta}}$ encircles poles of $h(\underline{\theta},\underline{z})$ (see below). The scalar function

$$h(\underline{\theta},\underline{z}) = \prod_{i=1}^{n}\prod_{j=1}^{m} \phi_j(\theta_i - z_j) \prod_{1\le i<j\le m} \tau_{ij}(z_i - z_j) \tag{18}$$

depends only on the S-matrix (see below), whereas the p-function $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ depends on the operator.

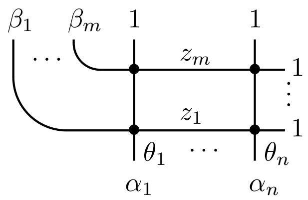
In case of higher rank r the **"nested Bethe ansatz state"** Ψ is of the form

$$\Psi_{\alpha_1 \dots \alpha_n}(\underline{\theta}, \underline{z}) = L_{\beta_1 \dots \beta_m}(\underline{z}) \Phi^{\beta_1 \dots \beta_m}_{\alpha_1 \dots \alpha_n}(\underline{\theta}, \underline{z}) \,. \tag{19}$$

For the co-vector valued function L which belongs to rank $r-1$ one makes an ansatz analogous to (17). Nesting means that one repeats this up to $SU(2)$, respectively $O(4)$ or $O(3)$. The number of Bethe ansatz levels is equal to the rank, i.e. $\mathrm{rank}(SU(N)) = N-1$ and $\mathrm{rank}(O(N)) = [N/2]$.

- For $SU(N)$ the basic Bethe ansatz co-vectors Φ of equation (19) may be depicted as

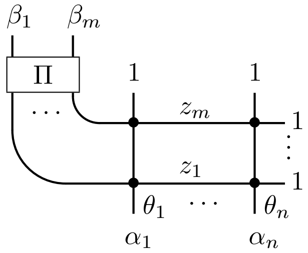
$\Phi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) =$



with $\begin{cases} 2 \le \beta_i \le N \\ 1 \le \alpha_i \le N \,. \end{cases}$

  It means that $\Phi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ is a product of S-matrix elements as given by the picture where at all crossing points of lines there is an S-matrix (3) and the sum over all indices of internal lines is to be taken.
- For $O(N)$ the basic Bethe ansatz co-vectors Φ is more complicated

$\Phi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) =$



with $\begin{cases} \beta_i = 2, \dots, (0), \dots, \bar{2} \\ \alpha_i = 1, 2, \dots, (0), \dots, \bar{2}, \bar{1} \end{cases}$

  The matrix Π maps the $O(N)$ S-matrix to the $O(N-2)$ one where the rank decreases by 1

$$S^{O(N-2)}_{ij} \Pi_{\dots ij \dots} = \Pi_{\dots ji \dots} S^{O(N)}_{ij} \,.$$

For $SU(N)$ the matrix Π is trivial because the S-matrix elements do not depend on N (for a suitable normalization and parameterization).

We concentrate here on the results for $O(N)$, the results for $SU(N)$ have been published[10–12] previously. For general N the functions ϕ_j and τ_{ij} in (18) depend on whether $i, j = e, o$ are even or odd. The form factor equations (ii) and (iii) imply equations for $\tilde{\phi}_j(z) = a(z)\phi_j(z)$ and $\tau_{ij}(z)$

$$\text{(ii)} : \begin{cases} \tilde{\phi}_j(z) = \tilde{b}(z+2\pi i)\tilde{\phi}_j(z+2\pi i) \\ \tau_{ij}(z-2\pi i)/\tilde{b}(2\pi i - z) = \tau_{ij}(z)/\tilde{b}(z) \end{cases}$$

$$\text{(iii)} : \tilde{\phi}_e(-z)\tilde{\phi}_o(-z-i\pi+i\eta)F(z)F(i\pi+z) = 1\ , \qquad \eta = \frac{2\pi}{N-2}$$

with $\tilde{b}(z) = b(z)/a(z)$. The equation for τ_{ij} from (iii) is more complicated[35] and skipped here. For $O(3)$ the functions ϕ_j do not depend on j and solutions for $\tilde{\phi}$ and τ are simple

$$\tilde{\phi}(z) = \frac{1}{z}\ , \quad \tau(z) = z^2.$$

In order that the form factors $F^{\mathcal{O}}(\underline{\theta})$ satisfy the form factor equations (i) - (iii) the p-function $p^{\mathcal{O}}(\underline{\theta}, \underline{\underline{z}})$ have to satisfy some simple equations. e. g.

$$\begin{aligned} p^{\mathcal{O}}(\underline{\theta}, \underline{\underline{z}}) &= p^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, \dots, \underline{\underline{z}}) \\ &= p^{\mathcal{O}}(\underline{\theta}, \dots, z_i^{(l)} + 2\pi i, \dots) \end{aligned}$$

For $SU(N)$ there are some additional phase factors because of the anyonic statistics of the fields and particles.

4. Examples:

4.1. *The chiral $SU(N)$-Gross-Neveu model*

The classical Lagrangian density is

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha\, i\gamma\partial\, \psi_\alpha + \frac{1}{2}g^2 \left(\left(\sum_{\alpha=1}^{N} \bar{\psi}_\alpha \psi_\alpha \right)^2 - \left(\sum_{\alpha=1}^{N} \bar{\psi}_\alpha \gamma^5 \psi_\alpha \right)^2 \right) \tag{20}$$

where ψ_α is an $SU(N)$ isovector N-plet of fermi fields. The quantum version of this model exhibit anyonic statistics.[24]

The p-function which gives the exact $SU(N)$ form factors for the field component $\psi^{(\pm)}(x) = \psi_1^{(\pm)}(x)$ is[10]

$$p^{\psi_1^{(\pm)}}(\underline{\theta}, \underline{z}) = \exp \pm\frac{1}{2}\left(\sum_{i=1}^{m} z_i - \left(1 - \frac{1}{N}\right) \sum_{i=1}^{n} \theta_i \right) \tag{21}$$

and the 1-particle form factor is

$$\langle\, 0 \,|\, \psi^{(\pm)}(0) \,|\, \theta \,\rangle_\alpha = \delta_{\alpha 1}\, e^{\mp\frac{1}{2}\left(1-\frac{1}{N}\right)\theta}\,.$$

The 3-particle form factor given by (16), (17) and (21) can be expressed in term of Meijer's G-functions. The $1/N$ expansion of the exact result for the operator

$\mathcal{O}(x) = -i\,(i\gamma\partial - m)\,\psi(x)$ is[12]

$$\begin{aligned}{}_{out}^{\;\gamma}\langle\,\theta_3\,|\,\mathcal{O}(0)\,|\,\theta_1,\theta_2\,\rangle^{in}_{\alpha\beta} &= \frac{2i\pi}{N}m \\ &\times\left(\delta^1_\alpha\delta^\gamma_\beta\frac{\sinh\theta_{23}}{\theta_{23}}\left(\frac{1}{\cosh\frac{1}{2}\theta_{23}}-\gamma^5\frac{1}{\sinh\frac{1}{2}\theta_{23}}\right)u(\theta_1)-(1,\alpha\leftrightarrow 2,\beta)\right)\end{aligned} \tag{22}$$

which agrees with the $1/N$ expansion in terms of Feynman graphs starting from the Lagrangian (20).

4.2. *The nonlinear $O(N)$ σ-model*

The model is defined by the Lagrangian and the constraint

$$\mathcal{L} = \frac{1}{2}\sum_{\alpha=1}^{N}\left(\partial_\mu\varphi_\alpha\right)^2 \;, \qquad g\sum_{\alpha=1}^{N}\varphi_\alpha^2 = 1 \tag{23}$$

where $\varphi_\alpha(x)$ is an isovector N-plett set of bosonic fields.

The p-function for the field $\varphi(x) = \varphi_1(x)$ is

$$p^\varphi(\underline{\theta},\underline{z}) = 1$$

and the 1-particle form factor is

$$\langle 0|\varphi(0)|\theta\rangle_\alpha = \delta_{\alpha 1}$$

The exact 3-particle form factor of $\varphi(x)$ for $O(3)$ can be calculated from

$$K^\varphi_{\underline{\alpha}}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} dz_2\, \tilde{h}(\underline{\theta},\underline{z})p^\varphi(\underline{\theta},\underline{z})\,L(z_{12})\tilde{\Phi}_{\underline{\alpha}}(\underline{\theta},\underline{z})$$

with $L(z) = \frac{(z-i\pi)}{z(z-2\pi i)}\tanh\frac{1}{2}z$. as[35]

$$F^\varphi_{\alpha\beta\gamma}(\underline{\theta}) = \left(\theta_{23}\delta^1_\alpha\mathbf{C}_{\beta\gamma} - (\theta_{13}-2\pi i)\,\delta^1_\beta\mathbf{C}_{\alpha\gamma} + \theta_{12}\delta^1_\gamma\mathbf{C}_{\alpha\beta}\right)G(\theta_{12})G(\theta_{13})G(\theta_{23})$$

where $G(\theta) = \frac{(\theta-i\pi)}{\theta(\theta-2\pi i)}\tanh^2\frac{1}{2}\theta$. This agrees with results of Balog et al.[36] obtained using different techniques. The $1/N$ expansion of 3-particle form factor of the operator $\mathcal{O}(x) = i(\Box+m^2)\varphi(x)$ is[35]

$$\begin{aligned}&F^{\mathcal{O}}{}_{\alpha\beta\gamma}(\theta_1,\theta_2,\theta_3) \\ &= -\frac{8\pi i}{N}m^2\left(\delta^1_\alpha\mathbf{C}_{\beta\gamma}\frac{\sinh\theta_{23}}{i\pi-\theta_{23}} + \delta^1_\beta\mathbf{C}_{\alpha\gamma}\frac{\sinh\theta_{13}}{i\pi-\theta_{13}} + \delta^1_\gamma\mathbf{C}_{\alpha\beta}\frac{\sinh\theta_{12}}{i\pi-\theta_{12}}\right)\end{aligned}$$

which agrees with the $1/N$ expansion in terms of Feynman graphs starting from the Lagrangian (23).

Acknowledgments

The authors have profited from discussions with A. Fring, R. Schrader, F. Smirnov and A. Belavin. H.B. thanks R. Flume, R. Poghossian and P. Wiegmann for valuable discussions. H.B. and M.K. were supported by Humboldt Foundation and H.B. is also supported by the Armenian grant 11-1_c028. A.F. acknowledges support from DAAD (Deutscher Akademischer Austausch Dienst) and CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).

References

1. W. Heisenberg, *Zeitschrift für Naturforschung* **1**, 608 (1946).
2. G. Chew, *W. A. Benjamin, Inc. New York* (1961).
3. B. Schroer, T. T. Truong and P. Weisz, *Phys. Lett.* **B63**, 422 (1976).
4. M. Karowski, H. J. Thun, T. T. Truong and P. Weisz, *Phys. Lett.* **B67**, 321 (1977).
5. M. Karowski and H. J. Thun, *Nucl. Phys.* **B130**, 295 (1977).
6. A. B. Zamolodchikov and A. B. Zamolodchikov, *Annals Phys.* **120**, 253 (1979).
7. M. Karowski and P. Weisz, *Nucl. Phys.* **B139**, 455 (1978).
8. B. Berg, M. Karowski and P. Weisz, *Phys. Rev.* **D19**, 2477 (1979).
9. F. Smirnov, *Adv. Series in Math. Phys. 14, World Scientific* (1992).
10. H. M. Babujian, A. Foerster and M. Karowski, *J. Phys.* **A41**, p. 275202 (2008).
11. H. M. Babujian, A. Foerster and M. Karowski, *SIGMA* **2**, paper 082, 16 pages (2007).
12. H. M. Babujian, A. Foerster and M. Karowski, *Nucl. Phys.* **B825**, 396 (2010).
13. H. M. Babujian, A. Foerster and M. Karowski, *Theor. Math. Phys.* **155**, 512 (2008).
14. H. M. Babujian, A. Foerster and M. Karowski, *J. Phys.* **A45**, p. 055207 (2012).
15. D. J. Gross and A. Neveu, *Phys. Rev.* **D10**, p. 3235 (1974).
16. A. M. Polyakov, *Phys. Lett.* **B59**, 79 (1975).
17. K. Pohlmeyer, *Commun. Math. Phys.* **46**, 207 (1976).
18. R. Z. Bariev, *Phys. Lett.* **A55**, 456 (1976).
19. B. M. McCoy, C. A. Tracy and T. T. Wu, *Phys. Rev. Lett.* **38**, 793 (1977).
20. M. Sato, T. Miwa and M. Jimbo, *Proc. Japan Acad.* **53**, 6 (1977).
21. V. P. Yurov and A. B. Zamolodchikov, *Int. J. Mod. Phys.* **A6**, 4557 (1991).
22. G. Lechner, *Commun.Math.Phys.* **277**, 821 (2008).
23. B. Berg, M. Karowski, V. Kurak and P. Weisz, *Nucl. Phys.* **B134**, 125 (1978).
24. R. Koberle, V. Kurak and J. A. Swieca, *Phys. Rev.* **D20**, 897 (1979).
25. B. Berg and P. Weisz, *Nucl. Phys.* **B146**, 205 (1978).
26. E. Abdalla, B. Berg and P. Weisz, *Nucl. Phys.* **B157**, 387 (1979).
27. V. Kurak and J. A. Swieca, *Phys. Lett.* **B82**, 289 (1979).
28. H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cim.* **1**, 205 (1955).
29. H. Babujian, A. Foerster and M. Karowski, *Nucl. Phys.* **B736**, 169 (2006).
30. M. Karowski, *Nucl. Phys.* **B153**, 244 (1979).
31. H. Babujian and M. Karowski, *Nucl. Phys.* **B620**, 407 (2002).
32. M. Karowski, *in: W. Rühl (Ed.), Field theoretic methods in particle physics, Plenum, New York, (1980)* , 307 Presented at Kaiserslautern NATO Inst. 1979.
33. H. M. Babujian, A. Fring, M. Karowski and A. Zapletal, *Nucl. Phys.* **B538**, 535 (1999).
34. K. M. Watson, *Phys. Rev.* **95**, 228 (1954).
35. H. M. Babujian, A. Foerster and M. Karowski, *to be published* .
36. J. Balog and M. Niedermaier, *Nucl. Phys.* **B500**, 421 (1997).